Economics 205, Fall 2002: Quiz 3, Possible Answers

Comments. 36 possible (12 points each problem); High: 36, Low: 21; Median: 32. Scores were high. There were some problems that led to minor deductions but could signal serious misunderstandings. Please review your quiz and these answers carefully. On question one: Several people apparently did not recall the definition of homogeneity or Euler's Theorem. A few wrote the wrong equation for a plane. A few got confused with notation and used y_1 and y_2 to stand for both a particular point on the plane and a variable (see suggested answer). On question two: some, perhaps unwilling to believe that there would be no critical points, incorrectly asserted that there were solutions to $Df = \mathbf{0}$. On three, some forgot that eigenvectors must be nonzero. These mistakes are serious, but I hope that they are easy to recognize and correct.

1. (a) $f'(x) = 3x^2$ and $Dg(y_1, y_2) = \begin{bmatrix} 1 + y_2 e^{y_1 y_2} & y_1 e^{y_1 y_2} \end{bmatrix}$.

   (b) $D(f \circ g)(y_1, y_2) = Df(g(y_1, y_2) = 3(y_1 + e^{y_1 y_2})^2 \begin{bmatrix} 1 + y_2 e^{y_1 y_2} & y_2 e^{y_1 y_2} \end{bmatrix}$

   (c) $f(\cdot)$ is homogeneous of degree 3 because $f(\lambda x) = \lambda^3 f(x)$. Euler's Theorem states that in this case $xf'(x) = 3f(x)$, which is obviously true. The other two functions are not homogeneous. In order to have $g(\lambda y) = \lambda^n g(y)$ you would need $\lambda y_1(1 - \lambda^{n-1}) = \lambda^n e^{y_1 y_2} - e^{\lambda^2 y_1 y_2}$ for all λ, y_1, y_2. But if $y_1 = 0$, the equation becomes $\lambda^n = 1$, which obviously cannot hold globally. Similarly, the composite function cannot be homogeneous.

   (d) The equation is:

   $$z - g(y_1, y_2) = Dg(y_1, y_2) \begin{bmatrix} u - y_1 \\ v - y_2 \end{bmatrix}$$

   ((u, v, z) is a point on the plane.)

2. (a) $Df(x, y, z) = \begin{bmatrix} y & x & 1 \end{bmatrix}$. Hence $Df(\cdot) \neq \mathbf{0}$ and there are no critical points.

   (b) $Df(x, y) = \begin{bmatrix} e^{x-y^2} & -2ye^{x-y^2} \end{bmatrix}$. Since $e^{x-y^2} \neq 0$, $Df(x, y) \neq \mathbf{0}$ and again there are no critical points.

3. For all (x, y, z) (including $(1, 1, 1)$), $D^2 f(x, y, z) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix is symmetric, so it must be diagonalizable. The matrix can be decomposed into the upper 2×2 matrix and the lower 1×1 matrix. It should be clear that one eigenvalue is 0 with associated eigenvector $(0, 0, 1)$. The other eigenvalues are $-1, 1$ with associated eigenvectors $\frac{1}{\sqrt{2}}(1, -1, 0)$ and $\frac{1}{\sqrt{2}}(1, 1, 0)$. (I normalized to make them unit vectors.) It follows that $P^{-1} D^2 f(1, 1, 1) P = D$, where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and (since P is orthogonal) $P^{-1} = P^t$.