Economics 205 Exercises

Prof. Watson, Fall 2006
(Includes examinations through Fall 2003)

Part 1: Basic Analysis

1. Using ε and δ, write in formal terms the meaning of $\lim_{x \to a} f(x) = c$, where $f : \mathbf{R} \to \mathbf{R}$.
2. Write the definition of $\sup X$ for some set $X \subset \mathbf{R}$.
3. Compute:
   (a) $\lim_{n\to\infty} \frac{6n^2+2}{12n^3}$
   (b) $\lim_{n\to\infty} \frac{(2n+3)n}{n^2+1}$.
4. Indicate whether the following limits exist and compute those that do exist.
   (a) $\lim\limits_{x\to\infty} \frac{3x^2-2x+1}{(x+3)^2}$,
   (b) $\lim\limits_{x\to 0} \frac{x(x+1)^2}{(x-1)^3}$, and
   (c) $\lim_{x\to 0} \frac{(x+3)^2-9}{x^3}$.
5. Let $X \equiv [0, 2]$ and $Y \equiv [0, 1]$. Define function $f : X \to Y$ by $f(x) = 2x - x^2$, for every $x \in X$.
   (a) Graph the function f.
   (b) Is f *onto*? Explain briefly.
   (c) Is f *one-to-one*? Explain briefly.
   (d) Is there a function $g : [0, 1] \to [0, 1]$ such that $g \circ f$ is one-to-one? Find such a function g or prove that one does not exist.
   (e) Is there a function $g : [0, 2] \to [0, 2]$ such that $f \circ g$ is one-to-one? Find such a function g or prove that one does not exist.
6. Consider the sequence $\{a_n\}$ defined by $a_1 \equiv 1$ and $a_{n+1} \equiv a_n + 1/2^n$. Is this a Cauchy sequence? (Try to demonstrate it.) If so, compute its limit.
7. Consider the sequence $\{x_n\}$ defined inductively by $x_1 = 2$ and $x_{n+1} = \frac{1}{x_n^2}$, for all n.
   (a) Write a few terms of this sequence.
   (b) Does this sequence converge?
   (c) Does $\{x_n\}$ have a convergent subsequence? If so, describe it.

8. Consider the sequence $\{a_n\}$ that is defined inductively by $a_1 = \frac{1}{2}$ and, for each $n \in \mathbf{P}$, $a_{n+1} = \frac{3-a_n}{2}$.

   (a) Find an expression for a_n as a function of n only (not written as a function of a_{n-1}). Hint: Define the sequence $\{b_n\}$ by $b_n \equiv a_n - 1$ for every $n \in \mathbf{P}$; then write b_{n+1} as a function of b_n.
   (b) What is the limit of $\{a_n\}$?
   (c) Calculate $\sup X$ and $\inf X$, where $X \equiv \{a_n \mid n \in \mathbf{P}\}$.

9. Let $\{a_n\}$ be defined by $a_n = 1/n(n+1)$ for all n.

   (a) Does $\{a_n\}$ converge? If so, what is its limit?
   (b) Let $s_k \equiv \sum_{n=1}^{k} a_n$ for $k \in \mathbf{P}$. Write the first few elements of the sequence $\{s_k\}$. Does $\{s_k\}$ converge? If so, what is its limit? (Hint: $1/n(n+1) = 1/n - 1/(n+1)$.)
   (c) Use the conclusion of the previous part to prove that $t_k \equiv \sum_{n=1}^{k} 1/n^2$ converges. (Hint: compare $1/n^2$ to $2/n(n+1)$.)

10. Find $\sup\{f(x) \mid x \in \mathbf{R}\}$ for $f(x) = \begin{cases} x^3 & x < 2 \\ 3 - x & x \geq 2 \end{cases}$.

11. Consider the function $f(x) \equiv \begin{cases} x^2 & x \geq 0 \\ \frac{x}{|x|} & x < 0 \end{cases}$.

    (a) Is f continuous (on the entire real line)?
    (b) Does $\lim_{x \to 0^+} f(x)$ exist? If so, compute it.
    (c) Does $\lim_{x \to 0^-} f(x)$ exist? If so, compute it.
    (d) Does $\lim_{x \to 0} f(x)$ exist? If so, compute it.

