Econ 205 - Slides from Lecture 12

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Monotone Comparative Statics

- Loyalty to Stanford
- Comparative Statics Without Calculus
- Optimizer Set Valued
- No concavity
- No differentiability

Motivating Example

$$x^*(\theta) \equiv \arg\max f(x,\theta), \text{ subject to } \theta \in \Theta; x \in S(\theta)$$

This problem is equivalent to

$$x^*(\theta) \equiv \arg\max \phi(f(x,\theta)), \text{ subject to } \theta \in \Theta; x \in S(\theta)$$

for any strictly increasing $\phi(\cdot)$.
$\phi(\cdot)$ may destroy smoothness or concavity properties of the objective function.

Formulation

- Begin with problems in which $S(\theta)$ is independent of θ and both x and θ are real variables.
- Assume existence of a solution.
- Don't assume uniqueness.
- Generalize Notion of Increasing.

Strong Set Order

Definition

For two sets of real numbers A and B, we say that $A \geq_s B$ ("A is greater than or equal to B in the *strong set order*") if for any $a \in A$ and $b \in B$, $\min\{a, b\} \in B$ and $\max\{a, b\} \in A$.

Comments

1. According to this definition $A = \{1, 3\}$ is not greater than or equal to $B = \{0, 2\}$.
2. Includes the standard definition when sets are singletons.
3. $x^*(\cdot)$ is non-decreasing in μ if and only if $\mu < \mu'$ implies that $x^*(\mu') \geq_s x^*(\mu)$.
4. If $x^*(\cdot)$ is nondecreasing and $\min x^*(\theta)$ exists for all θ, then $\min x^*(\theta)$ is non decreasing.
5. An analogous statement holds for $\max x^*(\cdot)$.

Supermodular

Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is *supermodular* or *has increasing differences* in $(x; \mu)$ if for all $x' > x$, $f(x'; \mu) - f(x; \mu)$ is nondecreasing in μ.

- If f is supermodular in $(x; \mu)$, then the incremental gain to choosing a higher x is greater when μ is higher.
- Supermodularity is equivalent to the property that $\mu' > \mu$ implies that $f(x; \mu') - f(x; \mu)$ is nondecreasing in x.

Differentiable Version

When f is smooth, supermodularity has a characterization in terms of derivatives.

Lemma

A twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ *is supermodular in* $(x; \mu)$ *if and only if* $D_{12}f(x; \mu) \geq 0$ *for all* $(x; \mu)$*.*

The inequality in the definition of supermodularity is just the discrete version of the mixed-partial condition in the lemma.

Topkis's Monotonicity Theorem

Supermodularity is sufficient to draw comparative statics conclusions in optimization problems.

Theorem (Topkis's Monotonicity Theorem)

If f is supermodular in $(x; \mu)$*, then* $x^*(\mu)$ *is non-decreasing.*

Proof.

Suppose $\mu' > \mu$ and that $x \in x^*(\mu)$ and $x' \in x^*(\mu')$.

1. $x \in x^*(\mu)$ implies $f(x;) - f(\min\{x, x'\}; \mu) \geq 0$.
2. This implies that $f(\max\{x, x'\}; \mu) - f(x'; \mu) \geq 0$ (you need to check two cases, $x > x'$ and $x' > x$).
3. By supermodularity, $f(\max\{x, x'\}; \mu') - f(x'; \mu') \geq 0$,
4. Hence $\max\{x, x'\} \in x^*(\mu')$.

5. $x' \in x^*(\mu')$ implies that $f(x'; \mu') - f(\max\{x, x'\}, \mu) \geq 0$,
6. or equivalently $f(\max\{x, x'\}, \mu) - f(x'; \mu') \leq 0$.
7. This implies that $f(\max\{x, x'\}; \mu') - f(x'; \mu') \geq 0$,
8. which by supermodularity implies $f(x; \mu) - f(\min\{x, x'\}; \mu) \leq 0$
9. and so $\min\{x, x'\} \in x^*(\mu)$.

□

Comment

Don't be surprised.

Theorem follows from the IFT whenever the standard full-rank condition in the IFT holds.

At a maximum, if $D_{11} f(x^*, \mu) \neq 0$, if must be negative (by the second-order condition), hence the IFT tells you that $x^*(\mu)$ is strictly increasing.

Example

A monopolist solves $\max p(q)q - c(q, \mu)$ by picking quantity q. $p(\cdot)$ is the price function and $c(\cdot)$ is the cost function, parametrized by μ.
Let $q^*(\mu)$ be the monopolist's optimal quantity choice. If $-c(q, \mu)$ is supermodular in (q, μ) then the entire objective function is.
It follows that q^* is nondecreasing as long as the marginal cost of production decreases in μ.

