Economics 205, Fall 2007: Final Examination, Possible Answers

Comments. The first two questions went well, with deductions typically stemming from failure to provide justification or incorrectly specifying where derivatives were defined. On question 3, some people forgot to include the line $y = 0$ in the graph and there were a few conceptual errors with the implicit function theorem. Questions 4 and 5 were disappointing to me. The modal student forgot to consider the possibility of a boundary solution in Question 4. Instead, a mechanical computation correctly identified the valid solution (for large w), but wrote down negative values for x_1 otherwise. This answer received a 15 point deduction. If you ignored boundaries on the next problem, you obtained the right answer and received a smaller deduction. On both questions 4 and 5 and accurate, well described picture substitutes for checking boundary conditions and second-order conditions. In general, however, you lost points for not supplying an argument about boundary values and why the critical point was a maximum. People had a lot of time coming up with reasonable graphs for this question. Please practice some optimization problems. This is important. Questions 6 and 7 were more theoretical and created problems for most people. On Question 6, I was pleased that some people attempted to gain intuition by looking at lower-order cases. Although this does not translate into lots of points, it is an excellent strategy. Several people tried to argue that since $\mathbf{A}^t\mathbf{A}$ was non-negative (which is false!) or has a non-negative diagonal (which is true), then it was positive semi definite. A non-negative diagonal is a necessary condition for a matrix to be positive semi-definite, but it is not necessary. In both Questions 6 and 7 there were some questionable proof technique (proof by restating the question, proof by assertion, proof of something completely unrelated), but this is to be expected on examinations. One problem that I noticed was a failure to distinguish between necessary and sufficient conditions.

1. (a) $h'(x) = 1 - \frac{1}{x^2}$, valid for $x \neq 0$ by standard rules of differentiation. $h(\cdot)$ is not defined (and hence not differentiable) when $x = 0$.

   (b) $h'(x) = \sqrt{1-x^2} - x^2(1-x^2)^{-\frac{1}{2}}$. This part requires $-1 < x < 1$ for the square root to be differentiable.

   (c) $h'(x) = e^{x^e}(x^e)' = e^{x^e}\left(e^{e\log(x)}\right)' = e^{x^e}x^e(e/x)$, valid for all x. (The first equation is the chain rule. The second is the definition of x^e.)

   (d) The derivative of x^x is $x^x(1+\log(x))$. So the derivative of e^{x^x} is $e^{x^x}(x^x(1+log(x)))$ by the chain rule. This requires $x > 0$.

   (e) If you write $\log h(x) = \log(2+x^2)\log(x)$, you can write

   $$\frac{h'(x)}{h(x)} = \frac{\log(2+x^2)}{x} + \frac{2x\log(x)}{2+x^2}.$$

   So

   $$h'(x) = h(x)\left(\frac{\log(2+x^2)}{x} + \frac{2x\log(x)}{2+x^2}\right).$$

   You need $x > 0$ for powers of x to make sense.

(f) $Dh = DfDg$ by the chain rule. Here

$$Df = \begin{bmatrix} 2uv & u^2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2(x+y)(x^3-xy) & (x+y)^2 \\ 1 & 2 \end{bmatrix}.$$

and

$$Dg = \begin{bmatrix} 1 & 1 \\ 3x^2-y & -x \end{bmatrix}.$$

2. (a) $\lim_{x\to 3} \frac{x}{5} = \frac{3}{5}$. (Justified because $x/5$ is continuous.)

   (b) $\lim_{x\to 1} \frac{(x-1)(3x-5)}{x^3-1} = \lim_{x\to 1} \frac{3x-5}{x^2+x+1} = -\frac{2}{3}$ The first equation comes from factoring (the simplification is legitimate for $x \neq 1$). The second is justified because the simplified function is continuous at $x = 1$. You could also use L'Hopital's Rule.

   (c) $\lim_{x\to 1} \frac{1-x^2}{\log(3x)} = 0$ by continuity.

3. (a) The graph consists of the graph of $y = -x^3$ and the line $y = 0$.

   (b) The critical points satisfy: $Df(x,y) = \begin{bmatrix} 3x^2y & x^3+2y \end{bmatrix} = 0$. So we need $(x,y) = (0,0)$.

   The matrix of second derivatives at $(0,0)$ is

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

   This matrix is positive semi definite, so the critical point cannot be a local maximum. Otherwise we cannot be sure from the test alone (but if you look at the function you can see that it is both positive and negative in a neighborhood of zero, so the critical point can be neither a local minimum nor a local maximum).

   (c) $Df(x,y) = \begin{bmatrix} 3x^2y & x^3+2y \end{bmatrix}$. Therefore $Df(0,2) = \begin{bmatrix} 0 & 4 \end{bmatrix}$ the equation of the plane is $z - 4 = (0,4)\cdot(x, y-2)$ or $4y - z = 4$.

   (d) Need $L(t) = (x_0, y_0, z_0) + tv$ where $4y_0 - z_0 = 4$ (the point lies in the plane) and $v \cdot (0,4,-1) = 0$ (the direction of the line is orthogonal to the normal of the plane). For example, $L(t) = (0,1,0) + t(1,0,0)$.

   (e) $D_v f(0,2) = Df(0,2)v = 3.2$. (Note v as given is a unit vector.)

