Econ 172A, F2003: Midterm Examination I, Possible Solutions

Comments On the first question, some students drew the feasible set improperly. This should be simple to check even in an exam. On the second question, some students "over used" the substitutions $x = u - v$. If you make this substitution for a variable that is constrained to be nonnegative, like x_3, then you must also include $u_3 - v_3 \geq 0$ or you have changed the problem. A few duals appeared to follow some, but not all of the rules for generating a dual.

On the third question, there were major deductions for students who wrote down arrays that were not in the proper form (as described in class and lecture notes).

Grading Notes The distribution of grades was: median 78.5; high 100; low 27. For general questions about what this means for your final grade, see FAQ on webpage. (At this point I will not translate a numerical score into a letter grade.)

Here is an outline of the rules used to assign points. If you think that your exam has been incorrectly graded, follow rules for regrading.

1. (a) lines bounding constraint set: 5 points; feasible set: 6 points;
   (b) for each part, 2 points for correctly identify the solution(s); 1 point for giving the right value: 9 points total;
   (c) 5 points total, with deductions for failure to justify answers;
   (d) identify the solution (3 points) and state that it's a corner (2 points): 5 points total.
2. (a) min to max: 2 points; modify equation (requires several steps): 7 points; transform unconstrained variable: 4 points; non-negativity: 2 points.
   (b) min to max: 2 points; coefficients on constraints: 4 points; signs of inequalities: 2 points; non-negativity: 2 points.
3. Correct Setup in the first table: 4 points; identification of correct pivot location: 5 points; correct algebra for finding new basis: 4 points; further calculations: 8 points; guess: 2 points; explanation: 2 points.
4. No partial credit

Answers

1. Find x_1 and x_2 to solve:

$$\max x_0$$

$$\begin{array}{rrcrcr} \text{subject to} & 5x_1 & + & 3x_2 & \leq & 45 \\ & x_1 & - & x_2 & \geq & 1 \\ & x_1 & & & \geq & 2 \end{array}$$

On the other form, I multiplied the coefficient of x_1 by two everywhere in the problem. This has the effect of rescaling the x_1 variable: all x_1 values in the graph are cut in half. This changes the scale of the graph, but otherwise the answers remain the same.

(a) Graphically represent the feasible set of this problem. See next page.

(b) Graphically solve the problem for the following values of x_0:

i. $x_0 = x_1 - x_2$; no solution, unbounded: for example let $x_1 = 2$ and x_2 be really small.

ii. $x_0 = -x_1 + x_2$; non-unique solution - all points on segment connecting $(2, 1)$ to $(6, 5)$; value: -1.

iii. $x_0 = 3x_1 + 4x_2$; solution $(6, 5)$; value: 38.

I should draw level sets for each objective function, but I do not like drawing by computer, so I'll just give the answer. Since you know that solutions occur at corners, you can check the answer by plugging the corners into the objective function and picking the highest value.

(c) Which of the following points can be a solution to a linear programming problem for some (linear) choice of x_0 and the constraint set given above? The answer to the question is that any feasible point can be a solution (if, for example, the objective function is $x_0 = 0$). So you need only check which of the points are feasible.

i. $(x_1, x_2) = (6, 5)$; yes, it is a corner.

ii. $(x_1, x_2) = (2, 0)$; yes, it is feasible (solution to $\max -x_1$).

