Economics 205, Fall 2010
Quiz III Possible Answers

September 10, 2009

Comments. Range: 50-98, Median: 73, Mean: 75. Larry says that most did well on the first question. He said that some people confused the definition of convexity of a set with convexity of a function. The definitions are related, of course, but different. He reported that there were problems figuring out what equations needed to be differentiated to answer the last parts of question two. I still maintain that it is easier and more intuitive to do these problems directly rather than attempting to force things into the implicit function theorem formula. The second question illustrates an important technique.

1. Let $K = \{(x, y) : x^2 + y^2 \leq 4\}$.

   (a) Prove that K is convex.
   One answer: $h(z) = z^2$ is a convex function (second derivative positive), so
   $$(\lambda z + (1-\lambda)z')^2 \leq \lambda z^2 + (1-\lambda)(z')^2.$$
   It follows that if $(x, y), (x', y') \in K$, and $(u, v) = (\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y')$, then
   $$u^2 \leq \lambda x^2 + (1-\lambda)(x')^2,$$
   $$v^2 \leq \lambda y^2 + (1-\lambda)(y')^2,$$
   and hence $u^2 + v^2 \leq 4$.

   (b) Show that $(3, 1) \notin K$.
   $$9 + 1 > 4$$

   (c) Find the equation of a hyperplane that separates K from $(3, 1)$.
   The proof of the separating hyperplane theorem use the direction of the line that connects $(3, 1)$ to the point in K closest to $(3, 1)$. This point turns out to be $(x, y) = \sqrt{.4}(3, 1)$. So the hyperplane would have normal in the direction $(3, 1) - (x, y)$ and pass through a point on the segment connecting (x, y) to $(3, 1)$. A simpler to describe separating hyperplane is $x = 2.5$. Every point in K is in $\{(x, y) : x < 2.5\}$, while $3 > 2.5$.

   (d) Show that $(x_0, y_0) = (0, 2)$ satisfies $x^2 + y^2 = 4$.
   $$0 + 4 = 4.$$

(e) Is it possible to solve the equation $x^2+y^2 = 4$ for y as a differentiable function of x for (x, y) near $(0, 2)$. If so, write $y = Y(x)$ and find $Y'(2)$.

It is possible because at this point the derivative of $x^2 + y^2$ with respect to y is not zero. $Y'(2) = 0$.

(f) Is it possible to solve the equation $x^2+y^2 = 4$ for x as a differentiable function of y for (x, y) near $(0, 2)$. If so, write $x = X(y)$ and find $X'(2)$.

It is not possible because at this point the derivative of $x^2 + y^2$ with respect to x is zero.

2. A monopoly firm can influence demand by advertising. If the firm buys a units of advertising, it can sell q units at the price $P(a, q) = a(15 - q)$. The price of a units of advertising is αa^2 dollars. It costs the monopolist βq^2 to produce q units.

(a) Write the profit function of the firm.

$$P(a, q)q - \alpha a^2 - \beta q^2.$$

(b) Show that when $\alpha = 5$ and $\beta = .5$ the solution to the monopolist's profit maximization problem is to set $a = 10$ and $q = 5$.

First-order conditions:

$$a(15 - 2q) - 2\beta q = 0 \text{ and } (15 - q)q - 2\alpha a = 0.$$

You can check (by differentiating again) that the objective function is strictly concave, so first-order conditions characterize a local maximum. Profits are zero on the boundary (q or $a = 0$), so the equations describe a global maximum.

These equations are satisfied at the given point (check by substitution).

(c) Is it possible to describe how the profit maximizing values of q and a change as α and β change? If so, compute the derivatives of q and a as functions of α and β near $(q, a, \alpha, \beta) = (5, 5, 5, 2.5)$. Derivatives with respect to α:

$$(15-2Q)D_1Q-2\alpha D_1A = 2A \text{ and } -2(A+\beta)D_1Q+(15-2Q)D_1A = 0.$$

or

$$5D_1Q - 10D_1A = 10 \text{ and } - 15D_1Q + 5D_1A = 0$$

so $D_1Q(5, 2.5) = -.4$ and $D_1A(5, 2.5) = -1.2$

Similarly, derivatives with respect to β:

$$5D_2Q - 10D_2Q = 0 \text{ and } - 15D_2Q + 5D_2A = 10.$$

so $D_2Q(5, 2.5) = -.8$ and $D_2A(5, 2.5) = -.4$ (So it is possible.)

(d) If α increases to 5.01 and β decreases to 2.48 will the monopolist's output increase?

The question asks for $.01(D_1Q - 2D_2Q) = .01(-.4 - 2(-.8)) = .01(1.2) > 0$, so the answer is yes.