Formulation Problems: Possible Answers

1. Let x_1 = number of cans of cheap mix; x_2 = number of cans of party mix; and x_3 = number of cans of deluxe mix. These are what you want to find. Finding the revenue is easy. The problem provides information about price per can. The objective is to maximize $.9x_1 + 1.1x_2 + 1.3x_3$.

   Figuring out the constraints requires some care. The idea is to figure out how much of each of the ingredients you use when you produce x. Since each can contains 12 ounces, you need to change units. Remember that one pound contains 16 ounces. A can of cheap mix contains 80% peanuts and 20% cashews. Hence it contains .6 (= (.8)(.75)) pounds peanuts and .15 (= (.2)(.75)) pounds cashews. Similarly, one can of the party mix contains .375(= (.5)(.75)) pounds peanuts; .225(= (.3)(.75)) pounds cashews; and .15(= (.2)(.75)) pounds almonds. The deluxe mix contains .375(= (.2)(.75)) pounds peanuts; .375(= (.5)(.75)) pounds cashews; and .225(= (.3)(.75)) pounds almonds. We get the total amount of each ingredient used by adding up the total amount used in each mix. For example, the total amount of peanuts used is equal to the peanuts used in x_1 cans of cheap + peanuts in x_2 cans of party + peanuts in x_3 cans of deluxe. From the information above, this translates into

   $$.6x_1 + .375x_2 + .15x_3.$$

   Following the same steps for the other ingredients leads to the complete formulation.

   Find $x = (x_1, x_2, x_3)$ to solve:

   $$\begin{array}{lrcrcrcr} \text{max} & .9x_1 & + & 1.1x_2 & + & 1.3x_3 & & \\ \text{subject to} & .6x_1 & + & .375x_2 & + & .15x_3 & \leq & 150 \\ & .15x_1 & + & .225x_2 & + & .375x_3 & \leq & 100 \\ & & & .15x_2 & + & .225x_3 & \leq & 50 \\ & & & & & x & \geq & 0 \end{array}$$

2. Let x_k be the percentage contribution of the kth ingredient. The problem is to find x_i, $i = 1, \ldots, N$ such that

   $$x_i \geq x_j \text{ whenever } i < j,$$

   $$\sum_{i=1}^{N} x_i = 1$$

   and

   $$x = (x_1, \ldots, x_n) \geq 0$$

   The first set of restrictions ranks the ingredients. As written, x_1 is the main ingredient; x_N makes the smallest contribution. (It would be enough to assume that $x_i \geq x_{i+1}$ for $i = 1, \ldots, N-1$.) The second line describes the contributions as fractions of the weight of the product. (I think of percentages as numbers between zero and one. That is, $x_1 = .5$ means that 50% of the item comes from ingredient one.) Finally, the non-negativity constraint states that weights are not negative numbers.

The objective is to either $\min x_k$ or $\max x_k$.

You can really solve this one. It is easy to minimize x_k for $k > 1$: you can make $x_k = 0$ (and, for example, $x_1 = 0$). This won't work for the first ingredient, because it must be larger than all of the others. The smallest contribution that the first ingredient can make is $\frac{1}{N}$ (all ingredients give equal contribution). To maximize the contribution of the kth ingredient, set $x_i = 0$ for $i > k$ and $x_i = \frac{1}{K}$ for $i < k$.

3. The problem asks you to figure out how many of each kind of computer you need to produce, so name your variables x_L for the number of Lemons and x_B for the number of Bananas. The total profit will be $1800x_L + 1200x_B$. The statement of the problem contains two simple constraints on the number of computers produced: $x_L \leq 20$ and $x_B \geq 10$. Finally, there is the labor constraint. We know how many hours it takes to produce a Banana (18). So (assuming constant returns to scale) it takes $18x_B$ hours to produce x_B Bananas. The total labor needed to produce x is $25x_L + 18x_B$. Full employment requires that this quantity is equal to 800.

