Economics 205 Quiz 1

Joel Watson, Fall 2006

1. Consider an arbitrary function $f : [0,1] \to [0,1]$. Define the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ inductively by (i) $x_1 = 0$, and (ii) for each $n \in \mathbf{P}$, $x_{n+1} = f(x_n)$.

(a) Given the assumptions made above, must it be the case that $\{x_n\}$ converges? If so, explain why. If not, provide a counterexample.

No – consider $f(x) \equiv 1 - x$.

(b) Given the assumptions made above, must it be the case that $\{x_n\}$ has a convergent subsequence? If so, explain why. If not, provide a counterexample.

Yes, f's codomain is bounded, so the sequence is also bounded. The Bolzano-Weierstrass Theorem implies the existence of a convergent subsequence.

(c) Suppose that $f(x) = (x+1)/3$. Write the first few elements of $\{x_n\}$ and calculate $\lim_{n\to\infty} x_n$.

$x_1 = 0$. $x_2 = \frac{1}{3}$. $x_3 = \frac{4}{9}$. $x_4 = \frac{13}{27}$. $x_5 = \frac{40}{81}$

$x_n = \dfrac{\sum_{k=0}^{n-2} 3^k}{3^{n-1}}$. $\quad x_n \to \frac{1}{2}$

2. Calculate the following limits. In the case in which one or both of the limits does not exist, state this.

(a) $\lim_{n\to\infty} x_n$, where the sequence $\{x_n\}$ is defined by $x_n = \frac{(2n^2+3)(4n+1)}{4n^3}$ for all $n \in \mathbf{P}$

$x_n = \dfrac{8n^3 + 2n^2 + 12n + 3}{4n^3} = 2 + \dfrac{1}{2n} + \dfrac{3}{n^2} + \dfrac{3}{4n^3} \longrightarrow 2$

(b) $\lim_{x\to 1} \frac{(x-1)^2}{\ln x} = \lim_{x\to 1} \dfrac{2x-2}{\left(\frac{1}{x}\right)} = \lim_{x\to 1} \dfrac{(2x-2)x}{1} = (2-2)(1) = 0$

L'Hopital's rule

3. For both cases below, write expressions for $h'(x)$.

(a) $h(x) \equiv [f(x)]^{-2}[1 - f(x)]$

$$h'(x) = f'(x)\left[f(x)^{-2} - 2f(x)^{-3}\right]$$

(b) $h(x) \equiv f(3x + g(x))$

$$h'(x) = f'(3x + g(x))\,(3 + g'(x))$$

4. Prove by induction: $1 + 2 + 3 + \ldots + n = n(n+1)/2$.

(a) At $n = 1$, left side is 1 and right side is 1, so the equation holds.

(b) Consider any k and presume that the equation holds for $n = k$.
Examine the equation for $n = k+1$:

$$\underbrace{1+2+3+\cdots+k}_{A} + (k+1) = (k+1)(k+2)\tfrac{1}{2} = \underbrace{(k+1)k\tfrac{1}{2}}_{B} + k+1$$

By presumption, $A = B$, so this equality holds as well.

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and satisfies $f(0) = 0$ and $f'(0) = 3$. Prove that there is a number $\sigma > 0$ such that $f(x) > 0$ for every $x \in (0, \sigma)$.

We have that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = f'(0) = 3$

This means that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $|x| < \delta \Rightarrow \left|\frac{f(x)}{x} - 3\right| < \varepsilon$.

Consider $\varepsilon = 3$ and let σ be that value of δ for which holds.
then restrict attention to $x > 0$ and we have that $0 < x < \sigma$ implies $\left|\frac{f(x)}{x} - 3\right| < 3$, which further implies that $\frac{f(x)}{x} > 0$ and so $f(x) > 0$.

Economics 205 Quiz 2

Joel Watson, Fall 2006

1. Consider the function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) \equiv x^2 - 8x + 6 \ln x$.

(a) Calculate the first four derivatives of f evaluated at the point $c = 1$.

$f^{(1)}(x) = 2x - 8 + 6x^{-1}$; $f^{(2)}(x) = 2 + (-1)6x^{-2}$; $f^{(3)}(x) = 2 \cdot 6 x^{-3}$

$f^{(4)}(x) = -6 \cdot 6 x^{-4}$; $f^{(1)}(1) = 2 - 8 + 6 = 0$; $f^{(2)}(1) = -4$; $f^{(3)}(1) = 12$

$f^{(4)}(1) = -36$.

