Economics 205, Fall 2002: Final Examination, Possible Answers

Comments on the Exam. Grades: 432 possible; high: 413; median: 342; low: 236. I was generally happy with the answers to questions 3-8, satisfied with the answers to 1 and 9, and disappointed with the answers to 2.

1. Common mistakes: drawing a graph that asymptotes on the coordinate axes; leaving out details in the statement of the mean-value theorem; and transposing Dg is the final part. Major mistakes: writing down an expression in part e that makes no sense (dividing by a vector or claiming that a number is equal to a vector) and getting the dimension of Df or $Df \circ g$ wrong.

2. I was surprised by the amount of problems caused by this question. Many people gave me an equation for a plane as an answer to the first part. Many could not keep straight that the direction that plays a role in the equation for a line is the direction of the line, while the "direction" in the equation for a plane is the normal direction. On part c, you got full or nearly full credit for setting up the problem, even if you made algebraic errors.

3. There are diagonalizable matrices that are neither symmetric nor have distinct e-values. Hence you must give another reason why the matrix in part a is not diagonalizable.

4. I deducted points if you gave an answer for parts a and b, because positive definitiveness (or, more importantly, the relationship between positive definiteness and e-values) is only defined for symmetric matrices.

5. This typically went well, although some people either failed to write down D or wrote down the rows of P inconsistently with the rows of D.

6. I gave almost all of the points to almost everyone on this question. A few people messed up the graph. Several people were causal about the mechanics of solving the optimization. I was lenient.

7. Most people get full credit on this one. I subtracted a lot of points on papers that had an incorrect expression for the derivative of price (typically leaving out an important term) and big reductions for answers that made no attempt to describe the conditions in economic terms.

8. Many people had the right idea, but omitted crucial details. The most common omission was to invoke the intermediate value theorem without making an argument to show why, for all $K > 0$, there exists x such that $g(x) > K$.

9. I graded fairly harshly on this one. I wanted something resembling a logical argument on the first two parts. Proofs by assertion (or proofs using

derivatives, unless they were especially careful and noted that differentiability was not assumed) received little partial credit. The third part was silly, and most people could find a counter example. The fourth part tricked people – just because $[0, 1]$ is compact (closed and bounded) does not mean all subsets will be. Also, on this part, several people wrote something about f not being defined. I did not understand this comment because the problem statement guarantees that f is defined.

I computed the total points in the course using the formula:

$$\text{Course Total} = \max\{F, .75F + Q, \frac{5}{6}F + BQ\}$$

where F is your final exam score (432 possible); Q is your quiz total (108 possible); and BQ is the total of your best two quizzes (72 possible). After rounding, the extra options only helped two or three people's final course score. The summary statistics are exactly the same as for the final (432 possible; high: 413; median: 342; low: 236).

The Answers

1. (a) Graph drawn separately.

   (b) The equation is

   $$z + 1 = Df(1,1)\begin{bmatrix} x-2 \\ y-2 \end{bmatrix}.$$

   Since $Df(1,1) = \frac{1}{4}\begin{bmatrix} 1 & 1 \end{bmatrix}$, the equation becomes: $x + y - 4z = 8$.

   (c) $f(\lambda x, \lambda y) = \lambda^{-1} f(x, y)$ is homogeneous of degree negative one. Euler's Theorem says that $-f(x, y) = D_1 f(x, y)x + D_2 f(x, y)y$. Since $Df(x, y) = \begin{bmatrix} \frac{1}{x^2} & \frac{1}{y^2} \end{bmatrix}$, Euler's Theorem holds.

   (d) If $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous and differentiable on the interior of a segment connecting a to b, then there exists $t \in (0, 1)$ such that

   $$f(b) - f(a) = Df(tb + (1-t)a)(b - a).$$

   (e) $f(2, 2) = -1, f(1, 1) = -2, Df(x, x) = \begin{bmatrix} \frac{1}{x^2} & \frac{1}{x^2} \end{bmatrix}$, so

   $$f(2,2) - f(1,1) = Df(x,x)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   when $x = y = \sqrt{2}$. (Since $\sqrt{2} \in (1, 2)$ the mean-value theorem really works.)