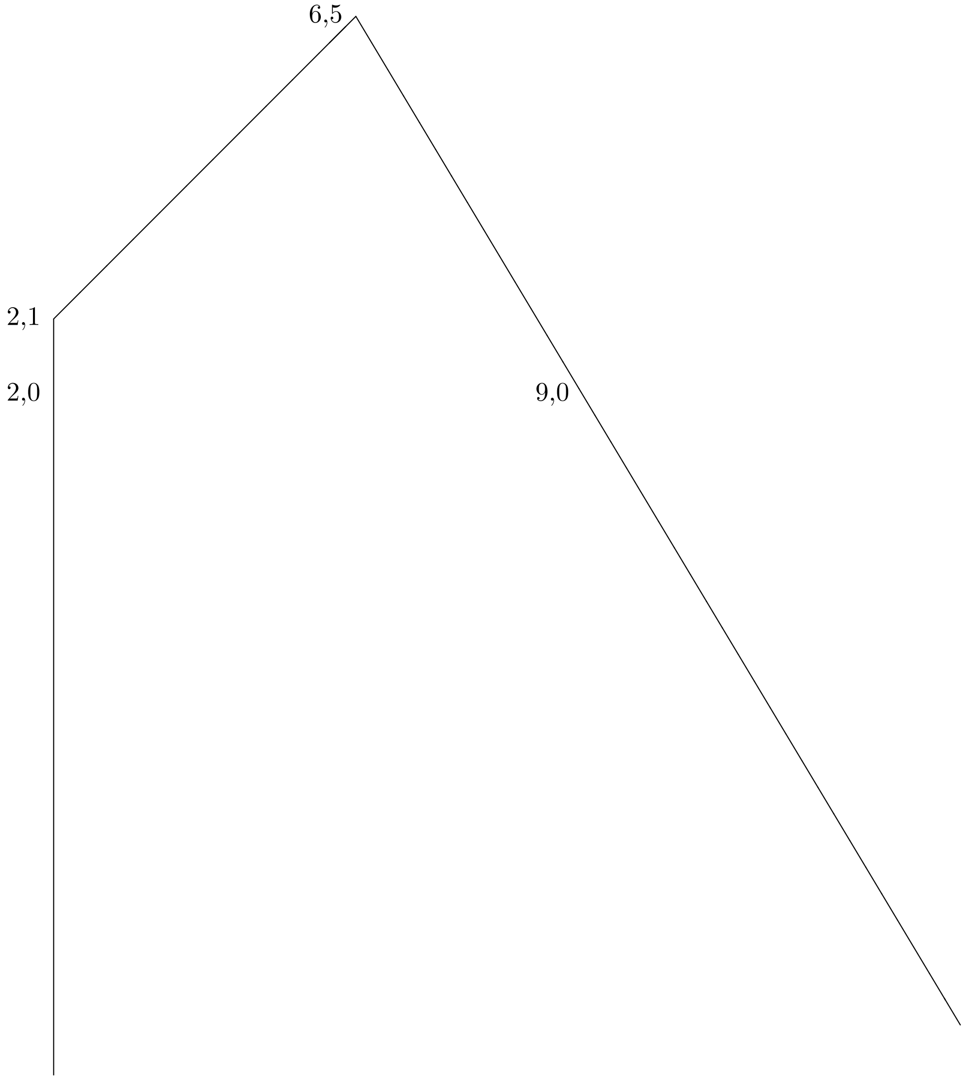
iii. $(x_1, x_2) = (2, 2)$; no, it violates constraint.

iv. $(x_1, x_2) = (9, 0)$; yes, it is feasible (solution to $\max 5x_1 + 3x_2$).

v. $(x_1, x_2) = (3, 2)$; yes, it is feasible (solution to $\max x_1 - x_2$).

(d) Which of the points in question 3 can be a **unique** solution to a linear programming problem for some (linear) choice of x_0 and the constraint set given above?

The answer is (a). Only (a) is a corner. (c) can never be solutions (it is not infeasible). If (b), (d), or (e) were solutions, then everything on the boundary segment contain the point would also be a solution.



This picture is unbounded below.

2. I asked you to write the problem in standard form and find the dual.

(a)
$$
\begin{array}{lrcrcrcrcr}
\max & -u_1 & + & v_1 & + & x_2 & + & x_3 & & \\
\text{subject to} & -5u_1 & + & 5v_1 & - & 3x_2 & + & 3x_3 & \leq & -45 \\
 & u_1 & - & v_1 & - & x_2 & & & \leq & 1 \\
 & -u_1 & + & v_1 & + & x_2 & & & \leq & -1 \\
 & u_1 & & v_1 & & x_2 & & x_3 & \geq & 0
\end{array}
$$

(b)
$$
\begin{array}{lrcrcrcr}
\min & -45y_1 & + & y_2 & - & y_3 & & \\
\text{subject to} & -5y_1 & + & y_2 & - & y_3 & \geq & -1 \\
 & 5y_1 & - & y_2 & + & y_3 & \geq & 1 \\
 & -3y_1 & - & y_2 & + & y_3 & \geq & 1 \\
 & 3y_1 & & & & & \geq & 1 \\
 & y_1 & & y_2 & & y_3 & \geq & 0
\end{array}
$$

For the other form, here are the answers.