(b) Using your observation of the pattern that develops in part (a), write an expression for $f^{(k)}(1)$ (the kth derivative of f at $c = 1$) for $k \geq 3$.

$f^{(k)}(1) = (-1)^{k-1} \cdot 6 \cdot (k-1)!$

$f^{(k)}(x) = (-1)^{k-1} \cdot 6 (k-1)! \, x^{-k}$

(c) Write the third degree Taylor polynomial of f, centered at the point $c = 1$, as a function of x.

$P_3(x) = f(1) + f'(1)(x-1) + f''(1)(x-1)^2 \cdot \frac{1}{2} + f'''(1)(x-1)^3 \frac{1}{6}$

$= -7 + 0 + -4 \cdot \frac{1}{2}(x-1)^2 + 12 \cdot \frac{1}{6}(x-1)^3$

$= -7 - 2(x-1)^2 + 2(x-1)^3$

(d) Suppose you want to use a Taylor polynomial to estimate the value $f(2)$. Write the error term of the nth degree Taylor polynomial, as a function of n and t, for $x = 2$. (Center the polynomial at $c = 1$.) Find a convenient upper bound on the absolute value of the error term and use it to determine a value of n (as small as you can find) such that the error is smaller than 1/10.

$E_n(x) = \frac{f^{(n+1)}(t)(x-1)^{n+1}}{(n+1)!}$ $\cancel{E_n(2) = \frac{(-1)^n \cdot 6 \cdot (n!) \, t^{-n-1} (x-1)^{n+1}}{(n+1)!}}$

Because $t \in [1,2]$

$\cancel{|E_n(2)| = \frac{6(n!) \, t^{-n-1}(x-1)^{n+1}}{(n+1)!}} = \cancel{\frac{6}{n} t^{-n-1} (x-1)^{n+1}}$

$E_n(2) = \frac{(-1)^n \cdot 6 \cdot (n!) \, t^{-n-1} (2-1)^{n+1}}{(n+1)!} = \frac{(-1)^n \cdot 6}{n \, t^{n+1}}$. Note: $|E_n(2)| \leq \frac{6}{n}$

Pick $n > 60$.

(e) Determine the critical points of f.

$f'(x) \equiv 0 \iff 2x - 8 + 6x^{-1} = 0$, $2x^2 - 8x + 6 = 0$, $x^2 - 4x + 3 = 0$.

$\iff (x-1)(x-3) = 0$. $x^* = 1, 3$.

(f) Determine the nature (local maximizer, minimizer, neither) of the critical points.

$f''(1) = -4 < 0 \Rightarrow x^* = 1$ is a local maximizer.

$f''(3) = 2 - 6\left(\frac{1}{9}\right) = \frac{4}{3} > 0 \Rightarrow x^* = 3$ is a local minimizer.

2. Use matrix algreba to find the vector x^* that solves the following system of equations: Ax=y, where $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $y = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$. Show your work.

$A^{-1}Ax = A^{-1}y$; $I_2 x = A^{-1}y$; $x^* = A^{-1}y$

$|A| = 3 \cdot 2 - (-1)(1) = 7$. $A^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{pmatrix}$

$A^{-1}y = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{3}{7} \end{pmatrix}\begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{14}{7} \\ \frac{21}{7} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = x^*$

3. Calculate $\int_0^1 xe^x dx$. By parts... Let $F(x) = x$, $G(x) = e^x$. $f(x) = 1$, $g(x) = e^x$

$\int_0^1 F(x)g(x)dx = [F(x)G(x)]_0^1 - \int_0^1 f(x)G(x)dx = [xe^x]_0^1 - [e^x]_0^1 = e - 0 - e + 1$
$= 1$

4. Consider the subset of $\mathbf{R}^3$ defined as $\{x \in \mathbf{R}^3 \mid x_1x_2^2 - e^{2x_3} = 2\}$. Find the plane tangent to this set at the point $\hat{x} = (3, 1, 0)$. Represent the plane as a subset of $\mathbf{R}^3$.

$f(x) \equiv x_1x_2^2 - e^{2x_3}$. This is the 2-value level set of f.

