Economics 205, Fall 2007: Final Examination

Instructions. Try to answer all seven problems. (Read all of the questions now and start on the ones that seem easiest. Do not be intimidated by the multi-part questions. It is often possible to answer later parts without information from earlier parts.) Think before you write. You should be able to do everything without much tedious computation. Make your answers as complete and rigorous as possible: give reasons for your computations and prove your assertions. Informal and intuitive arguments are better than nothing. There are a total of 216 points on the examination. I list the weights on the individual questions below.

	Score	Possible
1		30
2		15
3		36
4		35
5		35
6		30
7		35
Exam Total		216
Course Total		288
Grade in Course		

1. In each part, determine at which points the derivative of the function h exists. When it does exist, compute it. When it does not exist, explain why it does not exist.

   (a) $h(x) = x + \frac{1}{x}$.

   (b) $h(x) = x\sqrt{1-x^2}$.

   (c) $h(x) = e^{x^e}$.

   (d) $h(x) = e^{x^x}$.

   (e) $h(x) = x^{\log(2+x^2)}$.

   (f) $h(x, y) = f \circ g(x, y)$ where $g(x, y) = (x + y, x^3 - xy)$ and $f(u, v) = (u^2v, u + 2v)$.

2. Find the indicated limits. Justify your answers either by appealing to a general property of limits or by giving a proof.

   (a) $\lim_{x\to 3} \frac{x}{5}$.

   (b) $\lim_{x\to 1} \frac{(x-1)(3x-5)}{x^3-1}$.

   (c) $\lim_{x\to 1} \frac{1-x^2}{\log(3x)}$.

3. Consider the function $f(x, y) = x^3 y + y^2$.

(a) Graph $\{(x, y) : f(x, y) = 0\}$.

(b) Find all critical points of the function f. Use the second derivative of f to class all critical points (as local maxima or local minima).

(c) Find an equation of the hyperplane tangent to the graph of $z = f(x, y)$ at the point $(0, 2, 4)$.

(d) Find an equation of a line (there are many) that lies in the tangent hyperplane that you found in part c.

(e) Compute the directional derivative of f at $(0, 2)$ in the direction $v = (.6, .8)$.

(f) Verify that the graph of f intersects the plane $x + y + z = 9$ at the point $(x, y, z) = (1, 2, 6)$.

(g) Determine whether it is possible to solve for x and y as differentiable functions $X(z)$ and $Y(z)$ of z in a neighborhood of the point $(1, 2, 6)$ on the surface defined by the intersection of the graph of f and $x + y + z = 9$. If it is possible, then compute the derivatives of these functions.

4. A consumer has a utility function of the form:

$$U(x_1, x_2) = x_1 - \frac{1}{x_2}.$$

Let p_i be the price of the ith good and let the consumer's wealth be w. Assume that $p_1, p_2, w > 0$. The consumer's demand function, $d(p_1, p_2, w)$, is the solution to the problem:

$$\max U(x_1, x_2) \text{ subject to } p_1x_1 + p_2x_2 \leq w, x_1, x_2 \geq 0.$$

Find the consumer's demand. Identify the set of points at which d is differentiable and compute the derivative of d at these points.

5. Consider the problem: $\max x_1 + 2x_2 - x_2^3$ subject to the constraints: $x_1 + x_2 \leq 1$, $x_1, x_2 \geq 0$.

   (a) Graph the feasible set and several level sets of the objective function. Use the graph to provide an intuition about the location of the solution.

   (b) Solve the maximization problem. Use any method you wish, but explain your reasoning.

6. Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{A}^t$ denote its transpose.

   (a) Show that the matrix $\mathbf{A}^t\mathbf{A}$ is well defined for all m and n.

   (b) Show that $\mathbf{A}^t\mathbf{A}$ is symmetric.

   (c) Show that $\mathbf{A}^t\mathbf{A}$ is positive semi-definite.

   (d) Show that if $\mathbf{A}$ is $n \times n$, then $\mathbf{A}^t\mathbf{A}$ is positive definite if and only if $\mathbf{A}$ is non-singular.

   For this question you may use the fact that $(\mathbf{A}\mathbf{x})^t = \mathbf{x}^t\mathbf{A}^t$.

7. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$. The function is called *even* if $f(x) = f(-x)$ for all x.

   (a) Show (by example) that there exists a non-constant even function.

   (b) Prove that if f is a non decreasing even function, then f is constant.

   (c) Prove that if f is $2n$ times continuously differentiable, then the $(2k-1)$th derivative of f evaluated at zero is equal to zero for all $k < n$.

   (d) Suppose that f is an even function that is four times differentiable and $f^{''}(0) = 0$. Prove that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x^3} = 0.$$