Economics 200C: Problem Sets I and II
Answer Notes

1. Can a player have two strictly dominant strategies? Give an example or prove that this is impossible.

   No. If s_i and s_i' were both strictly dominant, $s_i \neq s_i'$, then you would have $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$ for all s_{-i}, which is impossible.

2. Can a player have two weakly dominant strategies? Give an example or prove that this is impossible.

   No. If s_i and s_i' were both weakly dominant, $s_i \neq s_i'$, then you would have $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for some s_{-i} and also $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$ which is impossible.

3. Can adding a strategy for Player 2 increase Player 1's security level? Can it decrease it? Can it increase Player 1's maximum equilibrium payoff? Can it increase Player 1's minimum equilibrium payoff?

   Adding a strategy for Player 2 cannot increase Player 1's security level (because Player 2 can always ignore the new strategy). It can decrease the security level (for example if the new strategy always gives Player 1 a payoff less than the old security level). It can increase Player 1's (minimum) equilibrium payoff:

	L	R
UP	$0,0$	$100,100$

   Imagine adding the second column: The equilibrium payoff (for both players) goes from 0 to 100.

4. Can a strategy be strictly dominated by a non-trivial mixed strategy, but not by a pure strategy? Give an example or prove that this is impossible.

   Yes. Example given in class.

5. Can adding a weakly dominated strategy change the set of Nash equilibria in a game? How about adding a strictly dominated strategy?

   Adding a weakly dominated strategy can change the set of Nash equilibria. For example:

	L	R
UP	$10,10$	$0,0$
DOWN	$0,0$	$0,0$

   Adding R makes $(DOWN, R)$ an equilibrium. Adding a strictly dominated strategy does not change the set of Nash equilibria.

6. Ten firms must decide whether to operate at location A or location B. If there are n firms in location A, then each of these firms earns n^2. If there are m firms at location B, then each of these firms earns $2m^2 - 14$. Describe the pure-strategy Nash equilibria of the game that arises if the firms simultaneously decide upon a choice of location. Write down (but do not solve) an equation that would characterize a symmetric (all firms play the same strategy) mixed-strategy equilibrium for the game. Show (if you can) that this equation has a solution.

   In general (that is, the algebra below is not needed), let $\pi_i(k)$ be the payoff at location i when k firms are there and assume that π_i is increasing. For an interior equilibrium, you need $\pi_A(n) \geq \pi_B(15-n)$ and $\pi_B(14-n) \geq \pi_A(n+1)$ simultaneously (the first condition states that a firm at A does not want to go to B; the second says that a firm at B does not want to go to A. Note that the equilibrium specifies that there are n firms at A and $14-n$ at B then when an A firm considers a deviation, it decides between being one of n firms at A or one of $14-n+1$ firms at B), but this is inconsistent with monotonicity.

   You can confirm this by algebra in the special case of the problem. If there are $0 < n < 14$ firms at A, then one of these firms must choose between getting n^2 and moving to B and getting $2(15-n)^2 - 14$ (if one firm moves from A to B then there will be $14-n+1$ firms at B). So NE requires:

$$n^2 \geq 2(15-n)^2 - 14$$

   in order for a firm at A to best respond. In addition, it must be true that

$$(n+1)^2 \leq 2(14-n)^2 - 14$$

   if the firms at B are best responding. After some algebra (possibly correct) these inequalities become:

$$n^2 - 58n + 377 \geq 0 \geq n^2 - 60n + 436.$$

   The first inequality requires that $n \geq 9$ (n is an integer). The second inequality requires that $n \leq 7$. Hence they cannot both hold. If $n = 0$ no firm would want to move from A. If $n = 14$ no firm would want to move to A. Hence there are two pure-strategy equilibria: all at A or all at B.

   For a symmetric mixed-strategy equilibrium, each firm goes to A with probability p. If $p \in (0,1)$, firms must be indifferent between going to A or B. If all of the other firms (independently) go to A with probability p, then the payoff for going to A is

$$\sum_{k=0}^{13} \binom{13}{k} p^k (1-p)^{13-k} (k+1)^2$$

and to B:

$$\sum_{k=0}^{13} \binom{13}{k} (1-p)^k p^{13-k} \left(2(k+1)^2 - 14\right).$$

We know that the first expression is higher when $p = 0$ and the second expression is higher when $p = 1$, so there must be an interior equilibrium by the intermediate value theorem.