$\nabla f(x) = (x_2^2, 2x_1x_2, -2e^{2x_3})$. $\nabla f(3,1,0) = (1, 6, -2)$

$\nabla f(3,1,0)\cdot(x_1-3, x_2-1, x_3-0) = 0 \iff (1,6,-2)\cdot(x_1-3, x_2-1, x_3) = 0$

$x_1 - 3 + 6x_2 - 6 - 2x_3 = 0$

$x_1 + 6x_2 - 2x_3 = 9$ $\{x \in \mathbb{R}^3 / x_1 + 6x_2 - 2x_3 = 9\}$

5. Suppose $f : \mathbf{R} \to \mathbf{R}_+$ and $g : \mathbf{R} \to \mathbf{R}_+$ are both concave down, where $\mathbf{R}_+$ denotes the set of nonnegative real numbers. Must it be the case that $h(x) \equiv f(x)g(x)$ is also concave down? Prove or provide a counterexample if you can.

f, g concave down with $f(x) \geq 0$, $g(x) \geq 0$ $\forall x \in \mathbb{R}$

$\Rightarrow f(x) = c$, $g(x) = b$ $\forall x \in \mathbb{R}$, some $b, c \in \mathbb{R}$ constants.

Then, $f(x)g(x) = h(x) = bc$ also concave down (trivially).

Economics 205 Quiz 3

Joel Watson, Fall 2006

1. Consider the function $f : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}$ defined by $f(x_1, x_2; a) \equiv 11ax_2 + x_1x_2 - ax_1^2 - x_2^2$. Think of a as a parameter in the problem of maximizing $f(x_1, x_2; a)$ by choice of x.

(a) Calculate the first- and second-derivative matrices of f with respect to $x = (x_1, x_2)$.

$$D_x f = (x_2 - 2x_1 a,\ 11a + x_1 - 2x_2)$$

$$D_x^2 f = \begin{pmatrix} -2a & 1 \\ 1 & -2 \end{pmatrix}$$

(b) Calculate the critical point(s) of f (with respect to x, as a function of a) and determine the nature of each.

$x_2 - 2x_1 a = 0 \iff x_2 = 2ax_1$ substitute into second F.O.C. to get $11a + x_1 - 4ax_1 = 0$

$$\Rightarrow x_1^* = \frac{11a}{4a-1} \quad ; \quad x_2^* = \frac{22a^2}{4a-1}$$

$|D_x^2 f| = 4a - 1$. If $a > \frac{1}{4}$ then this is a ~~local max~~ global maximum.

(c) Calculate $v'(3)$, where $v(a) \equiv \max_{x \in \mathbf{R}^2} f(x, a)$.

$v'(a) = 11x_2 - x_1^2$ (evaluated at optimal x_1, x_2)

$$v'(3) = 11 \cdot \frac{22a^2}{4a-1} - \left(\frac{11a}{4a-1}\right)^2$$

2. The equation $x^2 + xye^z = z - 6$ defines a curved surface in $\mathbf{R}^3$. Find the equation of the tangent plane to this surface at the point $(2, -5, 0)$.

$f(x, y, z) \equiv x^2 + xye^z - z + 6$

$\nabla f = (2x + ye^z,\ xe^z,\ xye^z - 1)$, $\nabla f(2, -5, 0) = (-1, 2, -11)$

The tangent plane to level set $f(x, y, z) = 0$ at $(2, -5, 0)$ is

$$\{(x, y, z) \mid (-1, 2, -11) \cdot (x-2, y+5, z) = 0\}$$

$$= \{(x, y, z) \mid -x + 2y - 11z + 12 = 0\}$$

3. Consider the function $F : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, z_1, z_2) = \begin{pmatrix} xz_1 - z_2^2 \\ z_1^3 z_2 - 16 \end{pmatrix}$. Consider whether the identity $F(x, z_1, z_2) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ implicitly defines $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ as a function of x in a neighborhood of $(x^0, z_1^0, z_2^0) = (2, 2, 2)$. Is this the case? If so, write $z = g(x)$ and calculate $Dg(2)$.

Identity: $F(x, g(x)) = \vec{0}$. $D_x F(2,2,2) + D_z F(2,2,2)\, Dg(2) = \vec{0}$

g is well defined near $(2,2,2)$ if $|D_z F(2,2,2)| \neq 0$.

