Economics 205, Fall 2002: Final Examination

Instructions.

1. Try to answer all nine problems. (Read all of the questions now and start on the ones that seem easiest.)
2. You have three hours to do the exam.
3. Think before you write. Doing so may save time.
4. Read each question carefully and answer the question that I ask.
5. Make your answers as complete and rigorous as possible. In particular, give reasons for your computations, prove your assertions. You may use any theorem to support your arguments, provided that you state the theorem and verify that its assumptions hold. Informal and intuitive arguments are better than nothing.
6. The table below gives the point values for each question, allocate your time appropriately.
7. Joel Watson, Room 310, will be available to answer questions between 10 and noon.

	Score	Possible
I		62
II		50
III		40
IV		40
V		20
VI		65
VII		60
VIII		55
IX		40
Exam Total		432
Course Total		432
Grade in Course		

1. Consider the function $f(x, y) = -\{\frac{1}{x} + \frac{1}{y}\}$ defined on $\{(x, y) : x, y > 0\}$.

   (a) Graph $\{(x, y) : x, y > 0, f(x, y) = -1\}$.

   (b) Find an equation of the hyperplane tangent to the graph of $f(x, y) = z$ at the point $(x, y, z) = (2, 2, -1)$.

   (c) Decide whether or not $f(\cdot)$ is homogeneous. If $f(\cdot)$ is homogeneous, then determine its degree of homogeneity and explicitly verify Euler's Theorem.

   (d) State the Mean-Value Theorem as it applies to functions from $\mathbf{R}^2 \to \mathbf{R}$.

   (e) Explicitly verify the Mean-Value Theorem by expressing $f(2, 2)$ in terms of $f(1, 1)$ and the derivative of $f(\cdot)$.

   (f) Let $g(u, v) = (u^2 + v^2 + 1, 10)$. Use the chain rule to compute all partial derivatives of $f \circ g(\cdot)$ when $u = v = 10$.

2. The sets $x + 2y + z = 4$ and $3x + y + 2z = 3$ intersect in a straight line.

   (a) Find the equation of the line of intersection.

   (b) Find the equation of the plane perpendicular to the line you found in part a and the point $(0, 0, 0)$.

   (c) Find the point on the line of intersection found in part a that is closest to the point $(0, 0, 0)$.

3. State which of the matrices A below are diagonalizable. You need not diagonalize the matrices, but you must justify your answer.

   (a) $A = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$.

   (c) $A = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$.

   (d) $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{bmatrix}$.

4. For each of the symmetric matrices above, state whether A is positive-definite, negative-definite, positive semi-definite, negative semi-definite, or indefinite.

5. Pick one of the diagonalizable matrices above and exhibit an invertible matrix P such that $P^{-1}AP = D$, for a diagonal matrix D.

6. Consider the optimization problem:

$$\max -4x^2 + 2y \text{ subject to } x \geq 0; y \geq 0; \text{ and } \; -4x + 2y \leq -1$$

(a) Graph the feasible set. (That is, $\{(x, y) : x \geq 0; y \geq 0; \text{ and } -4x + 2y \leq -1\}$.)

(b) Graph a level set of the function $f(x, y) = -4x^2 + 2y$.

(c) Using the graphs from parts a and b, identify the solution to the optimization problem.

(d) Solve the optimization problem using calculus techniques.

7. The demand for a good is a function $D(p, w)$ of price, p, and income, w. The supply of the good is a function $S(p)$ of price only. For fixed w, an equilibrium price is a value P such that $D(P, w) = S(P)$. Assume that D and S are differentiable functions. Identify economically sensible conditions that guarantee that $D(P, w) = S(P)$ implicitly defines P as a function of w. Find an expression for $P'(w)$ and interpret the sign of $P'(w)$.

8. Let $g(x) = e^x - 1 - x - \frac{x^2}{2}$. Assume $x > 0$.

   (a) Prove that $g(x) > 0$ for all $x > 0$.

   (b) Let $K > 0$. Prove that there exists a unique solution to the equation $g(x) = K$.

9. Prove or give a counterexample to the statements below. In each part, assume that the function $f : \mathbf{R}^n \to \mathbf{R}$ is continuous and $S \subset \mathbf{R}^n$.

   (a) If x^* solves: $\max f(x)$ subject to $x \in S$, then x^* solves $\max e^{f(x)}$ subject to $x \in S$.

   (b) If x^* solves: $\max f(x)$ subject to $x \in S$, then x^* solves $\min -f(x)$ subject to $x \in S$.

   (c) If x^* solves: $\max f(x)$ subject to $x \in S$, then x^* solves $\min f(x)$ subject to $-x \in S$.

   (d) If $S \subset [0, 1]$, then the problem: $\max f(x)$ subject to $x \in S$ has a solution.