# Econ 205 - Slides from Lecture 2

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# Limits of Functions

Take any function $f$,

$$f : X \longrightarrow \mathbb{R}$$

for $X = (c, d)$ (so note that it is an open set) and take $a \in X$.

## Definition

$y$ is called the *limit of f from the right at a* (right hand limit) if, for every $\varepsilon > 0$, there is a $\delta$ such that

$$0 < x - a < \delta \quad \Longrightarrow \mid f(x) - y \mid < \varepsilon$$

this is also denoted as

$$y = \lim_{x \to a^+} f(x)$$

*Note*: $x > a$, so $x$ is to the right of $a$.

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$$f : X \longrightarrow \mathbb{R}$$

for $X = (c, d)$ (so note that it is an open set) and take $a \in X$.

## Definition

$y$ is called the *limit of $f$ from the left at $a$* (left-hand limit) if, for every $\varepsilon > 0$, there is a $\delta$ such that

$$0 < a - x < \delta \quad \Longrightarrow \mid f(x) - y \mid < \varepsilon$$

this is also denoted as

$$y = \lim_{x \to a^-} f(x)$$

*Note*: $x < a$, so $x$ is to the left of $a$.

# LIMIT

## Definition

$y$ is called the *limit of $f$ at $a$* if

$$\begin{aligned} y &= \lim_{x \to a^-} f(x) \\ &= \lim_{x \to a^+} f(x) \end{aligned}$$

and we write

$$y = \lim_{x \to a} f(x)$$

$y$ is the limit of $f$ at $a$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 <\mid x - a \mid< \delta \qquad \Longrightarrow \qquad \mid f(x) - y \mid< \varepsilon$$

*Note*: The definition of the limit of a function at the point $a$ does not require the function to be defined at $a$.

# Examples

Consider the function

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

$$\begin{aligned} \lim_{x \to 0^-} f(x) \;&=\; 0 \;=\; \lim_{x \to 0^+} f(x) \\ \Rightarrow \;\; \lim_{x \to 0} f(x) \;&=\; 0 \end{aligned}$$

But the value of the function at the point 0 is not equal to the limit. The example shows that it is possible for $\lim_{x \to a} f(x)$ to exist but for it to be different from $f(a)$. Since you can define the limit without knowing the value of $f(a)$, this observation is mathematically trivial. It highlights a case that we wish to avoid, because we want a function's value to be approximated by nearby values of the function.

# Properties

**Theorem**

*Limits are unique. That is, if* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} f(x) = L'$*, then* $L = L'$*.*

**Proof.**

Assume that $L \neq L'$ and argue to a contradiction. Let $\varepsilon = \mid L - L' \mid /2$. Given this $\varepsilon$ let $\delta^* > 0$ have the property that $\mid f(x) - L \mid$ and $\mid f(x) - L' \mid$ are less than $\varepsilon$ when $0 < \mid x - a \mid < \delta^*$. (This is possible by the definition of limits.) Since

$$\mid f(x) - L \mid + \mid f(x) - L' \mid \geq \mid L - L' \mid \tag{1}$$

it follows that $\varepsilon \geq \mid L - L' \mid$, which is not possible. □

The theorem simply states that the function cannot be close to two different things at the same time. Inequality (inequality (1)) is called the triangle inequality. It says (in one dimension) that the distance between two numbers is no larger that the distance between the first number and a third number plus the distance between the second number and the third number. In general, it says that the shortest distance between two points in a straight line.

### Theorem

*A limit exists if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if*

$$0 <| x - a |< \delta \ implies \ that \quad | f(x) - f(a) |< \varepsilon$$

### Theorem

*If $f$ and $g$ are functions defined on a set $S$, $a \in (\alpha, \beta) \subset S$ and $lim_{x \to a} f(x) = M$ and $lim_{x \to a} g(x) = N$, then*

1. $lim_{x \to a} (f + g)(x) = M + N$
2. $lim_{x \to a} (fg)(x) = MN$
3. $lim_{x \to a} \left(\frac{f}{g}\right)(x) = M/N$ *provided* $N \neq 0$.