   (f) $Dg(u, v) = \begin{bmatrix} 2u & 2v \\ 0 & 0 \end{bmatrix}$, so $Dg(10, 10) = \begin{bmatrix} 20 & 20 \\ 0 & 0 \end{bmatrix}$, $Df(x, y) = \begin{bmatrix} \frac{1}{x^2} & \frac{1}{y^2} \end{bmatrix}$, so $Df(201, 10) = \begin{bmatrix} \frac{1}{201^2} & \frac{1}{10^2} \end{bmatrix}$. Since $g(10, 10) = (201^2, 10)$, and

   $$Df \circ g(10,10) = Df(g(10,10))Dg(10,10) = Df(201^2,10)Dg(10,10),$$

   it follows that $Df \circ g(10, 10) = \begin{bmatrix} \frac{20}{201^2} & \frac{20}{201^2} \end{bmatrix}$.

2. (a) Solve the equations simultaneously to get (for example): $x = .4 - .2z; y = 1.8 - .2z$, so an equation is

$$(x, y, z) = (.4, 1.8, 0) + t(-.6, -.2, 1).$$

(b) $-.6x - .2y + z = 0$.

(c) You want to solve: $\min x^2 + y^2 + z^2$ subject to (x, y, z) on the line above. The objective is therefore to solve:

$$\min(.4 - .6t)^2 + (1.8 - .2t)^2 + t^2.$$

This function is concave (you can check by taking two derivatives) and so you will have a global minimum when the derivative of the objective function with respect to t is zero. Computing you get $t = \frac{3}{7}$. Hence the closest point is $\frac{1}{7}(1, 12, 3)$.

Another way to answer this question is to observe that the answer must be the intersection of the plane in (b) and the line in (a). (Maybe you remember this fact; it is intuitive geometrically that the shortest distance between a line and a point is attained at the intersection of a line through $(0, 0, 0)$ that is perpendicular to the line.) If you plug the expression for (x, y, z) from (a) into the answer to (b) and solve for t, you again obtain $t = \frac{3}{7}$.

3. (a) Since the matrix is triangular, the eigenvalues are the diagonal entries – in this case 3 (with multiplicity two). $(0, 1)$ is an eigenvector. This matrix is not diagonalizable because it is triangular and has multiple eigenvalues. (If you could write $P^{-1}AP = D$, where D is a diagonal matrix with eigenvalues along the diagonal, then it must be that $P^{-1}AP = 2I$, so $A = 2I$, a contradiction.)

(b) An easy computation shows that this matrix has eigenvalues 2 and -2, with associated eigenvectors $(2, 1)$ and $(2, -1)$. Therefore if $P = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$, then $P^{-1}AP = 2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. You know that A is diagonalizable because the eigenvalues are distinct. For completeness, note that $P^{-1} = .25\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$. This matrix has one positive and one negative eigenvalue, so it is indefinite.

(c) This matrix is real and symmetric, so it must be diagonalizable. Computation shows that the eigenvalues are $1 + \sqrt{10}, 1 - \sqrt{10}$ and the associated eigenvectors are $(1, -3 + \sqrt{10})$ and $(1, -3 + \sqrt{10})$. You can diagonalize A using the matrix $P = \begin{bmatrix} 1 & 1 \\ -3 + \sqrt{10} & -3 - \sqrt{10} \end{bmatrix}$. With this choice of P, $P^{-1} = \frac{1}{2\sqrt{10}}\begin{bmatrix} 3 + \sqrt{10} & 1 \\ -3 + \sqrt{10} & -1 \end{bmatrix}$. A is symmetric. It has one negative and one positive eigenvalue ($\sqrt{10} > 3$). Therefore, A is indefinite.