   (f) Check that $(x,y,z) = (1,2,6)$ satisfies $x+y+z = 9$ and $f(x,y) = z$.

   (g) We know from the previous question that there really is an intersection. Implicit differentiation yields that $X'(6)$ must satisfy:

$$3x^2X'(z)y + x^3Y'(z) + 2yY'(z) = 1 \text{ and } X'(z) + Y'(z) = -1$$

   or

$$6X'(6) + Y'(z) + 4Y'(z) = 1 \text{ and } X'(z) + Y'(z) = -1.$$

   Solving: $X'(6) = 6$ and $Y'(6) = -7$.

4. You can solve using the Kuhn-Tucker Theorem, but I would prefer to note first that the budget constraint must be an equality (because the utility function is increasing in both arguments). Also, $x_2 > 0$ at the optimum. Hence you can set $x_1 = (w - p_2 x_2)/p_1$ and solve the problem:

$$\max \frac{w - p_2 x_2}{p_1} - \frac{1}{x_2} \text{ subject to } x_2 \leq \frac{w - p_1}{p_2}.$$

The constraint guarantees that $x_1 \geq 0$.

The solution to the problem is either when the derivative is equal to zero or the boundary condition holds. Differentiation yields:

$$x_2 = \sqrt{\frac{p_1}{p_2}}.$$

It follows that the solution is:

$$(x_1, x_2) = (\frac{w}{p_1} - \sqrt{\frac{p_2}{p_1}}, \sqrt{\frac{p_1}{p_2}}) \text{ if } w > \sqrt{p_1 p_2}$$

and

$$(x_1, x_2) = (0, \frac{w}{p_2}) \text{ if } w < \sqrt{p_1 p_2}.$$

As long as $w \neq \sqrt{p_1 p_2}$ the demand functions are differentiable. When $w < \sqrt{p_1 p_2}$ the derivative is of x_1 (with respect to price) is zero (note: two partial derivatives) and the derivative of x_2 is $\left[\begin{array}{cc} 0 & -wp_2^{-2} \end{array}\right]$.

When $w > \sqrt{p_1 p_2}$, the derivative of the demand for good 1 is:

$$\left[\begin{array}{cc} -wp_1^{-2} + \frac{1}{2}p_1^{-3/2}p_2^{1/2} & -\frac{1}{2}p_1^{-1/2}p_2^{-1/2} \end{array}\right]$$

while the derivative of the demand for the second good is:

$$\left[\begin{array}{cc} \frac{1}{2}p_1^{-1/2}p_2^{-1/2} & -\frac{1}{2}p_1^{1/2}p_2^{-3/2} \end{array}\right]$$

The derivative of demand with respect to w is easy to compute (note that demand is not differentiable when $w = \sqrt{p_1 p_2}$).

5. (a) Think of the objective function as a cubic for x_1 in terms of x_2. If you draw a level set that gives value one it will intersect both $(0, 1)$ and $(1, 0)$ and a higher level set would generate a tangency to $x_1 + x_2 = 1$ for $x_1, x_2 > 0$ (that is, a solution to the optimization problem in which $x_1 + x_2 = 1$ and $x_1, x_2 > 0$.

   (b) The objective function is increasing in x_1, so the $x_1 + x_2 \leq 1$ must bind. So replace x_1 in the objective function by $1 - x_2$ and solve:

$$\max 1 + x_2 - x_2^3$$

subject to $x_2 \in [0,1]$. So the solution is at the boundary or when $1-3x_2^2=0$. The interior solution satisfies the second-order condition and you can check that the function is increasing in x_2 at 0 and decreasing at 1, so the solution must be $x_2=\sqrt{1/3}$ and $x_1=1-x_2$.

6. (a) The product is well defined because $\mathbf{A}$ has the same number of rows as $\mathbf{A}^t$ has columns.
   (b) $\left(\mathbf{A}^t\mathbf{A}\right)^t = \mathbf{A}^t\mathbf{A}$.
   (c) $\mathbf{x}^t\mathbf{A}^t\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^t\,\mathbf{A}\mathbf{x} = \|\,\mathbf{A}\mathbf{x}\,\|^2 \geq 0$.
   (d) The equation holds in the previous part if and only if $\mathbf{A}\mathbf{x}=\mathbf{0}$.

7. (a) $f(x)=x^2$.
   (b) Suppose that $f(y) > f(x)$ for $y > x$. Then $f(-y)=f(y)>f(x)=f(-x)$ so $-y>-x$, which is a contradiction.
   (c) Repeated differentiation of the identity yields that $f^{(2k+1)}(x) = -f^{(2k+1)}(-x)$ for all k. Hence it suffices to show that if $g(x)=-g(-x)$ and g is continuous at zero, then $g(0)=0$. If g is continuous at zero and $g(0)\neq 0$, then there exists a $\delta>0$ such that $f(\delta)$ and $f(-\delta)$ have the same sign (as each other and as $f(0)$), but this is impossible because $f(\delta)=-f(-\delta)$.
   (d) By Taylor's Theorem $f(x)=f(0)+f'(0)x+f^{('')}(0)/2+f^{(3)}(0)/6+E_4$. By assumption and the previous parts we have $f(x)-f(0)=E_4$. The result follows because $\lim_{x\to 0} E_4/x^3 = 0$.