12. Suppose X and Y are subsets of real numbers.

    (a) Prove that if X and Y are closed, then $X \cap Y$ is closed.
    (b) Prove that if X and Y are both compact, then $X \cup Y$ is compact.

13. For each example below, state whether the set X is closed. You do not need to prove your answer.

    (a) $X = [0, 1]$.
    (b) $X = [0, 1] \cup [3, 4]$.
    (c) $X = [0, 4] \backslash [2, 3]$.
    (d) $X = \{x \mid x \geq 2\}$.
    (e) $X = [1, 3] \cap [0, 2)$.
    (f) $X = [\frac{1}{2}, 1] \cup [\frac{1}{3}, \frac{1}{2}] \cup [\frac{1}{4}, \frac{1}{3}] \cup [\frac{1}{5}, \frac{1}{4}] \cup \cdots$.

14. Consider the set $X = \{(-1)^n + \frac{1}{n} \,|\, n \in \mathbf{P}\}$.

    (a) Do $\sup X$ and $\inf X$ exist? If so, determine these values.

    (b) Do $\max X$ and $\min X$ exist? If so, determine these values.

15. Suppose you know that the sequence $\{a_n\}$ converges and that $a_{k^2} = \frac{1}{k}$ for each positive integer k. What do you know about $\{a_n\}$?

16. Give an example of a bounded function that is defined on a closed interval but has no maximum.

17. Prove that between any two distinct rational numbers there is another rational number.

18. Find the indicated limits. Justify your answer by mentioning a general property of limits or with a short proof (using εs and δs).

    (a) $\lim\limits_{x \to 2} \frac{x^2-1}{x^2}$

    (b) $\lim\limits_{x \to 0} f(g(x))$, where $f(y) = y$ and $g(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$.

19. Let $\mathbf{R}$ be the space (universe). Prove that if $X \subset \mathbf{R}$ is closed then X' (the complement of X) is open.

20. Prove that a continuous function $f : [0, 1] \to [0, 1]$ has a fixed point (a point x such that $f(x) = x$.

Part 2: Calculus of One Variable

1. Write the definition of the derivative of a function $f : X \to \mathbf{R}$ at a point a.

2. Compute the derivatives of $x \ln x$, $e^x \ln x$, and $x^2 + 3x - 4$.

3. Consider the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = \begin{cases} x^2 + 2x & x \geq 0 \\ 2x & x < 0 \end{cases}$.

   (a) Is f continuous at $x = 0$? Why?

   (b) Is f differentiable at $x = 0$? Prove your answer by constructing the appropriate limits.

4. Consider the function $f(x) = \begin{cases} x^3 + 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$.

   (a) Does $\lim_{x \to 0}$ exist? If so, compute it.

   (b) Does $f'(x)$ exist at x=0? If so, compute $f'(0)$.

5. Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable and satisfy $f(x) \neq 0$ for all x. What is the derivative of h, for:

   (a) $h(x) = (f(x))^a$,

   (b) $h(x) = f(x) + 3x^2$,

   (c) $h(x) = f(x) \ln[f(x)]^2$,

   (d) $h(x) = f(f(x))$, and

   (e) $h(x) = f(e^x)$.

6. Consider the function $h(x) = x \ln(x^2 + 1)$. Compute $h'(x)$.

7. Suppose $f : \mathbf{R} \to \mathbf{R}$ is differentiable and satisfies $f(x) \neq 0$ for all x. Which of the following is true? Why?

   (a) f attains a maximum.

   (b) Either $f(x) > 0$ for all x or $f(x) < 0$ for all x.

   (c) $h(x) = 1/f(x)$ is continuous.

