Econ 205 - Slides from Lecture 4

Joel Sobel

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Mean-Value Theorem

Theorem (Mean Value Theorem)

If f is real valued and continuous on $[a, b]$, and differentiable on (a, b), then $\exists$ a point $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c)$$

There must be a point between a and b such that the derivative at that point (i.e. slope of the tangent to the curve) is the same as that of the line connecting our two points.

MVT and First Order Approximations

One form:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Another form:

$$f(x) = f(x_0) + f'(c)(x - x_0)$$

Proof of MVT

Proof.

Define

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

We know that $g(x)$ is continuous on compact $[a, b]$. Thus $g(x)$ attains its maximum and minimum on $[a, b]$. Further, $g(a) = g(b)$, so either g has a local max at some point $c \in (a, b)$ or $g(x) \leq g(a)$ for all $x \in [a, b]$ so g has a local min at some point $c \in (a, b)$. In either case, there exists $c \in (a, b)$ such that $g'(c) = 0$. Thus

$$\begin{aligned} g'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ &= 0. \end{aligned}$$

□

First Differential Equation

Theorem

Suppose f is real valued and continuous on $[a, b]$, and differentiable on (a, b). If $f'(x) \equiv 0$ for $x \in (a, b)$, then f is constant.

The converse is true too: If f is constant, then it is differentiable and its derivative is always zero.

Proof.

By the Mean Value Theorem, for all $x \in [a, b]$,

$$f(x) - f(a) = f'(c)(x - a)$$

for some $c \in [a, x]$. Since the right-hand side is zero by assumption, we have $f(x) = f(a)$ for all x, which is the desired result. □

Theorem

Suppose that f is a continuous real-valued function defined on the interval $(-1,1)$. Further suppose that $f'(x)$ exists for all $x \neq 0$ and that $lim_{x\to 0} f'(x)$ exists. f is differentiable at $x = 0$ and f' is continuous at $x = 0$.

Proof.

By the Mean Value Theorem,

$$\frac{f(x) - f(0)}{x} = f'(c)$$

for c between 0 and x. Since f' is continuous at 0, $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$. It follows that

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{c\to 0^+} f'(c) = \lim_{c\to 0^-} f'(c) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x},$$

which establishes that $f'(0)$ exists and is equal to $\lim_{c\to 0} f'(c)$. □

Theorem

If

$$g : \mathbb{R} \longmapsto \mathbb{R}$$

is the inverse of

$$f : \mathbb{R} \longmapsto \mathbb{R}$$

and if f is strictly increasing, differentiable and $f' > 0$, then

$$g'(f(x)) = \frac{1}{f'(x)}$$

Theorem

(L'Hopital's Rule) *Consider functions f and g differentiable on $[a, b)$, where $g'(x) \neq 0$ on $[a, b)$.*
If either

1. $\lim_{x \to b^-} f(x) = 0,$ *and* $\lim_{x \to b^-} g(x) = 0$
   OR
2. $\lim_{x \to b^-} f(x) = \infty,$ *and* $\lim_{x \to b^-} g(x) = \infty$

and further

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$$

Then

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = L$$

L'Hopital's Rule is a useful way to evaluate indeterminate forms $(0/0)$.

You prove that the rule works by using a variation of the mean value theorem. In Case 1, what is going on is that $f(x) = f(b) + f'(c)(x - b)$ for some $c \in (x, b)$ and similarly $g(x) = g(b) + g'(d)(x - b)$. Since $f(b) = g(b) = 0$ (loosely), the ratio of f to g is the ratio of derivatives of f and g. The trouble is that these derivatives are evaluated at different points. The good news is that you can prove a version of the Mean-Value Theorem that allows you to take $c = d$. This enables you to prove the theorem.

Taylor's Theorem

1. We know about zero-th order approximations: Constants.
2. We know about first-order approximations: Linear functions with slope equal to derivative.
3. What about higher-order approximations?

An nth-order approximation is a polynomial of degree n, that is a function of the form

$$a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n.$$

Technically, the degree of a polynomial is the largest power of n that appears with non-zero coefficient. So this polynomial has degree n if and only if $a_n \neq 0$.
An n^{th} order approximation of the function f at x is a polynomial of degree at most n, A_n that satisfies

$$\lim_{y \to x} \frac{f(y) - A_n(y)}{(y - x)^n} = 0.$$

Higher-Order Derivatives

Definition

The *n*th derivative of a function f, denoted f^n, is defined inductively to be the derivative of $f^{(n-1)}$.
We say f is *of class* C^n on (a, b) $(f \in C^n)$ if $f^{(n)}(x)$ exists and is continuous $\forall\, x$.

One can check that, just as differentiability of f implies continuity of f, if f^n exists, then f^{n-1} is continuous.

Taylor's Theorem

Theorem (Taylor's Theorem)

Let $f \in C^n$ and assume that f^n exists on (a, b) and let c and d be any points in (a, b). Then there exists a point t between c and d and a polynomial A_n of degree at most n such that

$$f(d) = A_n(d) + \frac{f^{(n+1)}(t)}{(n+1)!}(d-c)^{n+1}, \tag{1}$$

where A_n is the Taylor Polynomial for f centered at c:

$$A_n(d) = \sum_{j=0}^{n} \frac{f^{(k)}(c)}{k!}(d-c)^k.$$

The theorem decomposes f into a polynomial and an error term $E_n = \frac{f^{(n+1)}(t)}{(n+1)!}(d-c)^{n+1}$.

$$f(x) = A_n(x) + E_n(x)$$

where

$$\lim_{d \to c} \frac{E_n}{(d-c)^n} = 0$$

The Taylor polynomial is, in fact, the nth order approximation of f at c.

Uses of the theorem

1. "Smooth" functions have polynomial approximations.
2. Approximations.
3. Conceptual properties of linear and quadratic approximations.

Aside: Exponential and Log

$f(x) = \log x$. [Always base e.] Defined on $(0, \infty)$. $g(y) = e^y$. Defined on $\mathbb{R}$.

Properties:

1. $f'(x) = \frac{1}{x} > 0$.
2. $\lim_{x \to 0^+} f(x) = -\infty$,
3. $f(1) = 0$, $f(e) = 1$
4. $\lim_{x \to \infty} f(x) = \infty$.
5. $g'(y) = g(y) > 0$.
6. $\lim_{x \to -\infty} g(y) = 0$.
7. $\lim_{x \to \infty} g(y) = \infty$
8. $g(0) = 1$.
9. $g \circ f(x) = x, f \circ g(y) = y$.

Example

- $f^{(k)}(x) = x^{-k}(-1)^{k-1}(k-1)!$
- so $f^{(k)}(1) = (-1)^{k-1}(k-1)!$
- Hence:

$$f(x) = \sum_{k=1}^{N} (-1)^{k-1} \frac{(x-1)^k}{k} + E_N$$

  where $E_N = (-1)^N \frac{(y-1)^N}{N+1}$ for y between 1 and x.

- $$\log(x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-1)^k}{k}.$$

  Sometimes this formula is written in the equivalent form:

$$log(y+1) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{y^k}{k}.$$