Economics 205, Fall 2000
Final Examination

September 18, 2000

Comments.

1. (a) $h'(x) = \log(x^2+1) + \frac{2x^2}{(}x^2+1)$.

   (b) $Dh(x,y) = Df(g(x,y))Dg(x,y)$, where $Df(u,v) = \begin{bmatrix} e^v + v & ue^v + u \\ 1 & -1 \end{bmatrix}$ so $Df(g(x,y) = \begin{bmatrix} e^{x^2} + x^2 & (x+y)(e^{x^2}+1) \\ 1 & -1 \end{bmatrix}$ and $Dg(x,y) = \begin{bmatrix} e^{x^2} + x^2 & (x+y)(e^{x^2}+1) \\ 1 & -1 \end{bmatrix}$. Therefore $DH(x,y) = \begin{bmatrix} e^{x^2} + x^2 + 2x(x+y)(e^{x^2}+1) & e^{x^2} + x^2 \\ 1-2x & 1 \end{bmatrix}$.

2. (a) Yes: Triangular with distinct e-values. Eigenvalues -1 and 2; corresponding eigenvectors $(0,-1)$ and $(3,1)$. So we can write $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ for $P = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$

   (b) Yes. Real and symmetric. Eigenvalues are $1+\sqrt{37}$ and $1-\sqrt{37}$ and $P = \begin{bmatrix} -1 & 3-\sqrt{37} \\ 3-\sqrt{37} & 1 \end{bmatrix}$

   (c) No. Standard example of non-diagonalizable matrix. If it were diagonalizable, then (since A is triangular), eigenvalues would both equal four and diagonalized matrix would be $4I$. But then $P^{-1}AP = 4I$ implies $A = I$, a contradiction.

3. See above.

4. (a) For sufficiently large x, $f(x) > 0$ (if you need a proof, note that $f(1) > 0$ and $f'(x) > 0$ for $x > 1$) and for sufficiently large x, $f(-x) < 0$ (try $x > 5$ it is easy to check that $f(-5) < 0$ and $f'(x) > 0$ for $x < -5$) hence the result follows from the intermediate value theorem.

   (b) Need $5x^4 + 20x^3 = 0$ so candidates for local extrema are 0 and -4. It is clear from the derivative that the function increases for $x < -4$, decreases for $-4 < x < 0$, and then increases again. Although $f''(0) = 0$, examination of $f'(x) = 5x^3(x+4)$ indicates that $x = 0$ must be a local minimum.

5. To answer the question, you must decide when the equation: $D(P, w) = S(P)$ implicitly determines P as a function of w. If $P(w)$ existed, then, differentiating the identity yields:

$$P'(w) = -\frac{D_2(D(P(w), w)}{(D_1(D(P(w), w)) - S'(P(w)))}.$$

It makes sense to assume that demand is decreasing in price ($D_1(D(p, w)) < 0$), supply is increasing in price ($S'(p) > 0$), and demand is increasing in income ($D_2(D(p, w)) > 0$). Under these assumptions, the implicit function theorem implies that one can find a differentiable $P(w)$ (in the neighborhood of a solution to the equilibrium equation), and that

$$P'(w) = -\frac{D_2(D(P(w), w)}{(D_1(D(P(w), w)) - S'(P(w)))},$$

which is positive by the assumptions above. Hence under "natural" economic assumptions, equilibrium price is increasing in income (the added income increases demand, price increases to reduce excess demand - both by increasing supply and reducing demand - until the market clears).

6. The problem is equivalent to $\min x^2+2y^2$ subject to $3x+2y = 4$. A critical point must satisfy: $(2x, 4y) = (3, 2)\lambda$. Solving yields $x = \frac{12}{11}$, $y = \frac{4}{11}$, and $\lambda = \frac{8}{11}$. You can see that this is a minimum because the objective function is strictly convex (second derivative matrix positive definite) or by noting that an interior minimum must exist.

7. (a) The objective function is strictly increasing in both x and y, so the constraint $x+y \leq w$ must hold as an equation. Hence we can substitute out for the constraint. The problem becomes: $\max w - y + \log y$ subject to $0 \leq y \leq w$. The critical point of the objective function is when $y = 1$. Since the objective function is strictly concave, this will be the global maximum if and only if it satisfies the constraints. Otherwise, the solution is $y = 1$. To summarize: if $w > 1$, the solution is $x = w - 1$ and $y = 1$. Otherwise, you have the boundary solution $x = 0$ and $y = w$.

   (b) X and Y are differentiable except at the point $w = 1$. $X'(w) = \begin{cases} 0 \text{ if } w < 1 \\ 1 \text{ if } w > 1 \end{cases}$ and $Y'(w) = \begin{cases} 1 \text{ if } w < 1 \\ 0 \text{ if } w > 1 \end{cases}$.

   (c) $V(w) = \begin{cases} \log w \text{ if } w < 1 \\ w - 1 \text{ if } w > 1 \end{cases}$, hence $V'(w) = \begin{cases} \frac{1}{w} \text{ if } w < 1 \\ 1 \text{ if } w > 1 \end{cases}$. In fact, $V'(1)$ exists and is equal to one.

8. The condition implies that f is continuous on its domain. Fix $b \in (0, 1)$ and note that the condition implies that for $a \neq b$,

$$|a - b| \geq \frac{|f(a) - f(b)|}{|a - b|}.$$

It follows that $f'(b) = 0$ for all $b \in (0,1)$, since for any $\varepsilon > 0$ there exists $\delta > 0$ (for example, $\delta = \varepsilon$) such that if $|a - b| < \delta$, then $\frac{|f(a)-f(b)|}{|a-b|} < \varepsilon$. Since $f(x) - f(0) = f'(b)x$ for some $b \in (0,1)$, it follows from the mean-value theorem that f is constant.

9. (a) Suppose $\Sigma_{i=1}^{k} \lambda_i x_i = 0$. Take the inner-product with x_j to conclude that $\lambda_j \|x_j\|^2 = 0$. Since $x_j \neq 0$, this implies that $\lambda_j = 0$. This argument works for all j, so we can conclude that $\lambda_j = 0$ for all j and hence the vectors are linearly independent.

   (b) There are many answers, but the simplest is the standard basis, $e_1, \ldots, e_n$ where e_i has a one as its ith component and zeroes otherwise.