$D_z F(2,2,2) = \begin{pmatrix} 2 & -4 \\ 24 & 8 \end{pmatrix}$. Note that $\left| \begin{pmatrix} 2 & -4 \\ 24 & 8 \end{pmatrix} \right| = 112 \neq 0$.

$$Dg(2) = -\left[D_z F(2,2,2) \right]^{-1} D_x F(2,2,2)$$

$$= \frac{-1}{112} \begin{pmatrix} 8 & 4 \\ -24 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} \\ \frac{3}{7} \end{pmatrix}.$$

$D_z F = \begin{pmatrix} x & -2z_2 \\ 3z_1^2 z_2 & z_1^3 \end{pmatrix}$

$D_x F = \begin{pmatrix} z_1 \\ 0 \end{pmatrix}$

4. Recall that a function $f : \mathbf{R}^n \to \mathbf{R}$ is *concave* if, for all $x, y \in \mathbf{R}^n$ and every $\lambda \in (0, 1)$, we have $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda) f(y)$. Prove that if f is concave and x^* is a local maximizer of f then x^* is a global maximizer of f.

Suppose not. Then $\exists y^*$ s.t. $f(y^*) > f(x^*)$. By concavity, we know that $f(\lambda x^* + (1-\lambda) y^*) \geq \lambda f(x^*) + (1-\lambda) f(y^*) > f(x^*)$ for all $\lambda \in (0,1)$.

Presuming x^* is a local maximizer means $\exists \varepsilon > 0$ such that $\|x - x^*\| < \varepsilon \Rightarrow f(x) \leq f(x^*)$. But we can find $\lambda_\varepsilon \in (0,1)$ such that $\| \lambda_\varepsilon x^* + (1-\lambda_\varepsilon) y^* - x^* \| < \varepsilon$ (that is, $\| (1-\lambda_\varepsilon)(y^* - x^*) \| < \varepsilon$) and then we have a contradiction.

1	2	3	4	5	6	7	8	9	10	11	12	Total
5	4	5	5	7	5	4	5	5	5	5	5	60

Name: ______________________

Economics 205 Final Examination

Prof. Watson, Fall 2006

You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets.

1. Consider the sequence $\{x_n\}$ that is defined inductively by $x_1 = \frac{1}{2}$ and, for every $n \in \mathbf{P}$, $x_{n+1} = \frac{1+x_n}{2}$.

(a) Write the first four numbers in this sequence.

$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}$

(b) Write x_n as a function of n (not as a function of x_{n-1}).

$$x_n = \frac{2^n - 1}{2^n}$$

(c) Does $\{x_n\}$ converge? If so, to what number?

Yes, to 1.

2. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Formally write what it means for f to be continuous at $x = 0$. Show that f is continuous at $x = 0$ by finding an appropriate δ value for each ε.

f continuous at zero means:
For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon$.

In the ~~ex~~ case here, the statement is:
For all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - 0| < \delta$ implies $|x \sin(\frac{1}{x})| < \varepsilon$.

Note that $\sin(\frac{1}{x}) \in [-1, 1]$, so letting $\delta = \varepsilon$ is sufficient.

3. For each of the following matrices, say whether or not it is invertible (that is, whether A^{-1} exists) and, if so, find A^{-1}.

(a) $A = \begin{pmatrix} 2 & 4 \\ 5 & 10 \end{pmatrix}$ (b) $A = \begin{pmatrix} 2 & 7 \\ 3 & 11 \end{pmatrix}$ (c) $A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{2}{3} \end{pmatrix}$.

(a) No,

$|A| = 0$

(b) Yes.

$|A| = 1$

$$A^{-1} = \begin{pmatrix} 11 & -7 \\ -3 & 2 \end{pmatrix}$$

(c) Yes. Note that $\begin{vmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{vmatrix} = \frac{1}{3}$, so the inverse of $\begin{pmatrix} 2 & -1 \\ -1 & \frac{2}{3} \end{pmatrix}$ is $\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$.

$$A^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 6 \end{pmatrix}$$

4. Consider the function $f : (0, \infty) \to \mathbf{R}$ that is given by $f(x) = x^2 - \ln x$.

(a) Write the second-degree Taylor polynomial for f centered at the point $c = 1$.

$$\begin{aligned} P_2(x) &= f(1) + f^{(1)}(1)(x-1) + f^{(2)}(1) \cdot \tfrac{1}{2} \cdot (x-1)^2 \\ &= 1 + 0 \cdot (x-1) + \tfrac{1}{2} \cdot 3 \cdot (x-1)^2 \\ &= 1 + \tfrac{3}{2}(x-1)^2 + x - 1 \end{aligned}$$

(a) Suppose you want to approximate $f(2)$ to within $1/100$ using a Taylor polynomial centered at $c = 1$. Write the expression for the error term of the nth degree Taylor polynomial and find a convenient value of n to achieve the desired bound.