   (d) $h(x) = 1/f(x)$ is bounded.

8. Evaluate the following limits if they exist; otherwise note noncovergence:

   (a) $\lim_{x \to 0} (e^x/x)^2$ and

   (b) $\lim_{x \to 0} \ln x^2 - \ln x$.

9. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuously differentiable, with $f(0) = 0$, and supppose $g : \mathbf{R} \to \mathbf{R}$ is continuous. Suppose you know that for all $x \neq 0$, $g(x) = -f(x) \ln[f(x)]$. What is $g(0)$?

10. Let $f(x) = \begin{cases} 1 & x = 1 \\ 0 & x = 0 \\ Q(x) & x \in (0,1] \end{cases}$, where $Q(x) = 1/n$ for that $n \in \mathbf{P}$ for which $1/(n+1) \leq x < 1/n$.

   (a) For what x is $f'(x)$ defined?
   (b) What is $\int_0^1 f(x)dx$? (Difficult – just simplify the summation.)

11. Consider the function $f(x) = xe^x$ defined on $\mathbf{R}$.

   (a) Write the second-degree Taylor polynomial for f at $c = 0$.
   (b) Provide an expression for the error term of the nth degree Taylor approximation of f at $c = 0$, using t and x variables.

12. Consider the function $f(x) = 1/x$, for $x > 0$. Write the expression for the error term of the nth-degree Taylor polynomial that is anchored at the point $c = 1$. Your answer should include variables x and t.

13. Write the second degree Taylor polynomial of f at $c = 1$, for the following functions. Also write the expression for the error term.

   (a) $f(x) = xe^x$,
   (b) $f(x) = x^5 - 3x^2 + 1$,
   (c) $f(x) = x^2 - \ln x$, and
   (d) $f(x) = 1/x$.

14. Consider $f(x) = 1/x$, where f is defined over the interval $(0, 2)$.

   (a) Write the third degree Taylor polynomial for f around the point $c = 1$.
   (b) Using the expression of the error term $E_n(h)$ for the n-term Taylor polynomial (also around point $c = 1$) from Taylor's Theorem, find an upper bound on $|E_n(h)|$.

15. Consider the function $f(x) = x \ln x$.

   (a) Write the third degree Taylor polynomial of f at point $c = 1$.
   (b) Write an expression for the absolute value of the error term for the nth degree Taylor polynomial of f, as a function of x and t.

16. For each function f below, for what $n \in \mathbf{P}$ is the error of the n-term Taylor approximation around $c = 0$, for $x \in (0, 1)$, within 0.0001?

   (a) $f(x) = e^{2x}$,
   (b) $f(x) = e^x$, and
   (c) $f(x) = x^2 - \ln(x + 1)$.

17. Consider the function $f(x) = 2e^x + x^3$.

    (a) What is the equation of the tangent line of f at the point $x = 0$? (Write your answer in the form $y = mx + b$, where m and b are constants.)

    (b) Write the third-degree Taylor polynomial of f anchored at $c = 0$.

18. Consider the function $f(x) = e^x$.

    (a) Write the error term of the nth-degree Taylor approximation of f at the point $a = 0$. (The last term has the nth derivative.)

    (b) For what value of n will the nth-order Taylor approximation be accurate within $e/24$ for $x \in (-1, 1)$?

19. Find the local maxima and minima of: (a) $x^4 - 18x^2 + 80$, (b) $-x^3 + 4x^2 - x - 6$, (c) $2e^{x+1} - e^{2x}$, and (d) $x/\ln x$.

20. Consider the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 42$.

    (a) Find the local maxima and minima of f and graph the function (noting the critical points).

    (b) Find the equation of the tangent line to f at $x = 1$.

    (c) Compute $g'(0)$, where $g(x) \equiv \ln[f(x)]^2$.

21. Consider the function $f(x) = e^{2x-x^2}$.

    (a) Compute the first and second derivatives of f.

