Economics 205 Final Examination Fall 2010

Comments on Course Grade. Total Points: 1200. High: 1134; Low: 558; Median: 913; Mean 900. Formula: Maximum of (Final, .75Final + Quizzes, 5/6Final + 2 Best Quizzes). Grading:

- Lowest B, 662.
- Lowest $B+$, 815.
- Lowest $A-$, 854
- Lowest A, 1023.

Comments on Final. High: 1130/1200; Low 492; Median 895; Mean: 882.

1. Some people were casual about justifying their steps and about the domain of definition.
2. Fine (except a few people did not know how to perform integration by parts).
3. OK.
4. Some people did extra work (you needed to diagonalize only one matrix).
5. Minor deductions for not justifying your method.
6. Part (b) had a typo (corrected below). You need strict monotonicity (not continuity) for uniqueness. Most responded to the poorly posed question by writing nonsense. We allocated all of the points in the problem to the other parts, so no one lost points for responses to (b). On part (c) several people forgot that CE was implicitly defined by the equation (they treated lhs as C instead of $u(C)$). This is a significant error and led to a significant deduction.
7. Here it was ok if you solved the problem using the objective function $3x^{1/3}x^{13} - wx^2 - wy^2$ (the answers below are for $3x^{1/3}x^{13} - wx^2 - wy^2$.
8. On the first part some people acted as is positive semi-definite requires a zero eigenvalue. Not true. Positive definite matrices and positive semi-definite (in the same way that positive numbers are non-negative).

1. In each part, determine at which points the derivative of the function h exists. When it does exist, compute it. When it does not exist, explain why it does not exist.

   (a) $h(x) = \log(1 + \log(1 + x))$.
   For continuity you need the arguments of the $\log$s to be positive. This means that you need $x > -1$ in order for $1 + \log(1 + x) > 0$ and $\log(1 + x) > -1$ in order for $1 + \log(1 + x) > 0$. This means you need $1 + x > e^{-1}$ or $x > e^{-1} - 1$.

   $$h'(x) = \frac{1}{(1 + x)(1 + \log(1 + x))}$$

   by the chain rule (since the derivative of $\log(1 + x) = 1/(1 + x)$).

(b) $h(x) = (e^{\log x})^2$.

This one is differentiable when $x > 0$ (composition of differentiable functions). Since the formula is just a complicated way of writing $h(x) = x^2, h'(x) = 2x$.

(c) $h(x) = \int_0^x f(y)dy$ for a continuous function f.

$h'(x) = f(x)$, always differentiable (by the fundamental theorem of calculus).

(d) $h(x) = \int_0^x f(x)dy$ for a continuous function f.

Here $h'(x) = f(x) + xf'(x)$ provided that f is differentiable at x. When $x \neq 0$, differentiability of f at x is a necessary condition for h to be differentiable at x. When $x = 0$, $h'(0) = 0$ without further assumptions. [$h(x) = xf(x)$ so

$$\lim_{x\to 0}(h(x) - h(0))/x = \lim_{x\to 0} f(x) = 04.$$

(e) $h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 \log x & \text{if } x > 0 \end{cases}$

We have

$$h'(x) = \begin{cases} 0 & \text{if } x < 0 \\ x + 2x \log x & \text{if } x > 0 \end{cases}$$

by rules of differentiation. Also $h'(0)$ exists and is equal to 0 because

$$\lim_{x\to 0^+}(x + 2x \log x) = 0.$$

The limit is zero by L'Hopital's Rule. The fact that the limit being zero implies differentiability at zero follows from a theorem presented in class (or can be established by a direct argument).

2. Use integration by parts to compute $\int_1^2 x^2 \log x \, dx$.

$= \frac{x^3}{3} \log x \mid_{x=1}^{x=2} - \int_1^2 \frac{x^2}{3} dx = 8/3 \log 2 - x^3/9 \mid_{x=1}^{x=2} = 8/3 \log 2 - 8/9$

3. Consider the function $f(x, y) = (\frac{1}{x^2} - \frac{1}{y})$ defined on $\{(x, y) : x, y > 0\}$.

(a) Graph $\{(x, y) : x, y > 0, f(x, y) = -1\}$.

(b) Find an equation of the hyperplane tangent to the graph of $f(x, y) = z$ at the point $(x, y, z) = (1, 1, 0)$.

