Econ 205 - Slides from Lecture 3

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Differentiation

Calculus ideas:

1. Linear functions are easy to deal with.
2. Not all functions are linear.
3. Lots of functions are “locally” linear.
4. So: sacrifice “global” analysis of linear functions with “local” analysis of a larger class of functions.

For this to work you need to know how to approximate functions with linear functions.
This is what differentiation is about.

Constant Approximations

Definition

The *zero-th order approximation* of the function f at a point a in the domain of f is the function $A_0(x) \equiv f(a)$.

Good news: It does approximate at a point.
Bad news: It does nothing else.

Linear Approximation

Replace A_0 with the best linear approximation to f.
A line that intersects the graph of f at the points $(x, f(x))$ and the point $(x+\delta, f(x+\delta))$. This line has slope given by:

$$\frac{f(x+\delta)-f(x)}{(x+\delta)-x} = \frac{f(x+\delta)-f(x)}{\delta}$$

When f is linear, the line with this slope is f.

Limiting Argument

When δ is small:

$$\lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{(x+\delta) - x} \qquad = \qquad \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

Note:

- The denominator of the expression does not make sense when $\delta = 0$.
- For the limit to make sense x must be an interior point of the domain of f.
- If the limit exists, we say that f is differentiable at x and call the limit the derivative of f at x. There are two "one-sided" versions of this definition.

One-sided versions

$$f : (a, b) \longrightarrow \mathbb{R}$$

and $x \in (a, b)$.

Define if they exist

$$f'_+ \equiv \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x}$$

$$(\lim_{\delta \to 0^+} \frac{f(x + \delta) - f(x)}{\delta})$$

and

$$f'_- \equiv \lim_{y \to x^-} \frac{f(y) - f(x)}{y - x}$$

$$(\lim_{\delta \to 0^-} \frac{f(x + \delta) - f(x)}{\delta})$$

Definition

If both left and right hand derivative exist and are equal at a point x, then the function is said to be differentiable at x and the common value is called the derivative.

There are many ways to denote the derivative of the function f at the point x: $f'(x)$, $Df(x)$, and $\frac{df}{dx}(x)$ are all common.

Example

$f(x) = x^2$

$$\begin{aligned} f'(x) &= \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \\ &= \lim_{y \to x} \frac{y^2 - x^2}{y - x} \\ &= \lim_{y \to x} \frac{(y - x)(y + x)}{y - x} \\ &= \lim_{y \to x} (y + x) \\ &= 2x \end{aligned}$$

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

So we get from this that

$$\begin{array}{ll} f'_+(0) = 1 & \text{right hand derivative} \\ f'_-(0) = -1 & \text{left hand derivative} \end{array}$$

No derivative exists since

$$f'_+(0) \quad \neq \quad f'_-(0)$$

First-Order Approximation

$$A_1(y) \equiv f(x) + f'(x)(y - x).$$

This is the equation of the line with slope $f'(x)$ that passes through $(x, f(x))$. It follows from the definition of the derivative that

$$lim_{y \to x} \frac{f(y) - A_1(y)}{y - x} = 0.$$

A_1 is a better approximation to f than A_0.

More Approximation

1. Linear approximation A_1 close to f when y is close to x.
2. It is also the case that the linear approximation is close to f if you divide the difference by something really close to zero $(y - x)$.

Differentiable implies Continuous

"It is harder to be differentiable than to be continuous."

Theorem

If

$$f : X \longrightarrow \mathbb{R}$$

where X is an open interval is differentiable at a point x, then f is continuous at x.

Proof.

$$
\begin{aligned}
\lim_{y\to x} f(y) - f(x) &= \lim_{y\to x} \frac{f(y)-f(x)}{y-x}(y-x) \\
&= f'(x)\cdot 0 \\
&= 0
\end{aligned}
$$

because

$$\lim_{y\to x} g(y)h(y) = \left(\lim_{y\to x} g(y)\right)\left(\lim_{y\to x} h(y)\right)$$

□

Simple Facts

Theorem

Suppose f and g are defined on an open interval containing point x and both functions are differentiable at x. Then, $(f+g)$, $f \cdot g$, and f/g are differentiable at x (the last of these provided $g'(x) \neq 0$).

1. $$(f+g)'(x) = f'(x) + g'(x)$$

2. $$(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

3. $$(\frac{f}{g})'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Proof.

