Economics 205, Fall 2000
Final Examination

September 18, 2000

Instructions. Try to answer all nine problems. (Read all of the questions now and start on the ones that seem easiest.) Think before you write. Make your answers as complete and rigorous as possible. In particular, give reasons for your computations and prove your assertions. Informal and intuitive arguments are better than nothing.

1. In each part, determine at which points the derivative of the function h exists. When it does exist, compute it. When it does not exist, explain why it does not exist.

   (a) $h(x) = x \log(x^2 + 1)$.

   (b) $h(x, y) = f \circ g(x, y)$, where $f(u, v) = (ue^v + uv, u - v)$ and $g(x, y) = (x + y, x^2)$.

2. State which of the matrices below are diagonalizable. You need not diagonalize the matrices, but you must justify your answer.

   (a) $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$

   (c) $\begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$

3. Select one of the matrices in the previous problem that you found to be diagonalizable. Call it A. Find a nonsingular matrix P such that $P^{-1}AP$ is diagonal.

4. Let $f(x) = x^5 + 5x^4 - 4$.

   (a) Prove that there exists at least one real solution to the equation $f(x) = 0$.

   (b) Graph the function, indicating the local extrema, and intervals in which the function is increasing or decreasing.

5. The demand for a good is a function $D(p, w)$ of price, p, and income, w. The supply of the good is a function $S(p)$ of price only. For fixed w, an equilibrium price is a value P such that $D(P, w) = S(P)$. Assume that D and S are differentiable functions. Identify economically sensible conditions that guarantee that $D(P, w) = S(P)$ implicitly defines P as a function of w. Find an expression for $P'(w)$ and interpret the sign of $P'(w)$.

6. Solve: $\max e^{-(x^2+2y^2)}$ subject to $3x + 2y = 4$.

7. Consider the function $u(x, y) = x + \log y$ defined for $x \geq 0$ and $y > 0$.

   (a) Graph of set of points that satisfy $x + y \leq w$ for $w > 0$ and level sets of the function u.

   (b) For each $w > 0$, solve: $\max u(x, y)$ subject to $x + y \leq w$, $x \geq 0$, and $y \geq 0$.

   (c) Denote the solution to the optimization problem $X(w)$ and $Y(w)$. Identify the points at which X and Y are differentiable and compute $DX(w)$ and $DY(w)$ at these points.

   (d) Let $V(w) = u(X(w), Y(w))$. Compute $DV(w)$ (whenever it exists).

8. Let $f : [0, 1] \to \mathbf{R}$ satisfy

$$|a - b|^2 \geq |f(a) - f(b)| \text{ for all } a, b \in [0, 1].$$

   Prove that f is constant.

9. A set of non-zero elements $\{x_1, \ldots, x_k\} \subset \mathbf{R}^n$ is said to be orthogonal if, for each $i, j = 1, \ldots, k$, if $i \neq j$, then $x_i \cdot x_j = 0$.

   (a) Prove that if the set $\{x_1, \ldots, x_k\}$ is orthogonal, then it is linearly independent.

   (b) Find an orthogonal basis for $\mathbf{R}^n$.