For the first part, given $\varepsilon > 0$, let $\delta_1 > 0$ be such that if $0 <| x - a |< \delta_1$ then $| f(x) - M |< \varepsilon/2$ and $\delta_2 > 0$ be such that if $0 <| x - a |< \delta_2$ then $| g(x) - N |< \varepsilon/2$. This is possible by the definition of limit. If $\delta = \min\{\delta_1, \delta_2\}$, then $0 <| x - a |< \delta$ implies

$$| f(x) + g(x) - M - N | \leq | f(x) - M | + | g(x) - N | < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

The first inequality follows from the triangle inequality while the second uses the definition of $\delta$. This proves the first part of the theorem.

For the second part, set $\delta_1$ so that if $0 <| x - a |< \delta_1$ then $| f(x) - M |< \sqrt{\varepsilon}$ and so on.
For the third part, note that when $g(x)$ and $N$ are not equal to zero

$$\frac{f(x)}{g(x)} - \frac{M}{N} = \frac{g(x)\,(f(x) - M) + f(x)\,(N - g(x))}{g(x)N}.$$

So given $\varepsilon > 0$, find $\delta > 0$ so small that if $0 <| x - a |< \delta$, then $\frac{|f(x)|}{|g(x)N|} < \frac{2|M|}{N^2}$, $| f(x) - M |< N\varepsilon/2$ and $| g(x) - N |< \frac{N^2}{4|M|}\varepsilon/2$.
This is possible provided that $N \neq 0$.

## Definition

We say $f : X \to \mathbb{R}$ is *continuous at a* if

$$f(a) = \lim_{x \to a} f(x)$$

- If the condition holds, then $f$ is said to be continuous at the point $a$.
- If $f$ is continuous at every point of $X$, then $f$ is said to be *continuous* on $X$.
- the limit exists and the limit of the function is equal to the function of the limit. That is,

  $$\lim_{x \to a} f(x) = f(\lim_{x \to a} x).$$

  I state and prove a formal version of this below.
- The function has to be defined at $a$ for the limit to exist.
- In order for a function to be continuous at $a$, it must be defined "in a neighborhood" of $a$ – that is, the function must be defined on an interval $(\alpha, \beta)$ with $a \in (\alpha, \beta)$. We extend the definition to take into account "boundary points" in a

## Lemma

*$f : X \longrightarrow \mathbb{R}$ is continuous at $a$ if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$0 <| a - x |< \delta \qquad \Longrightarrow \qquad | f(x) - f(a) |< \varepsilon$$

Note we could also write the conclusion as:

$$| a - x |< \delta \qquad \Longrightarrow \qquad | f(x) - f(a) |< \varepsilon$$

# Sequential Continuity

## Theorem

*$f : X \longrightarrow \mathbb{R}$ is continuous at $a$ if and only if for every sequence $\{x_n\}$ that converges to $a$, $\lim_{n\to\infty} f(x_n) = f(a)$.*

## Proof.

Use the formulation of continuity in the previous lemma. Let $\varepsilon > 0$ be given. Let $\delta > 0$ correspond to that $\varepsilon$. Take a sequence $\{x_n\}$ that converges to $a$. We know that there is $N$ such that if $n > N$, then $| a - x_n | < \delta$ implies that $| f(x) - f(a) | < \varepsilon$. This proves that continuity implies the sequential definition.

Conversely, suppose that $f$ is not continuous at $a$. It follows that there is $\varepsilon > 0$ such that for all $\delta > 0$ there is $x(\delta)$ such that $| x(\delta) - a | < \delta$ but $| f(x(\delta)) - f(a) | \geq \varepsilon$. Let $x_n = x(1/n)$. By construction, $\{x_n\}$ converges to $a$, but $| f(x_n) - f(a) | \geq \varepsilon$ so that $\{f(x_n)\}$ does not converge to $f(a)$. We have shown that if $f$ is not continuous at $a$ then there is a sequence $\{x_n\}$ that converges to $a$ such that $\{f(x_n)\}$ doesn't converge to $f(a)$. $\square$

Note that the condition in the theorem ($\lim_{n\to\infty} f(x_n) = f(a)$) is a statement about limits of sequences (not limits of functions).

# Algebra of Continuity

The sums, products, and ratios of continuous functions are continuous (in the last case the denominator of the ratio must be non-zero in the limit).