    (b) Identify the critical points of f, determine whether these are maximizers or minimizers (or neither), and graph the function.

22. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 9 - 12x + 9x^2 - 2x^3$.

    (a) Calculate $f'(x)$ and $f''(x)$.

    (b) Determine the critical points of f.

    (c) Determine the nature (local maximizer, minimizer, or neither) of each critical point.

    (d) Draw the graph of f (the set of points (x, y) such that $y = f(x)$). Label the x and y intercepts and the critical points. (Note that $f(3) = 0$.)

    (e) Does $f(x)$ attain a global maximum or minimum?

23. Find and determine the nature of the critical points of $f(x) = \frac{2+x}{x^2}$. Graph the function. Is the function defined on $\mathbf{R}$?

24. Consider the function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) \equiv 27 - x^2 - \frac{54}{x}$, for every $x > 0$.

    (a) Calculate the first and second derivatives of f as a function of x.

    (b) Determine the critical point x^* of f.

    (c) Determine the nature (maximizer, minimizer, neither) and scope (local or global) of the critical point x^*. Also calculate $f(x^*)$.

    (d) Graph f.

25. Suppose $f : [0,\infty) \to \mathbf{R}$ and $g : [0,\infty) \to \mathbf{R}$ satisfy $f''(x) < 0$, $g''(x) > 0$, $\lim_{x\to\infty} f'(x) = 0$, and $\lim_{x\to\infty} g'(x) = \infty$. Does $\max\{f(x) - g(x)\}$ exist?

26. Evaluate/simplify:

    (a) $\int_0^1 xe^x dx$ (by parts),
    (b) $\int \{f'(x)g(x) + [f(x) + 2g(x)]g'(x)]\}dx$,
    (c) $\int \frac{6-x}{(x-3)(2x+5)} dx$ (partial fractions method),
    (d) $\int_0^1 x\ln(x+3)dx$,
    (e) $\int x^n \ln x dx$,
    (f) $\int_0^3 \frac{2x}{x^2+2} dx$,
    (g) $\int_0^1 e^{\sqrt{x}} dx$,
    (h) $\int_e^{e^2} 4x \ln x dx$,
    (i) $\int \ln x dx$,
    (j) $\int_0^5 [x^3 - (x-1)^2] dx$,
    (k) $\int_e^{e^2} (4x \ln x) dx$,
    (l) $\int_0^2 (8x^3 - 2x) dx$,
    (m) $\int_0^1 (3x^2 - 6x + 1) dx$,
    (n) $\int_1^e \frac{\ln x}{x} \ln x \, dx$, and

27. Compute: (a) $\int_e^{e^2} (\ln x) dx$, (b) $\int_0^2 (3x^2 - 14x - 2x^{-2}) dx$.

28. Compute the following definite integrals.

    (a) $\int_2^4 \frac{x^2}{x^3-4} dx$
    (b) $\int_e^{e^3} \ln x \, dx$
    (c) $\int_0^9 \frac{e^{x^{1/2}}}{x^{1/2}} dx$.

29. In precise terms, what does it mean for the problem $\max_{x\in X} f(x)$ to have no solution? Give an example of a maximization problem that has no solution because the feasible set is unbounded.

30. Take some function $f(x)$, where x is a scalar, and suppose that f is $K+1$ times continuously differentiable throughout some neighborhood of the point x^*. Let f_k denote the kth derivative of f.

    (a) Suppose K is even and $f_1(x^*) = f_2(x^*) = \ldots = f_{K-1}(x^*) = 0$. Derive necessary and sufficient conditions on $f_K(x^*)$ for x^* to be a local maximum of f. (First think about whether our usual first- and second-order necessary and sufficient conditions are met.)
    (b) What are the necessary and sufficient conditions when K is odd?
    (c) Determine whether the functions $f(x) = x^3$ and $f(x) = -x^4$ have and local maxima, and identify them if they exist.

