Economics 205, Fall 2010
Quiz I, Possible Answers

August 27, 2010

Comments. Scores out of 100. Range: 46-96. Average: 81, Median: 83. Allocation: 40/40/30.

First question: OK, but some people did not note range of validity. Third question: This is where most people lost points.

Minimal (but positive!) deduction for not providing justification. Deductions for not explaining your answers will increase on future assessments.

1. Let f be a differentiable function. Calculate the derivative of the function h defined in each of the problems below. If you need additional assumptions, make them explicit:

   (a) $h(x) = \log f(x^2)$. $h'(x) = \frac{2xf'(x^2)}{f(x^2)}$ by the Chain Rule (twice). You can only do this computation when $f(x^2) > 0$.

   (b) $h(x) = f(\log x)$. $h'(x) = f'(\log x)/x$ by the chain rule and the rule for differentiating $\log x$. Need $x > 0$ for h to be defined.

   (c) $h(x) = e^{\log x} = x$, so $h'(x) = 1$

2. Calculate the limits indicated below.

   (a) $\lim_{n\to\infty} \frac{n^2-1}{3n^3+6} = 0$
   Given $\epsilon > 0$, let $N = 1/\epsilon$

   $$0 \le \frac{n^2-1}{3n^3+6} \le \frac{n^2-1}{n^3} < \epsilon.$$

   (b) $\lim_{x\to 5} \frac{x^2-2}{x+5} = 2.3$.
   (Limit of continuous function – ratio of polynomials with nonzero denominator. So by continuity, just evaluate.)

   (c) $\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x}$. Can use L'Hopital's Rule on last expression to obtain:

   $$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 0$$

3. Let $f : [0,1] \to \mathbf{R}$.

(a) Prove that if

$$|a-b|^{1/2} \geq |f(a)-f(b)| \text{ for all } a, b \in [0,1], \tag{1}$$

then f is continuous on $(0,1)$.

The inequality says that —f(a) - f(b)— is "sandwiched" between the constant function equal to zero and the function $g(x) = |a-x|^{1/2}$. Since g is continuous at a (accepted wisdom), $\lim_{b\to a} |f(a)-f(b)| = 0$. This is what we needed to show.

(b) Give an example of a non-constant function f that satisfies (1). [Prove that your example satisfies the condition.]

There are many possibilities. $f(x) \equiv x$ works because $|a-b|^{1/2} \geq |a-b|$ for all $a, b \in [0,1]$. (To prove this assertion you can note that the right-hand side of the inequality is non-negative, so the inequality is equivalent to

$$|a-b| \geq |a-b|^2$$

(square both sides). This inequality holds when $a = b$ and otherwise is equivalent to $1 \geq |a-b|$, which will hold when $a, b \in [0,1]$.