## Theorem

*For*

$$g : X \longrightarrow Y$$

*and*

$$f : Y \longrightarrow \mathbb{R}$$

*where both* $X$ *and* $Y$ *are open intervals of* $\mathbb{R}$*, if* $g$ *is continuous at* $a \in X$ *and if* $f$ *is continuous at* $g(a) \in Y$*, then*

$$f \circ g : X \longrightarrow \mathbb{R}$$

*is continuous at* $a$*.*

Let $\varepsilon > 0$ be given. Since $f$ is continuous at $g(a)$, there exists $\gamma > 0$ such that

$$| f(y) - f(g(a)) | < \varepsilon$$

if

$$| y - g(a) | < \gamma$$

and $y \in g(X)$.
Since $g$ is continuous at $a$, there exist $\delta > 0$ such that

$$| g(x) - g(a) |< \gamma$$

if

$$| x - a |< \delta$$

and $x \in X$.
It follows that

$$| (g(x)) - f(g(a)) |< \varepsilon$$

if

$$| x - a |< \delta$$

and $x \in X$. Thus $f \circ g$ is continuous at a.

# Use of Algebra of Continuity

These functions are continuous:

1. Constant functions ($f(x) \equiv c$).
2. Linear functions ($f(x) \equiv ax$).
3. Affine functions ($f(x) \equiv ax + b$).
4. Polynomials of degree $n$ ($f(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$, $a_n \neq 0$)
5. Rational functions ($f(x) \equiv P(x)/Q(x)$ for polynomials $P$ and $Q$) at any point $a$ where $Q(x) \neq 0$.

# Continuous Image of Closed Bounded Interval

### Theorem

*The continuous image of a closed and bounded interval is a closed and bounded interval. That is, if*

$$f : [a, b] \longrightarrow \mathbb{R}$$

*is continuous, then there exists $d \geq c$ such that $f([a, b]) = [c, d]$.*

The assumptions in the theorem are important. If $f$ is not continuous, then there generally nothing that can be said about the image. If the domain is an open interval, the image could be a closed interval (it is a point of $f$ is constant) or it could be unbounded even if the interval is finite (for example if $f(x) = 1/x$ on $(0, 1)$).

# Existence of Max

### Definition

We say that $x^* \in X$ maximizes the function $f$ on $X$ if

$$f(x^*) \geq f(x),$$

for every $x \in X$.

Similar definition for min of a fuction.

When the domain of $f$ is a closed, bounded interval, the image of $f$ is a closed, bounded interval, $f$ attains both its maximum (the maximum value is $d$) and its minimum. This means that if a function is continuous and it is defined on a "nice" domain, then it has a maximum.

# Intermediate Value Theorem

## Theorem

*If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, and $z$ is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = z$]*

In the statement of the theorem, "between" means either $z \in [f(a), f(b)]$ (when $f(a) \leq f(b)$ or $z \in [f(b), f(a)]$ if $f(b) < f(a)$.

## Proof.

$f([a, b])$ is an interval. Hence if two points are in the image, then all points between these two points are in the image. □

Theorem 13 is a method of solving showing that equations have solutions. It is common to conclude that there must be a $c$ for which the continuous function $f$ is zero ($f(c) = 0$) from the observation that sometimes $f$ is positive and sometimes $f$ is negative. The most important existence theorems in micro (the existence of a market clearing price system, for example) follow from (harder to prove) versions of this result.

# A Baby Fixed-Point Theorem

The point $x^* \in S$ is called a *fixed point* of the function $f : S \longrightarrow S$ if $f(x^*) = x^*$.

**Theorem**

*Let $S = [a, b]$ be a closed bounded interval and $f : S \longrightarrow S$ a continuous function. There exists $x^* \in S$ such that $f(x^*) = x^*$.*

**Proof.**

Consider the function $h(x) = f(x) - x$. Since $f(a) \geq a$, $h(a) \geq 0$. Since $f(b) \leq b$, $h(b) \leq 0$. Since $h$ is a continuous function on a closed bounded interval, there must be an $x^*$ such that $h(x^*) = 0$. It is clear that for this value, $f(x^*) = x^*$. □

# Assumptions are important

1. There are functions that don't attain max (due to "bad" domain or failure of continuity).
2. There are functions that don't have fixed points.