Note: $f^{(1)}(x) = 2x - x^{-1}$; $f^{(2)}(x) = 2 + x^{-2}$; $f^{(3)}(x) = -2x^{-3}$; $f^{(4)}(x) = 6x^{-4}$;

for $k \geq 3$, $f^{(k)}(x) = (-1)^k (k-1)!\, x^{-k}$. $f(1) = 1$

$f^{(1)}(1) = 2 - 1 = 0$. $f^{(2)}(1) = 3$. $f^{(3)}(1) = -2$. $f^{(4)}(1) = 6$.

$f^{(k)}(1) = (-1)^k (k-1)!$

$$E_n(x) = f^{(n+1)}(t) \cdot \frac{1}{(n+1)!}(x-c)^{n+1} = (-1)^{n+1}(n!)\, t^{-n-1}(x-c)^{n+1} \frac{1}{(n+1)!}$$

$$E_n(2) = (-1)^{n+1} t^{-n-1} (2-1)^{n+1} \frac{n!}{(n+1)!}$$

$|E_n(2)| = t^{-n-1}\left(\frac{1}{n+1}\right)$. t is between 1 and 2, so $\frac{1}{t^{n+1}} \in [0,1)$. $\boxed{\text{Pick } n = 100}$

5. Consider the function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = 30 + x^2 - 14x + 20 \ln x$.

(a) Calculate $f'(x)$ and $f''(x)$.

$$f'(x) = 2x - 14 + 20x^{-1}$$

$$f''(x) = 2 - 20x^{-2}$$

(b) Determine the critical points of f.

$$f'(x) = 0 \iff 2x^2 - 14x + 20 = 0 \ ;$$

$$x^2 - 7x + 10 = 0$$

$$(x-2)(x-5) = 0$$

$$x^* = 2, 5$$

(c) Determine the nature (local maximizer, minimizer, or neither) of each critical point.

$f''(2) = -3 < 0 \Rightarrow x^* = 2$ is a local maximizer

$f''(5) = \frac{30}{25} > 0 \Rightarrow x^* = 5$ is a local minimizer

(d) Calculate the tangent line to the graph of f at the point $x = 1$.

It's the Taylor polynomial of degree one, centered at $c = 1$.

$$y = f(1) + f'(1)(x-1)$$

$$y = 17 + 8(x-1)$$

$$y = 9 + 8x$$

6. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given by $f(x, y) = xy + 2x + 5y - x^2 - y^2$. Solve $\max f(x, y)$ by finding the global maximizer (x^*, y^*). Verify first- and second-order conditions.

$$\nabla f(x,y) = (y + 2 - 2x,\ x + 5 - 2y)$$

$$D^2 f(x,y) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\nabla f(x,y) \equiv \vec{0} \iff \left.\begin{aligned} y + 2 &= 2x \\ x &= 2y - 5 \end{aligned}\right\} \Rightarrow \begin{aligned} y + 2 &= 4y - 10 \\ y^* &= 4 \\ x^* &= 3 \end{aligned}$$

$D^2 f(3,4)$ is negative definite. In fact, $D^2 f$ is negative definite for <u>all</u> (x,y), so f is concave down and $(3,4)$ is a global maximizer.

7. Consider the functions $f : \mathbf{R}^3 \to \mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}$ given by
$f(x_1, x_2, x_3) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 e^{x_1} - x_1 x_3^2 \\ x_1 x_2 x_3 - x_2^3 \end{pmatrix}$ and $g(y_1, y_2) = y_1 y_2$.