31. Recall the definition of concavity: f is concave over the interval $[a, b]$ if, for every pair of points $x, y \in [a, b]$ and every number $\delta \in (0, 1)$, we have $f(\delta x + (1 - \delta)y) \geq \delta f(x) + (1 - \delta)f(y)$. Consider the following claim:

   If f is concave over $[a, b]$ then f is continuous at each point in the interval (a, b).

   Explain whether you think this claim is true or false. If you can, provide a counterexample or a proof.

32. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a concave function and $g(x) = ax + b$, with $a > 0$. Prove that $g \circ f$ is concave.

33. Consider functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$.

   (a) Prove that if g is concave and f is both increasing and concave then $f \circ g$ is concave.

   (b) Is the assumption that f is increasing necessary to reach the conclusion of part (a)? In not, explain why. If so, find a counterexample to the claim that f, g concave implies $f \circ g$ concave.

34. Consider a function $f : \mathbf{R} \to \mathbf{R}$. Decide whether the following claims are true or false. For those that are true, provide a proof. For those that are false, provide counterexamples to the claims.

   (a) If f is concave then f attains a maximum.

   (b) If f is continuous and X is closed then $f(X)$ is closed.

   (c) If f is continuous and, for some $X \subset \mathbf{R}$, $f(X)$ is open then X is open.

35. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a strictly increasing function and $g : \mathbf{R} \to \mathbf{R}$ is any arbitrary function. Define $h : \mathbf{R} \to \mathbf{R}$ by $h \equiv f \circ g$. Is it possible for h to have a maximum whereas g does not have a maximum? Is it possible for g to have a maximum whereas h does not have a maximum? Explain.

36. Is it true that $\ln(1 + x) \leq x$ for all $x \geq 0$? Prove your answer.

37. Prove, using the definition of continuity, that if $f : [a, b] \to \mathbf{R}$ is continuous and $f(a) < 0 < f(b)$, then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

38. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

   (a) Prove that f is differentiable at $x = 0$ and calculate $f'(0)$.

   (b) Find the formula for $f'(x)$ for $x \neq 0$..

   (c) Show that f' is not continuous at $x = 0$.

Part 3: Multiple Variables

1. For each of the following matrices, say whether or not it is invertible (that is, whether A^{-1} exists) and, if so, find A^{-1}.

   (a) $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$ **(c)** $A = \begin{pmatrix} 1 & 0 \\ \frac{-3}{2} & \frac{1}{2} \end{pmatrix}$.

2. Compute $Df(x)$ for $f : \mathbf{R}^3 \to \mathbf{R}^2$ given by $f(x) = \begin{pmatrix} x_1^2 + 2x_1x_2 - x_3 \\ x_2x_3 + x_3^2 \end{pmatrix}$.

3. Evaluate
$$\lim_{(x,y)\to(0,0)} |x|^{|y|}.$$

4. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) \equiv x^3y - y^2$.

   (a) Graph the zero-value level set — that is, the set of points in $\mathbf{R}^2$ given by $\{(x, y) \mid f(x, y) = 0\}$.
   (b) Determine the equation of the line tangent to this level set at the point $(1, 1)$.
   (c) Find the equation of the tangent plane to the graph of $z = f(x, y)$ at the point $(x^*, y^*, z^*) = (2, 1, 7)$.

5. Consider the function $f(x, y) = 3x^2 - 2y + xy$.

   (a) Compute $\nabla f(x, y)$.
   (b) Find the equation of the plane tangent to the graph of f at the point $(3, -10, 17)$.

6. Find the equation of the plane tangent to $z = e^{xy} + e^y$ at $(x, y, z) = (0, 0, 2)$.

7. Evaluate the following derivatives.

   (a) $h'(x)$, for $h(x) = [f(x^2)]^2$,
   (b) $h'(x)$, for $h(x) = f(x, x^2)$ and $f(y, z) = z/(z + y)$,
   (c) $Dh(2, 1)$, for $h = f \circ g$ where $f : \mathbf{R}^2 \to \mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}^2$ are defined by $f(y) = \begin{pmatrix} y_1^2 + 3 \\ y_1y_2 \end{pmatrix}$ and $g(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$.

