LECTURE NOTES ON MAY 8TH

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1. Principal Agent Model: Setup

Principal offers wage schedule $\{w_h, w_l\}$ to the agent without observing agent's effort level $\{e_1, e_2, e_3\}$. Agent's effort choice can lead to two possible outcomes $\{\pi_h = 10, \pi_l = 0\}$ with the following conditional probability distribution,

$$f(\pi_h|e_1) = \frac{2}{3}, \quad f(\pi_l|e_1) = \frac{1}{3}$$
$$f(\pi_h|e_2) = \frac{1}{2}, \quad f(\pi_l|e_2) = \frac{1}{2}$$
$$f(\pi_h|e_3) = \frac{1}{3}, \quad f(\pi_l|e_3) = \frac{2}{3}$$

The disutility of each effort level is denoted by g($\cdot$):

$$g(e_1) = \frac{5}{3}, \quad g(e_2) = \frac{8}{5}, \quad g(e_3) = \frac{4}{3}$$

Principal has risk neutral preference, $E(U_p) = (\pi_h - w_h)P_h + (\pi_l - w_l)P_l$; Agent has risk averse preference, $U_a(w) = \sqrt{w}$, and $\overline{u}$ denotes the agent's reservation utility level. The principal-agent problem amounts to a nested optimization problem: given wages, agent selects effort level to maximize their utility, and the principal chooses the wage schedule, which affects agent's choice of effort level in order to maximize expected profit.

The principal-agent problem falls into the broad category of informational problems resulting from *hidden actions*, in which the principal's inability to observe the agent's effort creates moral hazard. The motivation for the principal agent problem is as follows:

- To utilize expertise of the agent by the principal
- To share risk between different risk preferences. We need principal to be risk neutral for tractability (of our model) and agent to be risk averse for some interesting property.

2. Optimal Contract when Effort is Observable

We first consider the absence of hidden action scenario as the benchmark. If effort is observable, there is no problem with providing incentives and principal has control over effort level so that we don't have to worry about sequential rationality. The principal solves the following optimization problem:

$$\max_{\{e, w_h, w_l\}} \quad (\pi_h - w_h)f(\pi_h|e) + (\pi_l - w_l)f(\pi_l|e)$$

$$\text{(IR)} \qquad s.t \quad U(w_h)f(\pi_h|e) + U(w_l)f(\pi_l|e) - g(e) \geq \overline{u}$$

Two observations are made to simplify the problem:

Claim 2.1. IR Constraint binds.

Proof. If the constraint does not bind, the principal could lower the agent's wage level but still having him to accept the contract. □

Claim 2.2. Given the risk aversion of the agent and risk neutrality of the principal, principal should always set $w_h = w_l$. And that is to say principal provides full insurance to the agent of his/her income risk.

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Proof. Let e_i be fixed, the principal solves the following cost minimization problem $\forall e_i \in \{e_1, e_2, e_3\}$,

$$\min_{\{w_h, w_l\}} \quad w_h f(\pi_h|e_i) + w_l f(\pi_l|e_i)$$

$$\text{(IR)} \qquad s.t \quad U(w_h)f(\pi_h|e_i) + U(w_l)f(\pi_l|e_i) - g(e_i) \geq \overline{u}$$

Thus we have the following Lagrangian

$$L = w_h f(\pi_h|e_i) + w_l f(\pi_l|e_i) - \lambda[U(w_h)f(\pi_h|e_i) + U(w_l)f(\pi_l|e_i) - g(e_i) - \overline{u}]$$

First order conditions are

$$f(\pi_h|e_i) - \lambda \frac{\partial U}{\partial w}(w_h) f(\pi_h|e_i) = 0$$

$$f(\pi_l|e_i) - \lambda \frac{\partial U}{\partial w}(w_l) f(\pi_l|e_i) = 0$$

Therefore, we have $\frac{\partial U}{\partial w}(w_h)=\frac{\partial U}{\partial w}(w_l)$. By strict concavity assumption of utility U which implies strictly monotonic decreasing property of $\frac{\partial U}{\partial w}$, it must be the case that $w_h = w_l$ to make $\frac{\partial U}{\partial w}(w_h)=\frac{\partial U}{\partial w}(w_l)$ hold $\forall e_i \in \{e_1, e_2, e_3\}$. □

Under the two claims above, the problem is significantly simplified. We can proceed with calculating the expected output, wage and profit for $\{e_1, e_2, e_3\}$. To demonstrate, for $e = e_1, g(e_1) = 5/3$ (given in problem), $U(w_h) = 5/3$ since principal pays just enough for the agent to be indifferent between accepting and rejecting the contract. So $w_h = (5/3)^2$. Expected output = 2/3 * 10 + 1/3*0 = 20/3. Profit is simply the difference between expected output and payment.