7. Three voters ($i = 1, 2, 3$) must decide between two candidates, A and B. The candidate with the most votes wins. Voters 1 and 2 prefer candidate A to candidate B. Voter 3 prefers candidate B. Voters vote simultaneously. Show that there is an equilibrium in which candidate B wins. Show that this outcome disappears if voters avoid weakly dominated strategies.

   If all voters vote for B, then B wins independent of what any individual voters does. Hence voting for B is a best response for each voter. Voting for B is weakly dominated by voting for A for voters 1 and 2: When the other two votes are split, it leads to a superior outcome. Otherwise, it leads to the same outcome. Hence A will win if 1 and 2 avoid weakly dominated strategies.

8. How do risk attitutes determine play in matching pennies? Suppose that a risk-neutral ROW plays matching pennies against an opponent. The ROW player is indifferent between winning receiving nothing (for sure) and the lottery that pays one penny with probability one half and costs one penny with probability one half. The COLUMN player is indifferent between winning K and the lottery that pays one penny with probability one half and costs one penny with probability one half (K may be negative.) Normalize both players' von Neumann-Morgenstern utility function so that the payoff for losing is -1.

   (a) Compute the payoff for winning as a function of K.
   (b) Compute the equilibrium of the game for each K.
   (c) Now suppose that each player can "chicken out." If a player opts out and the other player pays either heads or tails, then the chicken plays the monetary amount c. If both players chicken out, then they each receive the payoff zero. Answer the first two parts (assuming still that ROW is risk neutral).

   Let U be the player's utility function for money. $.5U(1) + .5U(-1) = U(K)$. So if $U(-1) = -1$, then $U(1) = 1 + 2U(K)$. The equilibrium is still for each player to randomize equally. If a player can chicken out, then it is an equilibrium for the first player to randomize equally over his first two strategies and the second player to chicken out provided that $U(K) < U(-c)$. So if the second player is risk averse ($K < 0$), then the second player is willing to pay a positive amount to chicken out. If $K \geq -c$, then the equilibrium is the same as without the third strategy.

9. Mixed strategy equilibria arise in games with discontinuous payoffs and continuous strategy sets. For example, consider a game in which an auctioneer "sells" one dollar to the highest bidder. The high bidder wins the dollar, but every agent pays their bid. Concretely, assume that there are two bidders; a strategy for bidder i is a non-negative number b_i. The payoff to bidder i is $\pi_i(b_i, b_j) - b_i$, where

$$\pi_i(b_i, b_j) = \begin{cases} 1 & \text{if } b_i > b_j \\ .5 & \text{if } b_i = b_j \\ -1 & \text{if } b_j < b_i \end{cases}.$$

   (a) Find a symmetric equilibrium of this game. [First show that no symmetric, pure-strategy equilibrium exists. Next assume that the strategy is described by a cumulative distribution function $F(\cdot)$ with the property that if one player bids less than or equal to b with probability $F(b)$, then the other player is indifferent between all bids in the support of $F(\cdot)$. The indifferent condition leads to an equation that you can use to find $F(\cdot)$.]

   (b) Are there any asymmetric equilibria of the game (in pure or mixed strategies). Say what you can.

You cannot have a symmetric pure-strategy equilibrium. If one player is bidding b, the payoff to also bidding b is $.5 - b$. A deviant can do better either by bidding slightly more than b (which earns approximately $1 - b$). (If $b > .5$, then a player can do better by bidding 0 too.) Of course, there is an asymmetric pure-strategy equilibrium in which one player bids 1 and the other player bids 0. Each gets net payoff of zero.

To construct the mixed-strategy equilibrium, let $F(b)$ = the probability that one player bids less than or equal to b. Assume that the distribution is continuous. The payoff to the other player is $F(b) - b$ if she bids b (because she pays her bid and wins with probability $F(b)$. This should be constant for all b is the support of the distribution. The constant must be zero: it can't be negative (else a player would deviate and bid 0); it can't be positive (else no player would bid zero, but the lowest bid would then yield a negative payoff). It follows that $F(b) \equiv b$. That is, each player randomizes uniformly on $[0, 1]$.