1. Easy.
2. Let $h = fg$. Then

   $$h(y) - h(x) = f(y)[g(y) - g(x)] + g(x)[f(y) - f(x)]$$

   If we divide this by $y - x$ and note that $f(y) \longrightarrow f(x)$, and $g(y) \longrightarrow g(x)$ as $y \longrightarrow x$, then the result follows.
   Then the result follows.
3. Now let $h = f/g$. Then

   $$\frac{h(y) - h(x)}{y - x} = \frac{1}{g(y)g(x)} \left[g(x)\frac{f(y) - f(x)}{y - x} - f(x)\frac{g(y) - g(x)}{y - x} \right]$$

   Now let $y \longrightarrow x$.

□

Chain Rule

Theorem (Chain Rule)

Suppose that

$$g : X \longrightarrow Y$$

and

$$f : Y \longrightarrow \mathbb{R}$$

that g is differentiable at x and that f is differentiable at $y = g(x) \in Y$. Then

$$\begin{aligned}(f \circ g)'(x) &= f'(y)g'(x) \\ &= f'(g(x))g'(x)\end{aligned}$$

Define the function

$$h(z) = \begin{cases} \frac{f(g(z))-f(g(x))}{g(z)-g(x)} & \text{if } g(z)-g(x) \neq 0 \\ f'(g(x)) & \text{if } g(z)-g(x) = 0 \end{cases}$$

$$\lim_{z \to x} \frac{f(g(z)) - f(g(x))}{z - x} = \lim_{z \to x} \left(h(z) \cdot \frac{g(z) - g(x)}{z - x} \right)$$

$$= \left[\lim_{z \to x} h(z) \right] \left[\lim_{z \to x} \frac{g(z) - g(x)}{z - x} \right]$$

To complete the proof it suffices to show that $\lim_{z \to x} h(z) = f'(g(x))$.
Let $\varepsilon > 0$.
There is $\delta_1 > 0$ such that if

$$0 <\mid y - g(x) \mid< \delta_1$$

implies

$$\mid \frac{f(y) - f(g(x))}{y - g(x)} - f'(g(x)) \mid< \varepsilon.$$

There is $\delta > 0$ such that if

$$0 <\mid z - x \mid< 0,$$

then

$$\mid g(z) - g(x) \mid< \delta_1.$$

Hence if $0 <\mid z - x \mid< \delta$ and $g(z) \neq g(x)$, then $\mid \frac{f(g(z)) - f(g(x))}{g(z) - g(x)} - f'(g(x)) \mid< \varepsilon$. This means that if $0 <\mid z - x \mid< \delta$ and $g(z) \neq g(x)$, then $\mid h(z) - f(g(x)) \mid< \varepsilon$. To complete the proof just note that if $g(z) = g(y)$ then $h(z) = f'(g(x))$.

Definition

x^* is a global maximizer of the function f on the interval $[a, b]$ if $x^* \in [a, b]$ and $f(x^*) \geq f(x)$ for all $x \in [a, b]$. We say *globally* since it maximizes the function over the whole domain. However, we say x^* is a *local maximizer* of the function f if $\exists$ a segment (a, b) such that

$$f(x^*) \geq f(x), \qquad \forall\, x \in (a, b)$$

What we call global max here is normally just called max.

Definition

We say x^* is a *local minimizer* of the function f if $\exists$ a segment (a, b) such that

$$f(x^*) \leq f(x), \qquad \forall x \in (a, b)$$

- A maximum that occurs at the boundary of the domain is not a local maximum.
- A maximum that occurs in the interior of the domain is a local maximum.
- Assume f is defined on $[a, b]$. If f has a local max at $c \in (a, b)$ and if $f'(c)$ exists, then

$$f'(c) = 0.$$

First Order Condition

Proof.

For $\mid x - c \mid < \delta$, we have

$$f(x) - f(c) \leq 0$$

Therefore, if $x \in (c, c + \delta)$, then

$$\frac{f(x) - f(c)}{x - c} \leq 0 \tag{1}$$

while for $x \in (c - \delta, c)$

$$\frac{f(x) - f(c)}{x - c} \geq 0. \tag{2}$$

That is, right derivative is not positive and left derivative is not negative.

Since f is differentiable at c, both left and right derivatives exist and they are equal. □

Comments

1. Same argument to conclude that if c is a local minimum and $f'(c)$ exists, then $f'(c) = 0$.
2. The equation $f'(c) = 0$ is called a first-order condition.
3. Problems:
   3.1 Doesn't distinguish local max from local min.
   3.2 Local (not global)
   3.3 $f'(c) = 0$ can be neither local min nor local max.
4. If $f'(x) \neq 0$ for all $x \in (a, b)$, then max and min occur at boundary points.

Solving Optimization Problems

1. Solve the equation $f'(c) = 0$.
2. Call the set of solutions Z.
3. Evaluate $f(x)$ for all $x \in Z \cup \{a, b\}$.
4. The highest value of f over this set is the maximum.
5. The lowest value is the minimum.

