Economics 205 Quiz 1

Joel Watson, Fall 2006

1. Consider an arbitrary function $f : [0, 1] \to [0, 1]$. Define the sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ inductively by (i) $x_1 = 0$, and (ii) for each $n \in \mathbf{P}$, $x_{n+1} = f(x_n)$.

(a) Given the assumptions made above, must it be the case that $\{x_n\}$ converges? If so, explain why. If not, provide a counterexample.

(b) Given the assumptions made above, must it be the case that $\{x_n\}$ has a convergent subsequence? If so, explain why. If not, provide a counterexample.

(c) Suppose that $f(x) = (x + 1)/3$. Write the first few elements of $\{x_n\}$ and calculate $\lim_{n\to\infty} x_n$.

2. Calculate the following limits. In the case in which one or both of the limits does not exist, state this.

(a) $\lim_{n\to\infty} x_n$, where the sequence $\{x_n\}$ is defined by $x_n = \frac{(2n^2+3)(4n+1)}{4n^3}$ for all $n \in \mathbf{P}$

(b) $\lim_{x\to 1} \frac{(x-1)^2}{\ln x}$

3. For both cases below, write expressions for $h'(x)$.

(a) $h(x) \equiv [f(x)]^{-2}[1 - f(x)]$

(b) $h(x) \equiv f(3x + g(x))$

4. Prove by induction: $1 + 2 + 3 + \ldots + n = n(n + 1)/2$.

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and satisfies $f(0) = 0$ and $f'(0) = 3$. Prove that there is a number $\sigma > 0$ such that $f(x) > 0$ for every $x \in (0, \sigma)$.

Economics 205 Quiz 2

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1. Consider the function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) \equiv x^2 - 8x + 6 \ln x$.

(a) Calculate the first four derivatives of f evaluated at the point $c = 1$.

(b) Using your observation of the pattern that develops in part (a), write an expression for $f^{(k)}(1)$ (the kth derivative of f at $c = 1$) for $k \geq 3$.

(c) Write the third degree Taylor polynomial of f, centered at the point $c = 1$, as a function of x.

(d) Suppose you want to use a Taylor polynomial to estimate the value $f(2)$. Write the error term of the nth degree Taylor polynomial, as a function of n and t, for $x = 2$. (Center the polynomial at $c = 1$.) Find a convenient upper bound on the absolute value of the error term and use it to determine a value of n (as small as you can find) such that the error is smaller than $1/10$.

(e) Determine the critical points of f.

(f) Determine the nature (local maximizer, minimizer, neither) of the critical points.

2. Use matrix algreba to find the vector x^* that solves the following system of equations: Ax=y, where $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $y = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$. Show your work.

3. Calculate $\int_0^1 xe^x dx$.

4. Consider the subset of $\mathbf{R}^3$ defined as $\{x \in \mathbf{R}^3 \mid x_1 x_2^2 - e^{2x_3} = 2\}$. Find the plane tangent to this set at the point $\hat{x} = (3, 1, 0)$. Represent the plane as a subset of $\mathbf{R}^3$.

5. Suppose $f : \mathbf{R} \to \mathbf{R}_+$ and $g : \mathbf{R} \to \mathbf{R}_+$ are both concave down, where $\mathbf{R}_+$ denotes the set of nonnegative real numbers. Must it be the case that $h(x) \equiv f(x)g(x)$ is also concave down? Prove or provide a counterexample if you can.

Economics 205 Quiz 3

Joel Watson, Fall 2006

1. Consider the function $f : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}$ defined by $f(x_1, x_2; a) \equiv 11ax_2 + x_1x_2 - ax_1^2 - x_2^2$. Think of a as a parameter in the problem of maximizing $f(x_1, x_2; a)$ by choice of x.

(a) Calculate the first- and second-derivative matrices of f with respect to $x = (x_1, x_2)$.

(b) Calculate the critical point(s) of f (with respect to x, as a function of a) and determine the nature of each.

(c) Calculate $v'(3)$, where $v(a) \equiv \max_{x \in \mathbf{R}^2} f(x, a)$.

2. The equation $x^2 + xye^z = z - 6$ defines a curved surface in $\mathbf{R}^3$. Find the equation of the tangent plane to this surface at the point $(2, -5, 0)$.

3. Consider the function $F : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, z_1, z_2) = \begin{pmatrix} xz_1 - z_2^2 \\ z_1^3 z_2 - 16 \end{pmatrix}$. Consider whether the identity $F(x, z_1, z_2) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ implicitly defines $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ as a function of x in a neighborhood of $(x^0, z_1^0, z_2^0) = (2, 2, 2)$. Is this the case? If so, write $z = g(x)$ and calculate $Dg(2)$.

4. Recall that a function $f : \mathbf{R}^n \to \mathbf{R}$ is *concave* if, for all $x, y \in \mathbf{R}^n$ and every $\lambda \in (0, 1)$, we have $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$. Prove that if f is concave and x^* is a local maximizer of f then x^* is a global maximizer of f.