(a) Compute $Df(0, 2, 1)$. Note that your answer will be a 2×3 matrix.

$$Df(x) = \begin{pmatrix} x_2 e^{x_1} - x_3^2 & e^{x_1} & -2x_1x_3 \\ x_2 x_3 & x_1 x_3 - 3x_2^2 & x_1 x_2 \end{pmatrix}$$

$$Df(0,2,1) = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -12 & 0 \end{pmatrix}$$

(b) Compute $Dh(0, 2, 1)$, where $h(x_1, x_2, x_3) \equiv [g \circ f](x_1, x_2, x_3)$. Note that your answer will be a 1×3 matrix.

Use chain rule: $Dg(Y_1, Y_2) = (Y_2 \;\; Y_1)$

$Dg(f(0,2,1)) = (-8 \;\; 2)$

$Dh(0,2,1) = Dg(f(0,2,1))Df(0,2,1) = (-8 \;\; 2)\begin{pmatrix} 1 & 1 & 0 \\ 2 & -12 & 0 \end{pmatrix}$

$= (-4 \;\; -32 \;\; 0)$

8. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = 7xy - y^3 e^x + 1$.

(a) Does the identity $f(x, y) = 0$ define y as a function of x near the point $(x^0, y^0) = (0, 1)$? If so, and letting g denote the implicit function (so we have $y = g(x)$), calculate $g'(0)$.

Yes. The identity is: $f(x, g(x)) = 0$. Thus,

$\frac{\partial f}{\partial x}(0,1) + \frac{\partial f}{\partial y}(0,1)\, g'(0) = 0$ and so $g'(0) = \dfrac{-\frac{\partial f}{\partial x}(0,1)}{\frac{\partial f}{\partial y}(0,1)}$.

Note that $\frac{\partial f}{\partial x} = 7y - y^3 e^x$; $\frac{\partial f}{\partial y} = 7x - 3y^2 e^x$.

$g'(0) = \dfrac{-(7-1)}{-3} = 2$. ($g$ well defined because $\frac{\partial f}{\partial y}(0,1) \neq 0$.)

(b) Determine the plane tangent to the graph of f at the point $(x^0, y^0, z^0) = (0, 1, 0)$.

It's the 0-value level set of $h(x,y,z) = 7xy - y^3 e^x + 1 - z$ at $(0,1,0)$. (Also, it's the 1st degree Taylor polynomial of f...)

$\nabla h = (7y - y^3 e^x,\; 7x - 3y^2 e^x,\; -1)$. $\nabla h(0,1,0) = (6, -3, -1)$

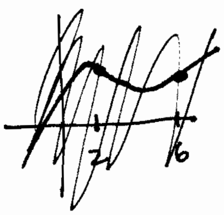
$\{(x,y,z) \mid \nabla h(0,1,0) \cdot (x-0, y-1, z-0) = 0\}$

$= \{(x,y,z) \mid (6,-3,-1)\cdot(x, y-1, z) = 0\} = \{(x,y,z) \mid 6x - 3y - z + 3 = 0\}$.

9. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3 - 9x^2 + 15x + 74$.

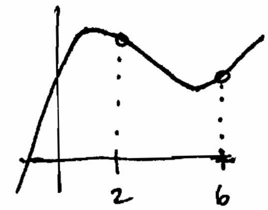
(a) Solve $\max f(x)$ subject to $x \in [2, 6]$. Find the maximizer and note conditions you check.

$f'(x) = 3x^2 - 18x + 15$, $\quad f'(x) = 0 \iff \hat{x} = 1, 5$



$f''(x) = 6x - 18.$ $\quad f''(1) = -12$ so local max here.

$f''(5) = 12$ " " min ".



$x = 1$ not feasible.

Check end points: $f(2) = 76$. $f(6) = 56$

$$\boxed{x^* = 2}$$

(b) Solve $\min f(x)$ subject to $x \geq 0$. Find the minimizer and note conditions you check.

We know $\hat{x} = 5$ is a local minimizer and f increases at higher values of x.

$f(5) = 49$. $\quad f(0) = 74$ (boundary check)

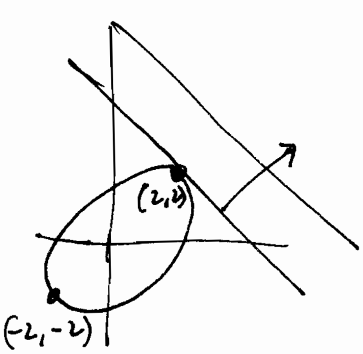
$\Rightarrow \quad x^* = 5$

10. Solve the problem $\max x + y$ subject to $x^2 + y^2 - xy - 4 = 0$ and determine the maximizer (x^*, y^*). Clearly show the conditions that you check.