Equation is:

$$(z - 0) = (-2, 1) \cdot (x - 1, y - 1)$$

or

$$2x - y + z = 1.$$

(c) Decide whether or not $f(\cdot)$ is homogeneous. If $f(\cdot)$ is homogeneous, then determine its degree of homogeneity and explicitly verify Euler's Theorem.

Not homogeneous. To satisfy $f(\lambda x, \lambda y) = \lambda^k f(x, y)$ you would need

$$x^{-2}(\lambda^{-2} - \lambda^k) = -y^{-1}(\lambda^{-1} - \lambda^k)$$

for all x, y. This means that both sides of the equation must be constant (one side doesn't depend on x, the other side doesn't depend on y, and they are equal). Consequently, the constant must be zero, but this leads to $\lambda = \lambda^2$, which is not true for all λ.

(d) Find the directional derivative of f in the direction $\mathbf{w} = \frac{1}{\sqrt{2}}(1, 1)$ at the point $(x, y) = (1, .5)$.

$Df(1, .5) = [-2 \;\; 4]$, so the directional derivative is $\sqrt{2}$.

(e) Let $g(u, v) = (u^2 + v^2 + 1, u + .5)$. Use the chain rule to compute all partial derivatives of $f \circ g(\cdot)$ when $u = v = 0$.

$g(0, 0) = (1, .5)$ so we can use $Df(1, .5)$ from the previous part. $DG(0, 0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ so $Df \circ g(0, 0) = [4 \;\; 0]$.

4. Consider the following matrices:

(a) $A = \begin{pmatrix} 5 & 3 \\ 4 & 3 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}$ **(c)** $A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{2}{3} \end{pmatrix}$.

(a) $A = \begin{pmatrix} 5 & 3 \\ 4 & 3 \end{pmatrix}$ is invertible (determinant is 3). The inverse is

$$A = (\frac{1}{3}) \begin{pmatrix} 3 & -3 \\ -4 & 3 \end{pmatrix}$$

The leading principal minors are both positive, so the matrix is positive definite.

The eigenvalues are distinct (I didn't compute them, but the characteristic polynomial is not a perfect square), so the matrix is diagonalizable.

(b) $A = \begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}$ is not invertible (determinant is 0). The eigenvalues are 0 and 9 and the associated eigenvectors are $(3, -2)$ and $(3, 1)$. Since the eigenvectors are nonnegative (but one is zero), the matrix is positive semi-definite. The diagonalization can be done by taking

$P = \begin{pmatrix} 3 & 3 \\ 1 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix}$ $P^{-1} = (1/9) \begin{pmatrix} 2 & 3 \\ 1 & -3 \end{pmatrix}$

$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{2}{3} \end{pmatrix}$ is invertible (determinant is 1/6). The inverse is

$$A^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 6 \end{pmatrix}.$$

The quadratic form is positive definite because the leading principal minors are all positive. The matrix is diagonalizable because it is symmetric.

5. A piece of cheese is located at $(12, 10)$ in $\mathbb{R}^2$. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

   (a, b) solves $\min(12 - x)^2 + (10 - y)^2$ subject to $y = -5x + 18$. Treated as an equality constrained problem leads to first-order conditions:

   $$2(12 - x) = 5\lambda \text{ and } 2(10 - y) = \lambda.$$

   Solving yields $(2, -8)$, so the sum is -6.

6. An agent has utility function $u(x)$ where x is income. The agent has initial wealth w. With probability p the agent suffers a loss of l dollars, reducing her wealth to $w - l$. The agent's *expected utility* is $pu(w-l)+(1-p)u(w)$. The agent's *certainty equivalent* C is implicitly defined by the equation

$$u(C) = pu(w - l) + (1 - p)u(w). \quad (1)$$

   (a) Show that if u is continuous, there exists a certainty equivalent. That is, there is a solution to equation (1).

   This is the intermediate value theorem. For concreteness, take $u(w) > w(w - l)$. It follows that the continuous function $u(x)-pu(w-l)-(1-p)u(w)$ is negative when $x = w - l$ and is positive when $x = w$. Hence it must be zero somewhere in between.

   (b) Show that if u is strictly increasing, then there exists a unique certainty equivalent. That is, there is one and only one C that solves equation (1).

   By contradiction: If there were two such values, C and C', $C' > C$, then we would have $u(C') = u(C)$, which violates the assumption.