Economics 205 Final Examination

Prof. Watson, Fall 2006

You have three hours to complete this closed-book examination. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets.

1. Consider the sequence $\{x_n\}$ that is defined inductively by $x_1 = \frac{1}{2}$ and, for every $n \in \mathbf{P}$, $x_{n+1} = \frac{1+x_n}{2}$.

(a) Write the first four numbers in this sequence.

(b) Write x_n as a function of n (not as a function of x_{n-1}).

(c) Does $\{x_n\}$ converge? If so, to what number?

2. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Formally write what it means for f to be continuous at $x = 0$. Show that f is continuous at $x = 0$ by finding an appropriate δ value for each ε.

3. For each of the following matrices, say whether or not it is invertible (that is, whether A^{-1} exists) and, if so, find A^{-1}.

(a) $A = \begin{pmatrix} 2 & 4 \\ 5 & 10 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 2 & 7 \\ 3 & 11 \end{pmatrix}$ **(c)** $A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{2}{3} \end{pmatrix}$.

4. Consider the function $f : (0, \infty) \to \mathbf{R}$ that is given by $f(x) = x^2 - \ln x$.

(a) Write the second-degree Taylor polynomial for f centered at the point $c = 1$.

(a) Suppose you want to approximate $f(2)$ to within $1/100$ using a Taylor polynomial centered at $c = 1$. Write the expression for the error term of the nth degree Taylor polynomial and find a convenient value of n to achieve the desired bound.

5. Consider the function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) = 30 + x^2 - 14x + 20 \ln x$.

(a) Calculate $f'(x)$ and $f''(x)$.

(b) Determine the critical points of f.

(c) Determine the nature (local maximizer, minimizer, or neither) of each critical point.

(d) Calculate the tangent line to the graph of f at the point $x = 1$.

6. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be given by $f(x, y) = xy + 2x + 5y - x^2 - y^2$. Solve $\max f(x, y)$ by finding the global maximizer (x^*, y^*). Verify first- and second-order conditions.

7. Consider the functions $f : \mathbf{R}^3 \to \mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}$ given by
$f(x_1, x_2, x_3) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_2 e^{x_1} - x_1 x_3^2 \\ x_1 x_2 x_3 - x_2^3 \end{pmatrix}$ and $g(y_1, y_2) = y_1 y_2$.

(a) Compute $Df(0, 2, 1)$. Note that your answer will be a 2×3 matrix.

(b) Compute $Dh(0, 2, 1)$, where $h(x_1, x_2, x_3) \equiv [g \circ f](x_1, x_2, x_3)$. Note that your answer will be a 1×3 matrix.

8. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = 7xy - y^3 e^x + 1$.

(a) Does the identity $f(x, y) = 0$ define y as a function of x near the point $(x^0, y^0) = (0, 1)$? If so, and letting g denote the implicit function (so we have $y = g(x)$), calculate $g'(0)$.

(b) Determine the plane tangent to the graph of f at the point $(x^0, y^0, z^0) = (0, 1, 0)$.

9. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3 - 9x^2 + 15x + 74$.

(a) Solve $\max f(x)$ subject to $x \in [2, 6]$. Find the maximizer and note conditions you check.

(b) Solve $\min f(x)$ subject to $x \geq 0$. Find the minimizer and note conditions you check.

10. Solve the problem $\max x + y$ subject to $x^2 + y^2 - xy - 4 = 0$ and determine the maximizer (x^*, y^*). Clearly show the conditions that you check.

11. Consider an arbitrary function $f : \mathbf{R}^2 \to \mathbf{R}$ for which the problem of maximizing $f(x, y)$ is *uniquely* solved by (x^*, y^*). That is, (x^*, y^*) is the only global maximizer of f. Let $w^* \equiv f(x^*, y^*)$ be the maximized value. Suppose that, for every $x \in \mathbf{R}$, the value $v(x) \equiv \max_{y \in \mathbf{R}} f(x, y)$ is well-defined; that is, the problem of maximizing f by choice of y (for a fixed x) has a solution.

(a) Prove that $\max_{x \in \mathbf{R}} v(x) = w^*$.

(b) Suppose that the sequence $\{x_k, y_k\}_{k=1}^{\infty}$ has the property that, for some $\alpha \in (0, 1)$ and for every integer k, $f(x_{k+1}, y_{k+1}) \geq \alpha f(x^*, y^*) + (1 - \alpha) f(x_k, y_k)$. Is it necessarily the case that $\{x_k, y_k\}$ converges to (x^*, y^*)? Explain why or provide a counterexample.

12. Suppose X is a compact subset of $\mathbf{R}^n$ and that $f : X \to \mathbf{R}$ is continuous. Prove that if $f(x) > 0$ for every $x \in X$ then there is a number $z > 0$ such that $f(x) > z$ for every $x \in X$.