$f(x,y) = x + y$. $\quad g(x,y) = x^2 + y^2 - xy - 4$

$\nabla f(x,y) = (1, 1)$. $\quad \nabla g(x,y) = (2x - y, 2y - x)$

F.O.C: $\nabla f(x,y) = \lambda \nabla g(x,y) \iff \left.\begin{array}{l} 1 = \lambda(2x - y) \\ 1 = \lambda(2y - x) \end{array}\right\} \Rightarrow x = y$



Substituting $x = y$ into $g(x,y) = 0$ yields critical points $(2,2)$ and $(-2,-2)$.

~~Note: f(2,2)~~ But ~~we need~~ also note that $\lambda > 0$ only for $(2,2)$. Further, f is concave down and g is concave up. Thus, $(2,2)$ is the global maximizer.

11. Consider an arbitrary function $f : \mathbf{R}^2 \to \mathbf{R}$ for which the problem of maximizing $f(x,y)$ is *uniquely* solved by (x^*, y^*). That is, (x^*, y^*) is the only global maximizer of f. Let $w^* \equiv f(x^*, y^*)$ be the maximized value. Suppose that, for every $x \in \mathbf{R}$, the value $v(x) \equiv \max_{y \in \mathbf{R}} f(x,y)$ is well-defined; that is, the problem of maximizing f by choice of y (for a fixed x) has a solution.

(**a**) Prove that $\max_{x \in \mathbf{R}} v(x) = w^*$. First let's show that $v(x) \geq w^*$ for ~~Suppose not. Case 1: max v(x)~~ some x. In particular, $v(x^*) \geq w^*$.
This is because y^* is feasible for the problem of $\max_y f(x^*, y)$.
Next suppose ~~max v(x)~~ for some $\hat{x}$, $v(\hat{x}) > w^*$. Then $\exists\, \hat{y}$ s.t. $f(\hat{x}, \hat{y}) > w^*$. But this would contradict that (x^*, y^*) maximizes f.
We thus conclude that $\max_x v(x) = v(x^*) = w^*$.

(**b**) Suppose that the sequence $\{x_k, y_k\}_{k=1}^{\infty}$ has the property that, for some $\alpha \in (0,1)$ and for every integer k, $f(x_{k+1}, y_{k+1}) \geq \alpha f(x^*, y^*) + (1-\alpha) f(x_k, y_k)$. Is it necessarily the case that $\{x_k, y_k\}$ converges to (x^*, y^*)? Explain why or provide a counterexample.

No. For example, consider $f(x,y) = \begin{cases} x & \text{if } x \in [0,1) \\ 1 & \text{if } x=2, y=2 \\ 0 & \text{otherwise} \end{cases}$

Take $\alpha = \frac{1}{2}$. $f(x^*, y^*) = 1$

Define: $x_0 \equiv 0$. For each k, let x_k be defined by:

$$1 - x_k = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot x_{k-1}.$$

$y_k = 0 \quad \forall k.$

Note that $x_0 = 0$, $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{4}$, ..., so $\{x_k, y_k\}$ satisfies the condition, but $(x_k, y_k) \to (1,0)$ ~~yet f(1,0)~~ $\neq (2,2)$. (A continuity issue.)

12. Suppose X is a compact subset of $\mathbf{R}^n$ and that $f : X \to \mathbf{R}$ is continuous. Prove that if $f(x) > 0$ for every $x \in X$ then there is a number $z > 0$ such that $f(x) > z$ for every $x \in X$.

Proof by contradiction:
Suppose that for every $z > 0$ $\exists\, x^z \in X$ for which $f(x^z) \leq z$.
We must show that this contradicts the assumptions of the question. Note that, writing $y_n \equiv x^{(\frac{1}{n})}$ (that is, $z = \frac{1}{n}$), we have $\{y_n\} \subset X$ and $f(y_n) \to 0$ by ~~cost~~ construction. Because X is compact, we can find a subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y$ for some $y \in X$. Noting that $y_{n_k} \to y$ and $f(y_{n_k}) \to 0$, by continuity of f we have $f(y_{n_k}) \to f(y)$.
This means that $f(y) = 0$, which contradicts that $f(x) > 0$ $\forall x \in X$.