Monotonicity

Definition

The function f is increasing if $x > y$ implies $f(x) \geq f(y)$. The function f is strictly increasing if $x > y$ implies $f(x) > f(y)$. The function is increasing in a neighborhood of x if there exists $\delta > 0$ such that if $y \in (x - \delta, x + \delta)$ then $x > y$ implies $f(x) \geq f(y)$. The function is strictly increasing in a neighborhood of x if there exists $\delta > 0$ such that if $y \in (x - \delta, x + \delta)$ then $x > y$ implies $f(x) > f(y)$.

There are analogous definitions for deceasing and strictly decreasing. A function that is either (strictly) increasing everyone or decreasing everywhere is called (strictly) monotonic.

and derivatives

Theorem

If f is a real valued function differentiable on (a, b), then

1. *If $f'(x) > 0$, then f is strictly increasing in the neighborhood of x.*
2. *If $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing.*
3. *If f is increasing in the neighborhood of x, then $f'(x) \geq 0$.*
4. *If f is increasing, then $f'(x) \geq 0$ for all $x \in (a, b)$.*

Warning, you can have functions that are strictly increasing but whose derivative is sometimes zero: $f(x) = x^3$ is again the standard example.

Graphing

Mean-Value Theorem

Theorem (Mean Value Theorem)

If f is real valued and continuous on $[a, b]$, and differentiable on (a, b), then $\exists$ a point $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(c)$$

There must be a point between a and b such that the derivative at that point (i.e. slope of the tangent to the curve) is the same as that of the line connecting our two points.

MVT and First Order Approximations

One form:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Another form:

$$f(x) = f(x_0) + f'(c)(x - x_0)$$

Proof of MVT

Proof.

Define

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

We know that $g(x)$ is continuous on compact $[a, b]$. Thus $g(x)$ has a max at some point $c \in (a, b)$ and $g'(c) = 0$. Thus

$$\begin{aligned} g'(c) &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ &= 0. \end{aligned}$$

□

First Differential Equation

Theorem

Suppose f is real valued and continuous on $[a, b]$, and differentiable on (a, b). If $f'(x) \equiv 0$ for $x \in (a, b)$, then f is constant.

The converse is true too: If f is constant, then it is differentiable and its derivative is always zero.

Proof.

By the Mean Value Theorem, for all $x \in [a, b]$,

$$f(x) - f(a) = f'(c)(x - a)$$

for some $c \in [a, x]$. Since the right-hand side is zero by assumption, we have $f(x) = f(a)$ for all x, which is the desired result. □

Theorem

Suppose that f is a continuous real-valued function defined on the interval $(-1, 1)$. Further suppose that $f'(x)$ exists for all $x \neq 0$ and that $lim_{x \to 0} f'(x)$ exists. f is differentiable at $x = 0$ and f' is continuous at $x = 0$.

Proof.

By the Mean Value Theorem,

$$\frac{f(x) - f(0)}{x} = f'(c)$$

for c between 0 and x. Since f' is continuous at 0, $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x)$. It follows that

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{c \to 0^+} f'(c) = \lim_{c \to 0^-} f'(c) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x},$$

which establishes that $f'(0)$ exists and is equal to $\lim_{c \to 0} f'(c)$. $\square$

Theorem

If

$$g : \mathbb{R} \longmapsto \mathbb{R}$$

is the inverse of

$$f : \mathbb{R} \longmapsto \mathbb{R}$$

and if f is strictly increasing and differentiable, then

$$g'(f(x)) = \frac{1}{f'(x)}$$

Theorem

(L'Hopital's Rule) *Consider functions f and g differentiable on $[a, b)$, where $g'(x) \neq 0$ on $[a, b)$.*
If either

1. $\lim_{x \to b^-} f(x) = 0, \qquad$ *and* $\qquad \lim_{x \to b^-} g(x) = 0$
   OR
2. $\lim_{x \to b^-} f(x) = \infty, \qquad$ *and* $\qquad \lim_{x \to b^-} g(x) = \infty$

and further

$$\lim_{x \to b^-} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$$

Then

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = L$$

L'Hopital's Rule is a useful way to evaluate indeterminate forms $(0/0)$.

You prove that the rule works by using a variation of the mean value theorem. In Case 1, what is going on is that $f(x) = f(b) + f'(c)(x - b)$ for some $c \in (x, b)$ and similarly $g(x) = g(b) + g'(d)(x - b)$. Since $f(b) = g(b) = 0$ (loosely), the ratio of f to g is the ratio of derivatives of f and g. The trouble is that these derivatives are evaluated at different points. The good news is that you can prove a version of the Mean-Value Theorem in which allows you to take $c = d$. This enables you to prove the theorem.