   We now have the complete formulation. Find $x = (x_L, x_B)$ to solve:

$$\begin{array}{lrcrcr} \max & 1800x_L & + & 1200x_B & & \\ \text{subject to} & 25x_L & + & 18x_B & = & 800 \\ & & & x_B & \geq & 10 \\ & x_L & & & \leq & 20 \\ & & & x & \geq & 0 \end{array}$$

4. This problem is difficult because it may be hard to figure out what the variables are. You need to know how many buses are running in each hour so let y_i be the number of buses running at the beginning of hour i. With this definition, you can figure out how to write the objective function. $y_i - b_i$ is the excess in the ith hour, so $(y_i - b_i)c_i$ is the amount you must pay in extra fees in hour i and your total excess charges are $\sum_{i=1}^{24}(y_i - b_i)c_i$. You are constrained to have $y_i \geq b_i$ for all i. So far, so good. The formulation is not complete because the y_i are not really the things that you choose. You choose how many buses to *put into* service at the start of each hour. These numbers (combined with the restriction that buses run for six consecutive hours) determine the y_i. So while it was convenient (and correct) to think of variables like y_i, you need other variables too. Let x_i be the number of buses put into service at the beginning of hour i. A little thought tells you that you can compute the y_i from x. Start with an easy one. How many buses are in service when

$i = 10$? In words, all the buses put into service in hours 5, 6, 7, 8, 9, and 10. (The buses put into service at the beginning of hour 4 end their shift at 10.) In symbols, $y_{10} = x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$. You might be tempted to write the general version of this expression:

$$y_i = x_{i-5} + x_{i-4} + x_{i-3} + x_{i-2} + x_{i-1} + x_i.$$

This expression is almost right. The only problem is how to define x_j when $j \leq 0$. (Check out the above formula when $i = 1$ for example. It should tell you how many buses are running from 1 until two. This should include buses put into service on the night before. Hence you must interpret the subscripts by associating -6 with 18, -5 with 19, -4 with 20, and so on (0 with 24). Having done so, the formula above makes sense. The problem then becomes to select x_i, y_i for $i = 1, \ldots, 24$ to solve:

$$\max \sum_{i=1}^{24} (y_i - b_i)c_i$$

$$\text{subject to } y_i \geq b_i$$

$$y_i = x_{i-5} + x_{i-4} + x_{i-3} + x_{i-2} + x_{i-1} + x_i$$

$$x_i \geq 0.$$

5. Define variables x_{ij} to be equal to the dollar amount invested in investment option i in year j. These variables are defined for $i = A, B$ (two possible investments) and $j = 1, 2, 3$ (three possible years). It is also useful to define variables y_j to be equal to the dollar amount of money available at the beginning of year j. y_j is defined for $j = 1, \ldots, 4$. With these definitions, the objective is to maximize y_4. Here are the constraints:

   (a) $x_{Aj} + x_{Bj} \leq y_j$ for $j = 1, 2, 3$.

   (b) $y_1 = 100000$.

   (c) $y_2 = y_1 - (x_{A1} + x_{B1}) + 1.70x_{A1}$.

   (d) $y_3 = y_2 - (x_{A2} + x_{B2}) + 3x_{B1} + 1.70x_{A2}$.

   (e) $y_4 = y_3 - (x_{A3} + x_{B3}) + 3x_{B2} + 1.70x_{A3}$.

   (f) $x_{ij} \geq 0$.

   Constraint (a) states that you cannot invest more than your wealth. Constraint (b) states your initial wealth. Constraints (c)-(e) update your wealth. Constraint (f) states that the investment amounts must be non-negative.

6. Let x_j be the tons of mixture j produced, where $j = 1$ indicates regular; $j = 2$ for extra; and $j = 3$ for super. Notice that from x you can determine the amounts of the various ingredients you use, but not vice versa. (That is, if I reported that the company used so many tons of potash, nitrates, and phosphates, you would in general not have enough information to figure out what its final output was.) Also let a_{ij} be the tons of ingredient i used in product j. The first index, i, represents nitrates, phosphates, and potash for $i = 1, 2, 3$, respectively. The second index, j, indicates regular, extra, and super, respectively as before.

