Economics 205, Fall 2009
Quiz I Answers and Comments

August 28, 2009

Comments The quiz had a total of 100 points. Twenty points each for the first two parts of problems one and for Question 3; ten points each for the third part of Question 1 and each part of Question 2. For the easy parts of Question 1 there were "fussy" deductions for providing no justification or for failure to discuss the "obvious ($x \neq 1$) parts of the domain. Don't let these deductions bother you, but remember to give justifications on future quizzes. You received half credit in 1 c for providing a function that did one of the two things required. Question 1c is about thinking calmly and clearly. For Question 2: It looked like a couple of people may not understand the chain rule (or maybe were a bit sloppy), but most answers were fine. Question 3 was mostly for my benefit (to see how many people were not intimidated by a simple proof). If you had trouble with the question, try to make sense of (one of my) answers below. You should also be able to get an intuition for the result by drawing a picture. I view 1c and 3 as bonus questions, so, in terms of measuring your chances of passing the class, so scores of 60 and above certain are fine.

1. Let f be defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 2x - 1, & \text{if } x > 1 \end{cases}.$$

   (a) Identify the set of points in the interval $(0, 2)$ at which f is continuous.
   All points. This is true for $x \neq 1$ because f is linear (locally). This is true for $x = 1$ because left and right hand limits exist and are equal.

   (b) Identify the set of points in the interval $(0, 2)$ at which f is differentiable.
   All points except $x = 1$. Not differentiable at one because left-hand limit is 1 and right-hand limit is 2 (they aren't equal).

   (c) Find a function $g : \mathbb{R} \to \mathbb{R}$ such that the composite function $g \circ f$ is both differentiable and onto. (Try to find one function g that leads to $g \circ f$ satisfying both properties. If you cannot do that, find two different functions.)
   There are lots of answers, one is

$$g(x) = \begin{cases} x, & \text{if } x \leq 1 \\ \frac{x+1}{2}, & \text{if } x > 1 \end{cases}.$$

This means that

$$g \circ f(x) = x,$$

which obviously satisfies the desired properties.

2. Let f be a differentiable function such that $f(x) > 0$ all x. Calculate the derivative of the function h defined in each of the problems below:

   (a) $h(x) = f(3e^2 + x)$. $h'(x) = f'(3e^2 + x)$.

   (b) $h(x) = \log x f^2(x)$, for $x > 0$. $h'(x) = 1/x + 2f'(x)/f(x)$

   (c) $h(x) = e^{f^2(x)}$. $h'(x) = 2f(x)f'(x)h(x)$.
   (Correct answers may look different.)

3. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) = -f(-x)$ for all $x \neq 0$. Show that if f is continuous at 0, then $f(0) = 0$.

   By continuity, $\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) = f(0)$. By definition of f, $\lim_{x\to0^+} f(x) = -\lim_{x\to0^-} f(x)$. Hence $f(0) = -f(0)$ and so $f(0) = 0$.

   Alternate proof: Suppose that $f(0) \neq 0$ and argue to a contradiction. For concreteness, assume $f(0) > 0$ and let $\epsilon = f(0)/2 > 0$. By definition of continuity, there is a $\delta > 0$ such that $| f(x) - f(0) | < \epsilon$ for $| x | < \delta$. Hence $f(x) > 0$ for $x \in (-\delta, \delta)$. But, by definition, if $f(a) > 0$, then $f(-a) < 0$, so it is impossible for f to be positive throughout $(-\delta, \delta)$.