In summary we have the following results for each effort level by setting $U(w_h) = U(w_l) = g(e_i)$, $\forall e_i \in \{e_1, e_2, e_3\}$:

Effort	Expected Output	Payment	Profit
e_1	$\frac{20}{3}$	$\frac{25}{9}$	$\frac{35}{9}$
e_2	5	$\frac{64}{25}$	$\frac{61}{25}$
e_3	$\frac{10}{3}$	$\frac{16}{9}$	$\frac{14}{9}$

Since $\frac{35}{9} > \frac{61}{25} > \frac{14}{9}$, principal will pick e_1 to maximize his/her profit and agent earns wage $\frac{25}{9}$.

3. Optimal Contract when Effort is Unobservable

The nonobservability of effort creates incentive problem since principal cannot deduce agent's effort level by outcome and agents have incentive to lower their effort level. This leads to inefficiencies and we cannot arrive to the first-best solution. The strategy in solving this problem involves 2 steps: first we solve the principal's problem which is to find the minimum cost to induce effort level e_i. We then have a cost function of each effort level. With that the second step solves the profit maximization at each effort level in order to find the optimal e_i for the principal to induce.

Now suppose the principal wants to induce effort level e_3, the principal problem can be written as:

$$\min_{\{w_h, w_l\}} \quad f(\pi_h|e_3)w_h + f(\pi_l|e_3)w_l$$

$$\text{(IR)} \qquad s.t \quad U(w_h)f(\pi_h|e_3) + U(w_l)f(\pi_l|e_3) - g(e_3) \geq \overline{u}$$

$$\text{(IC.1)} \qquad U(w_h)f(\pi_h|e_3) + U(w_l)f(\pi_l|e_3) - g(e_3) \geq U(w_h)f(\pi_h|e_2) + U(w_l)f(\pi_l|e_2) - g(e_2)$$

$$\text{(IC.2)} \qquad U(w_h)f(\pi_h|e_3) + U(w_l)f(\pi_l|e_3) - g(e_3) \geq U(w_h)f(\pi_h|e_1) + U(w_l)f(\pi_l|e_1) - g(e_1)$$

Claim 3.1. To induce the lowest effort e_3 at minimum expected cost, full insurance ($w_H = w_L$) is the solution.

Proof. To see this, we adopt the following strategy to solve the optimization problem: first we ignore the incentive constraints and the problem is reduced to the optimization problem we solved in complete information setting, i.e. principal minimizes cost subject to the IR constraint only. Next, we check whether the solution of the reduced problem satisfies the incentive constraints. If the solution satisfies the second and the third incentive constraints (IC.1) (IC.2), the minimizer of the reduced problem must also be the solution for the cost minimization problem of inducing e_3. And that is, $w_h = w_l$. On the other hand, when the agent is offered full insurance, his wage is unaffected by his effort and since $g(e_1) > g(e_2) > g(e_3)$, there is no incentive to provide effort level anything above e_3 and IC.1 and IC.2 are necessarily satisfied. □

The reduced problem:

$$\begin{aligned} \min_{\{w_h, w_l\}} \quad & f(\pi_h|e_3)w_h + f(\pi_l|e_3)w_l \\ s.t \quad & U(w_h)f(\pi_h|e_3) + U(w_l)f(\pi_l|e_3) - g(e_3) \geq \overline{u} \end{aligned}$$

The expected output, payment and profit is the same as in the complete information setting under e_3. Note that as long as the principal is risk-neutral, no further properties of the conditional outcome distribution is required to arrive to this result.

We then iterate the process for e_2 and e_1 as described further in Professor Sobel's handout to calculate the full cost function of effort level.