Economics 205, Fall 2007
Quiz II

Comments There were 24 points possible, with the questions receiving 6, 9, 3, and 6 points respectively. The high was 23, the low 13, and the median 18. Some people should review the material behind Question 1. Some people were not able to produce eigenvectors in Question 2. Question 4 caused problems for some because it was abstract and because you were expecting a Taylor's Theorem problem and were working hard to use Taylor's Theorem. I view the Mean-Value Theorem as a special case of Taylor's Theorem, which explains why the question "deserves" to be on the test. (Also, because of time constraints, I did not include a fourth part of the question. What can you say about the maximum of $f(.5)$ if you know that $f(0) = f(1) = 0$ and that $f^{(2)}(x) \in (-1, 2)$?)

1. Let $x = (4, 4, 0)$, $y = (1, 1, 1)$, and $z = (1, 1, 0)$.

   (a) Find an equation of the plane that passes through the points x, y, and z.
   Two directions in the plane are $(0, 0, 1)$ and $(3, 3, 0)$. So a normal to the plane is $(1, -1, 0)$. Hence an equation of the plane is: $(1, -1, 0) \cdot (u_1 - 1, u_2 - 1, u_3) = 0$ or $u_1 - u_2 = 0$.

   (b) Find an equation of a line that passes through x and is orthogonal to the plane that you found in part a.
   You need a line through x with direction $(1, -1, 0)$. You can write this as $u = (4, 4, 0) + t(1, -1, 0)$.

   (c) Find an equation of a plane that passes through $(0, 1, 1)$ and does not intersect the plane that you found in part a.
   The plane must have the same normal as the plane in the first part, so its equation is: $(1, -1, 0) \cdot (u_1, u_2 - 1, u_3 - 1) = 0$ or $u_1 - u_2 = -1$.

2. State which of the matrices below are diagonalizable. You need not diagonalize the matrices, but you must justify your answer. For each part determine at least one eigenvector and an associated eigenvalue of the matrix.

   (a) $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ The characteristic polynomial is $(2 - \lambda)(-1 - \lambda) = 0$. Hence there are two distinct eigenvalues, -1 and 2 and the matrix is therefore diagonalizable. (Note that since the matrix is triangular, you know that the diagonal entries are the eigenvalues.) An eigenvector associated with $\lambda = 2$ is $(3, 1)$. An eigenvector associated with $\lambda = -1$ is $(0, 1)$.

   (b) $\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$ This matrix is symmetric and therefore diagonalizable. The eigenvalues solve: $(4-\lambda)(-2-\lambda) = 1$ and so they are equal to $1+\sqrt{10}$ and $1-\sqrt{10}$. Associated eigenvectors are: $(1, -3 + \sqrt{10})$ and $(3 - \sqrt{10}, 1)$

   (c) $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$
   This matrix has eigenvalue 3 (multiplicity 3). The only eigenvectors are multiples of $(0, 1)$. The matrix is not diagonalizable. One reason: only one linearly independent eigenvector. Another reason: if the matrix were diagonalizable, then it would need to be similar to $3\mathbf{I}$, but this is not possible (if $\mathbf{A} = \mathbf{P}^{-1} (3\mathbf{I}) \mathbf{P}$ then $\mathbf{A} = 3\mathbf{I}$).

3. Determine whether the matrix $\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$ is positive (semi-)definite, negative (semi-)definite, or indefinite.

   You can check that the determinant is negative, so you must have one positive and one negative eigenvalue, so the matrix must be indefinite. Or you can use the computation from the previous problem.

4. (a) If $f'(x) \geq 0$, then f is nondecreasing. Since $f(0) = f(1) = 0$ the function must be constant on the interval. Hence $f(.5) = 0$ is the upper bound.

   (b) If $f(.5) > 0$, then f must have a local maximum at $x^* \in (0, 1)$. But then, $f''(x^*) < 0$, which is ruled out. Hence $f(.5) \leq 0$. It is possible to attain this bound when $f(x) \equiv 0$. So again $f(.5) \leq 0$.

   (c) By the mean value theorem $f(.5) = f(0) + .5f'(t)$ for $t \in (0, .5)$. This means that $f(.5) \leq 0 + .5f'(t) \leq 1$. Also, $f(.5) = f(1) - .5f'(t)$ for $t \in (.5, 1)$. So $f(.5) \leq .5$. The second bound is tighter. It is attained by a function that is equal to $.5 - x$ for $x \geq .5$. (Officially you probably should present a function that satisfies all of the conditions that attains this bound, but this is graphically "obvious" and not hard to do.)