   (c) Suppose that for given values (p_0, w_0, l_0) equation (1) has a solution C_0. State conditions on u and its derivatives under which you can locally solve equation (1) for C as a differentiable function $C = g(p, w, l)$ in a neighborhood of (p_0, w_0, l_0) with $g(p_0, w_0, l_0) = C_0$. Write down a formula for $Dg(p_0, w_0, l_0)$.

   By the implicit function theorem, $u'(C)Dg = D(pu(w - l) + (1 - p)u(w))$. This formula makes sense if $u'(C) \neq 0$.

   (d) State economically plausible conditions under which g is decreasing in p.

   Economically reasonable conditions are $u' > 0$ (strictly increasing utility for money) and $l > 0$ (a loss is really negative). This guarantees that $u(w) > u(w - l)$ and hence $D_1 g = -(u(w) - u(w - l))/u'(C) < 0$.

7. A profit-maximizing firm that uses two inputs to produce output has the production function $3x^{1/3}y^{1/3}$, where x is the amount of input 1 and y is the amount of input 2. The price of output is 1. The total cost of the inputs are wx and wy. The firm is constrained by the government to use no more than 1000 units of input 1. (Inputs x and y must be non-negative, but you may assume that the non-negativity constraints are not binding.)

   (a) How much of input 2 does it use?

   The firm solves:

$$\max 3x^{1/3}y^{1/3} - wx - wy \text{ subject to } x = 1000.$$

   The first order conditions are:

$$x^{-2/3}y^{1/3} - w = \lambda \text{ and } x^{1/3}y^{-2/3} - w = 0.$$

   Also, $\lambda \geq 0$ and $\lambda = 0$ if $y < 1000$. There are two possibilities. If $\lambda = 0$, then

$$x^{-2/3}y^{1/3} = w = x^{1/3}y^{-2/3}.$$

In this case, $x = y = 1/w^3$.

The other possibility is that the $x \leq 1000$ constraint binds. In this case,

$$y^{-2/3} = w/10 \text{ or } y = (10/w)^{3/2}.$$

and

$$\lambda = (.01)(10/w)^{1/2} - w.$$

So, when $w < .1$, the constraint binds and the solution is

$$(x, y, \lambda) = (1000, (10/w)^{3/2}, (.01)(10/w)^{1/2} - w)$$

and when $w > .1$ the solution is

$$(x, y, \lambda) = (w^{-3}, w^{-3}, 0)$$

(b) What is the most that the firm is willing to pay to have the right to increase the limit on input 1 by a tiny amount (from 1000 to $1000 + \Delta$ units), $\Delta > 0$?

This answer depends on w. When $w \geq .1$, there is surplus input 1 and the firm will not pay anything. When $w < .1$, the firm can use the additional input 1 and is willing to pay (at the margin) $\lambda\Delta$ for the opportunity.

8. Decide whether each of the statements below is true. If the statement is true, then prove it. If the statement is false, then given a counterexample.

(a) For any matrix $\mathbf{A}$, $\mathbf{A}^t\mathbf{A}$ is a positive semi-definite matrix.

This is true. $\mathbf{x}^t\mathbf{A}^t\mathbf{A}\mathbf{x} = ||\mathbf{A}\mathbf{x}||^2$, which is nonnegative.

(b) If $\mathbf{A}$ is a symmetric matrix that satisfies $\mathbf{A}^3 = \mathbf{I}$, then $\mathbf{A} = \mathbf{I}$.

This is true. We know $\mathbf{A}$ is diagonalizable and has real eigenvalues. So there is a $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. So $\mathbf{A}^3 = \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1} = I$. So $\mathbf{D}^3 = \mathbf{I}$. It follows that $\mathbf{D} = \mathbf{I}$ (since the only real solution to $\lambda^3 = 1$) so $\mathbf{A} = \mathbf{I}$.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies $f(x) = -f(-x)$, then $f(0) = 0$.

This is true. You do not need continuity. The formula implies that $f(0) = -f(0)$, which only is true if $f(0) = 0$. If you insist, by continuity, both one-sided limits exist and are equal. But $\lim_{x\to 0^+} f(x) = -\lim_{x\to 0^-} f(x)$ by assumption. Hence the limits must be equal to zero.

(d) If $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function that satisfies $f(0) = f(1) = f(2)$, then there exists $c \in [0, 2]$ such that $f''(c) = 0$.

By the mean value theorem, there exists $a_1 \in (0, 1)$ such that $f'(a_1) = 0$ and $a_2 \in (1, 2)$ such that $f'(a_2) = 0$. The result now follows from the mean value theorem applied to f'.