8. Consider the function $f(x, y, z) = 3x^2 + 2xy - z^2$.

   (a) Compute $\nabla f(x, y, z)$.
   (b) Find the equation of the plane tangent to the level set $f(x, y, z) = 7$ at the point $(2, 1, 3)$.
   (c) Find the equation of the hyperplane tangent to the graph of f at the point $(2, 1, 3)$.

9. Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}^2$ are defined by $f(x) = x_1^2(3x_1 + x_2^2)$ and $g(t) = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}$.

   (a) Compute $Df(x)$ and $Dg(t)$.

   (b) Is $f \circ g$ differentiable over R? (Briefly explain why you know.)

   (c) Compute the first derivative of $f \circ g$ using the chain rule.

10. Consider the functions $f : \mathbf{R}^2 \to \mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $f(w, z) = \begin{pmatrix} 2wz^2 \\ 2z - w \end{pmatrix}$ and $g(x, y) = \begin{pmatrix} xy \\ x + y \end{pmatrix}$. Define the function $h : \mathbf{R}^2 \to \mathbf{R}^2$ by $h(x, y) \equiv [f \circ g](x, y)$. Using the chain rule, calculate $Dh(1, 1)$.

11. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x) = x_1 - 7x_1^2 + 9x_1x_2 - 3x_2^2$.

   (a) Calculate $\nabla f(x)$ and $D^2 f(x)$.

   (b) Write the two-equation system that defines a critical point x^* using the matrix-algebraic form $Ax + b = (0, 0)'$, where A is a 2×2 matrix of constants, $x = (x_1, x_2)'$, and b is a column vector of constants. Identify the matrix A and the vector b.

   (c) For the matrix A in part (b), calculate A^{-1} and, using matrix algebra, find the critical point x^*.

   (d) Identify whether x^* is a maximizer, a minimizer, or neither.

12. Find and determine the nature of the critical points of $f(x, y) = 4x^3 + y^2 - 6xy + 6x$.

13. Solve $\max xy + 2x + 5y - x^2 - y^2$. Verify the first- and second-order conditions.

14. Consider the function $f(x, y) = x^3 - 3xe^{-y^2}$.

   (a) Compute $\nabla f(x, y)$.

   (b) Compute $D^2 f(x, y)$.

   (c) What is the function $g(x, y)$ that defines the tangent plane to the graph of f at the point $(x, y) = (2, 1)$? (Hint: g is the first-order Taylor polynomial of f at $(2, 1)$.)

   (d) Find the local minima and maxima of f. (Use necessary and sufficient conditions.)

15. Determine the nature of the critical points of $f(x, y) = 3xy - x^3 - y^3 + 1/8$.

16. Determine whether the function $f(x) = e^x$ is concave, convex, quasiconcave, and/or quasiconvex.

17. Consider the function $f(x, y) = e^{ax^{1/2} - y}$, where $x \geq 0$, and $y \in \mathbf{R}$. Determine, for each value of the parameter a whether f is quasiconcave, quasiconvex, or both.

18. Determine the sign definiteness of the following matrices: (a) $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} -3 & 4 \\ 4 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}$.

19. Which of the following functions are concave or convex (on $\mathbf{R}^n$)?

    (a) $f(x) = 3e^x + 5x^4 - \ln x$,
    (b) $f(x, y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$,
    (c) $f(x, y, z) = 3e^x + 5y^4 - \ln z$, and
    (d) $f(x, y, z) = Ax^a y^b z^c$, for $a, b, c > 0$.

20. For each of the following functions defined on $\mathbf{R}^2$, find the critical points and classify them as local maxima, local minima, saddle points, or "can't tell." Also determine which of the local maxima/minima are global maxima/minima.