   There are simple expressions for the x_j in terms of the a_{ij}, namely, $\sum_{i=1}^{3} a_{ij} = x_j$. Also, we can write down profit. Revenue is simply

$$750x_1 + 800x_2 + 900x_3.$$

   While cost is

$$800\sum_{j=1}^{3} a_{1j} + 400\sum_{j=1}^{3} a_{2j} + 1000\sum_{j=1}^{3} a_{3j}.$$

   Profit is the difference of these two quantities.

   We have the objective function. What are the constraints? The capacity constraint permits the firm to produce no more than 40 tons overall. It follows that

$$x_1 + x_2 + x_3 \leq 40.$$

   The firm cannot spend more than 25000; consequently cost is less than or equal to 25000. Finally, the composition constraints need to be met. For the regular mixture we know that the ingredients appear in a 3:6:1 ratio. This means that $\frac{a_{11}}{a_{21}} = \frac{3}{6}$ and $\frac{a_{11}}{a_{31}} = \frac{3}{1}$. These equations can be written: $6a_{11} - 3a_{21} = 0$ and $a_{11} - 3a_{31} = 0.$. Observe that the composition requirement determines the quantity of two of the ingredients knowing the quantity of the other one. That is, if you fix a_{1j}, then you can deduce a_{2j} and a_{3j}. Noting similar constraints for the other products, we can summarize the formulation. We need to find the a_{ij} and the x_j to maximize profit (revenue minus cost above) subject to:

   (a) $x_1 + x_2 + x_3 \leq 40$. (capacity)
   (b) $800\sum_{j=1}^{3} a_{1j} + 400\sum_{j=1}^{3} a_{2j} + 1000\sum_{j=1}^{3} a_{3j} \leq 25000$ (cost constraint)
   (c) $6a_{11} - 3a_{21} = 0$.
   (d) $a_{21} - 6a_{31} = 0$.
   (e) $4a_{12} - 4a_{22} = 0$.
   (f) $a_{22} - 4a_{32} = 0$.

(g) $4a_{13} - 6a_{23} = 0$.

(h) $3a_{23} - 4a_{33} = 0$.

(i) $a_{ij}, x_j \geq 0$

Without too much trouble, you can formulate this problem using only three variables: say the amount of nitrates used in each product. The mixing constraints would then determine the amount of other ingredients used in each product. Knowing these things, you can figure out the total weight of each product. Doing so yields the same problem. In my opinion, the formulation above is easier to understand (and easier to interpret). The lesson is that you can invent as many variables as you find useful. Extra variables probably will lead to extra constraints. On the other hand, naming all of the economically relevant terms may make the problem and its solution easier to interpret.

For the second part of the problem, the firm can earn ten percent return on the difference between 25000 and the cost of the ingredients. The constraints of the new problem don't change. Added to the objective function is the term: $.1(25000 - (800\sum_{j=1}^{3} a_{1j} + 400\sum_{j=1}^{3} a_{2j} + 1000\sum_{j=1}^{3} a_{3j}))$.

7. This is another problem in which a convenient choice of variables makes the formulation easier - but where that choice may not be obvious. You want to keep track of the amount invested in each of the four options; let x_i be the amount invested in option i (for $i = 1, \ldots, 4$). These variables describe a gambling strategy, but they do not directly tell you what your minimum return is. Denote the minimum return by v. Once you introduce this variable, you can write down your objective: you want to maximize v. The tricky part is figuring out how v relates to the gambler's plan. Note that it must be the case that v is no greater than the return in the event that outcome 1 takes place. This leads to the inequality:

$$-3x_1 + 4x_2 - 7x_3 + 15x_4 \geq v.$$

Reasoning similarly for the second and third outcomes, we can come up with a formulation. Find $x = (x_1, \ldots, x_4)$ and v to solve:

$$\begin{array}{lrcrcrcrcrcr}
\max & v & & & & & & & & & & \\
\text{subject to} & -3x_1 & + & 4x_2 & - & 7x_3 & + & 15x_4 & - & v & \geq & 0 \\
 & 5x_1 & - & 3x_2 & + & 9x_3 & + & 4x_4 & - & v & \geq & 0 \\
 & 3x_1 & - & 9x_2 & + & 10x_3 & - & 8x_4 & - & v & \geq & 0 \\
 & x_1 & + & x_2 & + & x_3 & + & x_4 & & & \leq & 500 \\
 & x & & & & & & & & & \geq & 0.
\end{array}$$

The first three constraints guarantee that v is smaller than or equal to the return of all three outcomes. These constraints say that v is no bigger than the minimum return. When you actually maximize v, it must be equal to the minimum return (otherwise, all of the first three constraints would not bind and you couldn't be maximizing v).

