Econ 172A, Fall 2007: Final Examination (I)

Instructions.

1. The examination has seven questions. Answer them all.
2. If you do not know how to interpret a question, then ask me.
3. Questions 1-6 require you to provide numerical answers. You must explain how you arrive at the numerical answer. If you properly use an algorithm introduced in the class that is appropriate for the problem, it is sufficient to say: "I used the algorithm introduced in class." (You still need to show the work necessary to demonstrate that you correctly carried out the steps in the algorithm.) If you use an alternate method, a detailed justification of why the algorithm works is necessary.
4. To be clear: It is not sufficient to write down a correct numerical answer to receive credit. You must explain how you arrived at your choice and why it is appropriate.
5. No justification is needed for Question 7.
6. The table below indicates how points will be allocated on the exam.
7. Work alone. You may not use notes, books, calculators, or any other electronic devices.
8. You have three hours.

	Score	Possible
I		27
II		12
III		30
IV		20
V		26
VI		15
VII		20
Exam Total		150
Course Total		300
Grade in Course		

1. Suppose a salesperson is located at Node 1 in the graph on the next page. Find the shortest route to each of the other locations. Be sure to explain how you arrived at your answers and why your method works. [Show your work on the next page.]

   (a) What is the shortest route from Node 1 to Node 2? What is the associated length of this route?

   (b) What is the shortest route from Node 1 to Node 3? What is the associated length of this route?

   (c) What is the shortest route from Node 1 to Node 4? What is the associated length of this route?

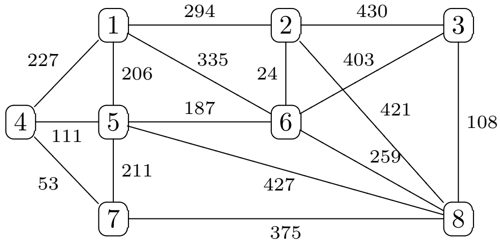
   (d) What is the shortest route from Node 1 to Node 5? What is the associated length of this route?

   (e) What is the shortest route from Node 1 to Node 6? What is the associated length of this route?

   (f) What is the shortest route from Node 1 to Node 7? What is the associated length of this route?

   (g) What is the shortest route from Node 1 to Node 8? What is the associated length of this route?

Network for Questions 1 and 2.



2. Find the minimal spanning tree for the network in Problem 1. Your answer should identify the minimum spanning tree and its cost. You must explain the method that you use to find the solution and why the method works.

3. The table below gives the capacities of the various routes between three warehouses and four markets. Show that for the given capacity constraints it is not possible to meet the market demands with the available supplies.

	Market				
Warehouse	1	2	3	4	Supplies
1	10	10	0	40	40
2	0	30	10	0	30
3	20	10	70	5	60
	20	20	70	20	
	Demands				

Now suppose that you were able to augment the capacity on one shipping route by twenty units. (For example, you could increase the shipping capacity from Warehouse 2 to Market 1 from 0 to 20 units.) Find a shipping route to augment (by 20 units) such that for the new problem it is possible to meet the given demands with the given supplies. Your answer should identify the route that you augment and construct a shipping route that meets the given demands with the given supplies. You must explain the method that you use to solve the problem and why the method works.

4. A building firm possesses four cranes $(1, 2, 3,$ and $4)$ each of which has a distance from four different construction sites (A, B, C, and D) as shown in the table:

	A	B	C	D
1	90	75	75	80
2	35	85	55	65
3	125	95	90	105
4	45	110	95	115

Place the cranes (one for each construction site) in such a way that the overall distance required for the transfer is as small as possible.

Show your work, but be sure to answer the following questions.

(a) Where should the cranes be placed to minimize total transfer distance?

(b) What is the minimum distance?

(c) What method did you use to solve the problem? Why does it work?

5. Consider the linear integer programming problem:

Find x_1 and x_2 to solve **P**:

$$\begin{array}{lrcrcl} \max & 2x_1 & + & 3x_2 & & \\ \text{subject to} & x_1 & + & 3x_2 & \leq & 9 \\ & 3x_1 & + & x_2 & \leq & 7 \\ & x_1 & - & x_2 & \leq & 1 \end{array}$$

where x_1 and x_2 must be nonnegative integers (but they may take on values different from 0 or 1).

Solve the problem using the branch-and-bound method.

(a) Give a finite upper bound for the value of the problem and explain how you found the upper bound.

(b) Give a finite lower bound for the value of the problem and explain how you found the lower bound.

(c) Perform one branching step. Describe how the upper and lower bounds that you found in parts (a) and (b) change.

(d) Continue the algorithm until you have a solution to the problem. Write down your solution (values for x_1 and x_2 and the value of the problem) and justify your answer.

6. Give an example of a knapsack problem that cannot be solved by the algorithm: Take the item with the highest ratio of value to weight that fits, reduce capacity by this weight, continue.

   Your answer should specify K (the number of items), C, the capacity of the knapsack, w_i and v_i, the weight and value of item i for $i = 1, \ldots, K$. You should also describe which items are selected according to the algorithm and an alternative way of packing the knapsack that is feasible but leads to a higher value.

   To avoid ambiguity, here is a full description of the algorithm:

   Step 1. Set remaining capacity, R, equal to C. Set the remaining items, I, equal to $\{1, \ldots, K\}$. Set the set of selected items S equal to the empty set.

   Step 2. If there are no remaining items $i \in I$ with $w_i \leq R$, stop. S is the set of selected items. If there is an item $i \in I$ with $w_i \leq R$, then go to Step 3.

   Step 3. Let k be the item that satisfies $k \in I$, with

$$\frac{v_k}{w_k} = \max_{i \in I} \frac{v_i}{w_i}.$$

   If there is more than one such item, let k be the one that weighs the least. Reduce R by w_k, subtract k from I, and add k to S. Return to Step 2.

7. For each of the statements below indicate whether the statement is always **TRUE**, by writing "TRUE" otherwise write "FALSE." No justification is required.

   The next three parts refer to a network in which there are $N > 1$ nodes and in which $c(i, j)$ is the cost of going from node i to node j, all pairs of nodes are connected by a single edge. Assume that $\infty > c(i, j) \geq 0$, that the edges go in both directions (that is $c(i, j) = c(j, i)$), but that otherwise the costs are distinct (if $i < j$, then $c(i, j) = c(i', j')$ if and only if $i = i'$ and $j = j'$).

   (a) The cheapest edge (that is, the edge in which $c(i, j)$ is smallest) is always part of the minimum spanning tree.

   (b) The most expensive edge (that is, the edge in which $c(i, j)$ is the largest) is never part of the minimum spanning tree.

   (c) If the collection of the $N - 1$ cheapest edges contains no cycles, then it is a minimum spanning tree.

   The next two parts refer to a network in which there is a source s and a sink n and in which $c(i, j) \geq 0$ is the capacity of the edge going from Node i to Node j. Denote a flow by (x_{ij}), where, for each pair of Nodes i and j, x_{ij} is the amount that flows from Node i to Node j.

   (d) There always exists a maximal flow (x_{ij}) in which either $x_{ij} = 0$ or $x_{ji} = 0$.

   (e) Let (S, N) be a minimum capacity cut in which $s \in S$ and $n \in N$. Suppose that one forms a new network by deleting the edge connecting Node i and Node j where $i \in S$ and $j \in N$. The maximum flow in the new network is exactly $c(i, j)$ less than the capacity of the maximum flow in the original network.