    (a) $xy^2 + x^3y - xy$,
    (b) $x^2 - 6xy + 2y^2 + 10x + 2y - 5$,
    (c) $x^4 + x^2 - 6xy + 3y^2$, and
    (d) $3x^4 + 3x^2y - y^3$.

21. In class, we developed a sufficient condition for x^* to be a local minimizer or maximizer of a twice continuously differentiable function f from $\mathbf{R}^n$ to $\mathbf{R}$. The second-order condition was that the Hessian be definite (positive or negative depending on whether x^* is a max or min). Is it true in these cases that x^* is a strict local maximizer/minimizer? (x^* is a strict maximizer if $f(x^*) > f(x)$ for all x near, but not equal to, x^*.) Prove your answer.

22. Consider the function $H(z; x, y) = xyz^3 - x^2z^2 + xz - 9z + 6$, where $x, y \in \mathbf{R}$ are parameters.

    (a) Suppose we want to find a local minimum of H by choice of z, for given values of x and y. What is the first-order condition? Do not try to solve this equation, but check that it holds for $x = 1$, $y = 1$, and $z = 2$.
    (b) Note that the first-order condition implicitly defines z as a function of x and y, $z = g(x, y)$, for x near 1, y near 1, and z near 2. Compute the gradient of g at the point $(1, 1)$.

23. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is maximized (globally) at $x \in \mathbf{R}^n$ and that $g : \mathbf{R} \to \mathbf{R}$ is an increasing function (that is, $g(a) \geq g(b)$ whenever $a \geq b$). Formally prove that x is the global maximizer of the function $h : \mathbf{R}^n \to \mathbf{R}$ where $h = g \circ f$ (that is, $h(x) = g(f(x))$ for all x).

24. Prove that any metric space which consists of only a finite number of points is compact. Demonstrate that the space $X \equiv \{x \in \mathbf{R} | x = 1/n \text{ for some } n \in \mathbf{P}\}$ is not compact.

25. Prove that any monotone (increasing) transformation of a real-valued quasiconcave function is also quasiconcave.

26. Solve $\max x_1 - x_2^2$ subject to $x_1^3 + x_2^2 = 0$.

27. Solve $\max y + xy$ subject to $x^2 + y^2 - 1 = 0$.

28. Consider the function $F(x, y, z) = zx^3 - 3y^2 + xyz^2 - 7$. Check that $F(2, 1, 1) = 0$. Note that the equation $F(x, y, z) = 0$ implicitly defines z as a function of x and y; that is, $z = g(x, y)$ in a neighborhood of $(2, 1, 1)$. Calculate $\nabla g(2, 1)$.

29. Take a function $f : X \to \mathbf{R}$, where X is an open subset of $\mathbf{R}^n$. We say that f is *homogeneous of degree p over X* if $f(\lambda x) = \lambda^p f(x)$ for every $\lambda \in \mathbf{R}$ and every $x \in X$ such that $\lambda x \in X$.

    (a) Prove that if such a function is differentiable at x then $\nabla f(x) \cdot x = pf(x)$. This is known as Euler's Theorem. (Hint: write the derivative of the composite function $g : \mathbf{R} \to \mathbf{R}$ where $g(\lambda) \equiv f(\lambda x)$.)

    (b) Check Euler's Theorem and find p for the function $f(x, y) = x^\alpha y^\beta$.

    (c) For what values of α and β is the function $f(x, y) = x^\alpha y^\beta - x$ homogeneous?

30. A function $g : \mathbf{R}^n \to \mathbf{R}$ is called *homogeneous of degree k* if, for every $x \in \mathbf{R}^n$ and $t \in \mathbf{R}$, it is the case that $g(tx) = t^k g(x)$. Prove that if g is differentiable and homogeneous of degree k then each of the partial derivatives of g is homogeneous of degree $k - 1$.

