Economics 205, Fall 2002: Quiz I Possible Answers

August 31, 2007

Comments 36 points possible; 12 points for each question. High: 36; Low: 13; Median: 22. First problem fine, except for those tricked by (c); second problem – several people make mistakes graphing, almost everyone was casual, but correct in the analysis of continuity and differentiability; third problem – a few got it correct; most had little to say, due to a mixture of the difficulty of the question and to time constraints.

1. Let f be a differentiable function. Calculate the derivative of the function h defined in each of the problems below:

   (a) $h(x) = f(3x^2), h'(x) = 6xf'(3x^2)$ by the chain rule.

   (b) $h(x) = xf^2(x), h'(x) = 2xf(x)f'(x) + f^2(x)$ by the product rule and the chain rule.

   (c) $h(x) = e^{f(3)^2}, h'(x) = 0$ (h is a constant function).

   (d) $h(x) = x \log x$, for $x > 0$. $h'(x) = 1 + \log x$.

2. (a) First graph $y = x^2$. The graph of f agrees with this one on integers. Between integers, f is a line segment that starts on the graph of x^2 and heads off from the point (n, n^2) at slope n. These segments are below x^2 when $x > 0$ and above x^2 when $x < 0$.

   (b) Intuition: The function clearly jumps only at non-zero integers. The only real issue is whether it is differentiable at 0. The function is continuous at $x = 0$ and for any non integer x. For x not equal to an integer, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies that y is also not an integer (let δ be smaller than the distance to the nearest integer). Hence, for purposes of checking continuity (or differentiability) f behaves like a linear function. When x is an integer and $y \in (0, 1)$, $f(x + y) - f(x - y) = 2xy + x - y$. Now, if f were continuous at x, then for every $\epsilon > 0$, there would be a $\delta > 0$ such that if $y \in (0, \delta)$ (and $y < 1$)

   $$|f(x + y) - f(x - y)| = |2xy + x - y| < \epsilon.$$

   This cannot happen if $x \neq 0$ ($2xy + x - y$ can be made close to x if y is small, and $|x|$ is greater than or equal to one). Hence the function is discontinuous at x integer, $x \neq 0$. When $x = 0$, the function is continuous (you can set $\delta = \epsilon$ in the definition of continuity).

   (c) The only thing to settle is whether f is differentiable at 0. The last answer demonstrates that f is differentiable at nonzero x and isn't at points of discontinuity. Well, it isn't. For $x > 0$, $\frac{f(x)-f(0)}{x} = 0$; for $x < 0$, $\frac{f(x)-f(0)}{x} = -1$; so there is no way for a limit to exist.

3. Fast intuition: the condition states that the ratio that determines the derivative must be really small. When $a \neq b$, it follows from the assumption that

   $$\frac{|f(a) - f(b)|}{|a - b|} \leq |a - b|.$$

   Taking limits of both sides (or pushing around epsilons, deltas, and the definition of derivative), this expression implies that $f'(x) = 0$ for all x. Consequently (really, it is a consequence of the mean value theorem), f must be constant.