(d) This matrix is real and symmetric. Note that it is block diagonal. One block is the upper corner: one eigenvalue is 4; an associated eigenvector is $(1,0,0)$. The other two eigenvalues come from the lower 2×2 matrix. Hence the eigenvalues and 2 and 6. Associated eigenvectors are $(0,1,-1)$ and $(0,1,1)$. You can diagonalize A by writing $P^{-1}AP = D$ where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \text{and } P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 & -.5 \\ 0 & .5 & .5 \end{bmatrix}$$

A is symmetric and all of its eigenvalues are positive, so A is positive definite.

4. Done above.

5. Done above.

6. I drew the graph separately. Here is the computation. You need to solve:

$$\begin{aligned} -8x &= -4\lambda_1 - \lambda_2 \\ 2 &= 2\lambda_1 - \lambda_3 \\ -4x + 2y &= -1 \\ \lambda_2 x &= 0 \\ \lambda_3 y &= 0 \end{aligned}$$

(the constraint must bind because the objective function is monotonic in y). By the picture, $x > 0$, so $\lambda_2 = 0, \lambda_1 = 2x, \lambda_3 = 4x - 2$. If $y = 0$, then $x = .25$ by the equality constraint. However, this would force $\lambda_3 = -1.5 < 0$, which is impossible. Therefore, $y > 0$ and $\lambda_3 = 0$. It follows that $(x,y) = (.5,.5), \lambda_1 = 1, \lambda_2 = \lambda_3 = 0$.

7. To answer the question, you must decide when the equation: $D(P,w) = S(P)$ implicitly determines P as a function of w. If $P(w)$ existed, then differentiating the identity yields:

$$P'(w) = -\frac{D_2(D(P(w),w)}{(D_1(D(P(w),w)) - S'(P(w)))}.$$

It makes sense to assume that demand is decreasing in price ($D_1(D(p,w)) < 0$), supply is increasing in price ($S'(p) > 0$), and demand is increasing in income ($D_2(D(p,w)) > 0$). Under these assumptions, the implicit function theorem implies that one can find a differentiable $P(w)$ (in the neighborhood of a solution to the equilibrium equation), and that

$$P'(w) = -\frac{D_2(D(P(w),w)}{(D_1(D(P(w),w)) - S'(P(w)))},$$

which is positive by the assumptions above. Hence equilibrium price is increasing in income under "natural" economic assumptions. The added income increases demand, price increases to reduce excess demand - both by increasing supply and reducing demand - until the market clears.

8. (a) Note that $g'(x) = e^x - 1 - x > 0$ for $x > 0$. (You can prove this from Taylor's Theorem, which states that $e^x = 1 + x + x^2 + .5x^2e^c$.) Hence g is strictly increasing for $x > 0$. The result follows since $g(0) = 0$.

   (b) By a, g is strictly increasing. Hence the equation has at most one solution. The equation has exactly one solution by the intermediate value theorem: g is continuous, $g(0) = 0$, and for any $K > 0$, there exists x such that $g(x) > K$. There are many ways to prove the last claim. One is to note that $g'' > 0$ so that $g(x) - g(1) \geq g'(1)(x-1) > .5(x-1)$ so that $g(2K+1) > K$.

9. (a) This is true, since e^x is increasing in x. ($f(x^*) \geq f(x)$ implies that $e^{f(x^*)} \geq e^{f(x)}$.)

   (b) This is true since multiplication by negative one is a decreasing function. ($f(x^*) \geq f(x)$ implies that $-f(x^*) \leq -f(x)$.)

   (c) This is false for many reasons. In particular, there is no reason why x^* should satisfy the constraint in the second problem. For example, if $f(x) = x, S = [0,1]$, then $x^* = 1$ solves $\max f(x)$ subject to $x \in S$, but -1 solves $\min f(x)$ subject to $-x \in S$.

   (d) This is false. For example, let $f(x) = x$, and $S = (0,1)$. This function does not attain its maximum (or minimum) on S.