Trick

It is sometimes useful to "invent" an objective function in order to apply the theorem. For example, if one wishes to compare the solutions to two different maximization problems, $\max_{x \in S} g(x)$ and $\max_{x \in S} h(x)$, then we can apply the theorem to an artificial function, f

$$f(x, \mu) = \begin{cases} g(x) \text{ if } \mu = 0 \\ h(x) \text{ if } \mu = 1 \end{cases}$$

so that if f is supermodular ($h(x) - g(x)$ nondecreasing), then the solution to the second problem is greater than the solution to the first.

Single-Crossing

Definition

The function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the *single-crossing condition* in $(x; \mu)$ if for all $x' > x$, $\mu' > \mu$

$$f(x'; \mu) - f(x; \mu) \geq 0 \text{ implies } f(x'; \mu') - f(x; \mu') \geq 0$$

and

$$f(x'; \mu) - f(x; \mu) > 0 \text{ implies } f(x'; \mu') - f(x; \mu') > 0.$$

Theorem

If f is single crossing in $(x;\mu)$, then $x^*(\mu) = arg\ \max_{x \in S(\mu)} f(x;\mu)$ is nondecreasing. Moreover, if $x^*(\mu)$ is nondecreasing in μ for all choice sets S, then f is single-crossing in $(x;\mu)$.

Unconstrained Extrema of Real-Valued Functions

Definition

Take $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

$\mathbf{x}^*$ is a *local maximizer* $\iff \exists\, \delta > 0$ such that $\forall\, \mathbf{x} \in B_\delta(\mathbf{x}^*)$,

$$f(\mathbf{x}) \leq f(\mathbf{x}^*)$$

$\mathbf{x}^*$ is a *local minimizer* $\iff \exists\, \delta > 0$ such that $\forall\, \mathbf{x} \in B_\delta(\mathbf{x}^*)$,

$$f(\mathbf{x}) \geq f(\mathbf{x}^*)$$

$\mathbf{x}^*$ is a *global maximizer* $\iff \forall\, \mathbf{x} \in \mathbb{R}^n$, we have

$$f(\mathbf{x}) \leq f(\mathbf{x}^*)$$

$\mathbf{x}^*$ is a *global minimizer* $\iff \forall\, \mathbf{x} \in \mathbb{R}^n$

$$f(\mathbf{x}) \geq f(\mathbf{x}^*)$$

First-Order Conditions

Theorem (First Order Conditions)

If f is differentiable at $\mathbf{x}^*$, and $\mathbf{x}^*$ is a local maximizer or minimizer then

$$Df(\mathbf{x}) = \mathbf{0}.$$

That is

$$\frac{\partial f}{\partial x_i}(\mathbf{x}^*) = 0,$$

$\forall\, i = 1, 2, \ldots, n.$

Define $h \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$h(t) \equiv f(\mathbf{x}^* + t\mathbf{v})$$

for any $\mathbf{v} \in \mathbb{R}^n$, $t \in \mathbb{R}$.
Take the case of a maximizer:
Fix a direction $\mathbf{v}$ ($\|\mathbf{v}\| \neq 0$).
We have

$$f(\mathbf{x}^*) \geq f(\mathbf{x}),$$

$\forall \mathbf{x} \in B_\delta(\mathbf{x}^*)$, for some $\delta > 0$. In particular for t small ($t < \delta \|\mathbf{v}\|$) we have

$$\begin{aligned} f(\mathbf{x}^* + t\mathbf{v}) &= h(t) \\ &\leq f(\mathbf{x}^*) \end{aligned}$$

Thus, h is maximized locally by $t^* = 0$.

Our F.O.C. from the $\mathbb{R} \longrightarrow \mathbb{R}$ case

$$\implies h'(0) = 0$$

So

$$\implies \nabla f(\mathbf{x}^*) \cdot \mathbf{v} = 0$$

And since this must hold for every $\mathbf{v} \in \mathbb{R}^n$, this implies that

$$\nabla f(\mathbf{x}^*) = \mathbf{0}$$

We know that if f is differentiable, then Df is represented by the matrix of partial derivatives. Hence $Df(\mathbf{x}^*) = \mathbf{0}$.

Definition

If $\mathbf{x}^*$ satisfies $Df(\mathbf{x}^*) = \mathbf{0}$, then it is a *critical point* of f.

Intuition

1. Like one-variable theorem.
2. If x^* is a local maximum, then the one variable function you obtain by restricting x to move along a fixed line through x^* (in the direction v) also must have a local maximum.
3. Hence all directional derivatives are zero.
4. The first-derivative test cannot distinguish between local minima and and local maxima, but an examination of the proof tells you that at local maxima derivatives decrease in the neighborhood of a critical point.
5. Critical points may fail to be minima or maxima.
6. One variable case: a function decreases if you reduce x (suggesting a local maximum) and increases if you increase x (suggesting a local minimum).
7. Generalization: this behavior could happen in any direction.
8. Also: the function restricted to direction has a local maximum, but it has a local minimum with respect to another direction.
9. Conclude: It is "hard" for critical point of a multivariable function to be a local extremum in the many variable case.