8. Define variables x_{ij} to be equal to the dollar amount invested in investment option i in year j. There variables are defined for $i = 1, 2, 3$ (three possible investments) and $j = 1, \ldots, 5$ (five possible years). It is also useful to define variables y_j to be equal to the dollar amount of money available at the beginning of year j. y_j is defined for $j = 1, \ldots, 6$. With these definitions, the objective is to maximize y_6. Here are the constraints:

   (a) $x_{1j} + x_{2j} + x_{3j} \leq y_j$ for $j = 1, \ldots, 5$.

   (b) $y_1 = 50000$.

   (c) $y_2 = y_1 - (x_{11} + x_{21} + x_{31}) + 1.09x_{11}$.

   (d) $y_3 = y_2 - (x_{12} + x_{22} + x_{32}) + (1.06)^2 x_{21} + 1.09x_{12}$.

   (e) $y_4 = y_3 - (x_{13} + x_{23} + x_{33}) + (1.10)^3 x_{31} + (1.06)^2 x_{22} + 1.09x_{13}$.

   (f) $y_5 = y_4 - (x_{14} + x_{24} + x_{34}) + (1.10)^3 x_{32} + (1.06)^2 x_{23} + 1.09x_{14}$.

   (g) $y_6 = y_5 - (x_{15} + x_{25} + x_{35}) + (1.10)^3 x_{33} + (1.06)^2 x_{24} + 1.09x_{15}$.

   (h) $x_{11} \leq 100000$.

   (i) $x_{31} \leq 50000$.

   (j) $x_{ij} \geq 0$.

   Constraint (a) states that you cannot invest more than your wealth. Constraint (b) states your initial wealth. Constraints (c)-(g) update your wealth. These constraints have the same general form. Available wealth in one period is equal to available wealth in the previous period, minus the amount placed into investments, plus the return on the investments of past years. At the start of year 2 you only get the return on the money placed in investment one at the start of year 1. At the start of year 3 you only get the return on money placed in investment one at the start of year 2 and in investment 2 at the start of year 1. And so on. Note two things. First, the constraints include numbers like x_{35} even though you know in advance (from common sense) that this number will be zero. Second, I have assumed that the returns in the table were annual percentage returns and took powers to compute returns over several years. This is an assumption, but it is a natural interpretation of what was described in the problem. Constraints (h) and (i) describe that maximum size of initial investment (I only impose them on the investments in year 1). Constraint (j) states that the investment amounts must be nonnegative.

9. The problem is to find fractions (or percentages) x_i where $i = 1, \ldots, 7$ represents one of the investment options ($i = 1$ means treasury bills, etc) to maximize return subject to the given constraints. The objective is to maximize

$$3x_1 + 12x_2 + 9x_3 + 20x_4 + 15x_5 + 6x_6 + 0x_7$$

   subject to

   (a) $4x_1 + 7x_2 + 8x_3 + 6x_4 + 10x_5 + 5x_6 + 0x_7 \leq 7$.

(b) $x_1 + 5x_2 + 4x_3 + 8x_4 + 6x_5 + 3x_6 + 0x_7 \leq 5$.

(c) $0x_1 + 18x_2 + 10x_3 + 32x_4 + 20x_5 + 7x_6 + 0x_7 \geq 10$.

(d) $x_7 \geq .1$.

(e) $\sum_{i=1}^{7} x_i = 1$.

(f) $x \geq 0$.

The constraints merely translate the requirements in the statement of the problem. The fifth constraint says that the fractions must add up to one.