31. Consider the function $F : \mathbf{R}^3 \to \mathbf{R}$ defined by $F(x_1, x_2, z) = x_1^3 x_2 - z^2$. Suppose we are interested in how the identity $F(x_1, x_2, z) \equiv 0$ implicitly defines z as a function of (x_1, x_2); that is, $z = g(x_1, x_2)$.

    (a) Is g well-defined (is it a function?) in a neighborhood of the point $(x_1^0, x_2^0, z^0) = (1, 4, 2)$? How do you know this?

    (b) If so, calculate $\frac{\partial g}{\partial x_1}(1, 4)$ and $\frac{\partial g}{\partial x_2}(1, 4)$.

32. Consider the problem of a firm, whose objective is to maximize its profit. The firm produces two goods, whose units of production are denoted x and y. The firm sells these goods on a competitive market, where the price of good x is 1 and the price of good y is p. The firm's cost of producing x and y is given by the twice continuously differentiable function $c(x, y)$. Thus, the firm's profit is $\pi = x + py - c(x, y)$, which it maximizes by choice of x and y.

    Assume that for every x, every y, and every vector $h \in \mathbf{R}^2$, $h'D^2c(x, y)h > 0$. (That is, c is strictly convex.) Also assume that, for each value p, π has a maximum. Denote the optimal values of x and y as $x^*(p)$ and $y^*(p)$, and define $g(p) = \begin{pmatrix} x^*(p) \\ y^*(p) \end{pmatrix}$.

    (a) Treating π as a function of x and y, write expressions for $D\pi(x, y)$ and $D^2\pi(x, y)$. What is the relationship between $D^2\pi(x, y)$ and $D^2c(x, y)$? What can you say about the definiteness of the quadratic form defined by the matrix $D^2\pi(x, y)$?

    (b) Show that the first order condition for the firm's maximization problem can be written in the form $F(x, y, p) = 0$. (What is F?)

    (c) Suppose that $x^*(5) = 2$ and $y^*(5) = 3$. Write an expression for $Dg(5)$, in terms of c.

    (d) Do you have enough information to evaluate the sign of $\frac{dy^*}{dp}(5)$? If so, what is the sign of this derivative?

    (e) What assumption on the function c guarantees that $\frac{dx^*}{dp}(5) < 0$?

    (f) Define $v(p) = \max_{x,y} x + py - c(x, y)$. What is $v'(5)$?

33. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = 6ay + xy - x^2 - y^2$, where a is a constant number.

    (a) Calculate the gradient of f (the partial derivatives with respect to x and y) and determine the critical point x^* (written in terms of a).

    (b) Calculate the 2×2 matrix of second-order partial derivatives and determine whether x^* is a maximizer or minimizer (or neither).

    (c) Now consider a as a parameter so, formally, f is a function from $\mathbf{R}^3$ to $\mathbf{R}$. Define $v(a) = \max_{(x,y)} f(x, y, a)$. Calculate $v'(a)$, written in terms of the parameter a.

34. For given data $(y_1, y_2, \ldots, y_n)$ and $(x_1, x_2, \ldots, x_n)$, find the values of a and b that minimize $\sum_{i=1}^{n}[y_i - (a + bx_i)]^2$, and prove your answer.

35. Consider the problem $\max 10x - x^2 - y^2$ subject to $y \geq x - 1$.

   (a) Write the appropriate first-order conditions for this problem.

   (b) Solve the problem, being careful to show that all sufficient conditions are satisfied.

   (c) Suppose the problem had two constraints: $x^2 \geq 1$ and $y \geq x - 1$. Does the nonlinear programming method characterize the solution? (Will the solution be found using the first-order conditions of the Lagrangean?) Why?

36. Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is continuous and quasiconcave, and $g : \mathbf{R}^n \to \mathbf{R}$ is continuous and quasiconvex. Are these conditions sufficient to guarantee the existence of a solution to $\max_x f(x)$ subject to $g(x) \leq 0$? Explain by providing a sketch of a proof or counterexample. In the latter case, can you find stronger conditions that are sufficient?