(a)
$$
\begin{array}{lrcrcrcrcr}
\max & -x_1 & + & u_2 & - & v_2 & + & x_3 & & \\
\text{subject to} & 5x_1 & + & 3u_2 & - & 3v_2 & - & 3x_3 & \leq & 45 \\
 & -5x_1 & - & 3u_2 & + & 3v_2 & + & 3x_3 & \leq & -45 \\
 & -x_1 & + & u_2 & - & v_2 & & & \leq & -1 \\
 & x_1 & & u_2 & & v_2 & & x_3 & \geq & 0
\end{array}
$$

(b)
$$
\begin{array}{lrcrcrcr}
\min & 45y_1 & - & 45y_2 & - & y_3 & & \\
\text{subject to} & 5y_1 & - & 5y_2 & - & y_3 & \geq & -1 \\
 & 3y_1 & - & 3y_2 & + & y_3 & \geq & 1 \\
 & -3y_1 & + & 3y_2 & - & y_3 & \geq & -1 \\
 & -3y_1 & + & 3y_2 & & & \geq & 1 \\
 & y_1 & & y_2 & & y_3 & \geq & 0
\end{array}
$$

3.

Row	*Basis*	x_1	x_2	x_3	x_4	x_5	x_6	*Value*
(0)	x_0	-2	4	6	-5	0	0	0
(1)	x_5	$<1>$	4	8	-2	1	0	2
(2)	x_6	-1	2	4	3	0	1	1

Row	*Basis*	x_1	x_2	x_3	x_4	x_5	x_6	*Value*
(0)	x_0	0	12	22	-9	2	0	4
(1)	x_1	1	4	8	-2	1	0	2
(2)	x_6	0	6	12	$<1>$	1	1	3

Row	*Basis*	x_1	x_2	x_3	x_4	x_5	x_6	*Value*
(0)	x_0	0	68	130	0	11	9	31
(1)	x_1	1	16	32	0	3	2	8
(2)	x_4	0	6	12	1	1	1	3

This is a solution because there are so negative numbers in row 0. The solution is: $x_1 = 8, x_4 = 3, x_2 = x_3 = 0$. The value is 31.

On the other form the problem was identical.

4. For each of the statements below, circle **TRUE** if the statement is always true, circle **FALSE** otherwise. No justification is required.

These problems refer to the linear programming problem (P) written in the form:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

and its dual

$$\min b \cdot y \text{ subject to } yA \geq c, y \geq 0.$$

(a) **TRUE** **FALSE**
If (D) is not feasible, then (P) is not feasible.
This is false ((P) could be unbounded).

(b) **TRUE** **FALSE**
Let u be a vector of ones (with the same number of components as b. If (P) has a solution, then

$$\max c \cdot x \text{ subject to } Ax \leq (b + u), x \geq 0$$

has a solution.
This is true. We added a positive number to each of the constraints, this makes the feasible set of the new problem bigger. Since it was non empty before (because the original problem had a solution), it continues to be non empty. In addition, changing the right-hand sides of (P) does not change the feasible set of the dual. So the new dual must be feasible. Since the new primal and new dual and feasible, they must both have solutions by the duality theorem.

(c) **TRUE** **FALSE**
If (P) has a solution and $\bar{c} \leq c$, then

$$\max \bar{c} \cdot x \text{ subject to } Ax \leq b, x \geq 0$$

has a solution ($\bar{c}$ may be different from c, but has the same number of components as c).
This is true. The new primal is feasible (because its feasible set is the same as the original (P)'s feasible set). The dual of the new problem is feasible, because the feasible set of the dual of the new problem is larger than the feasible set of the dual of (P). (The original dual must be feasible because the original primal had a solution.) Hence both the new problem and its dual are feasible, so both have solutions.

(d) **TRUE** **FALSE**

If a linear programming problem is infeasible, then it will continue to be infeasible if the objective function changes.
This is true. Changing the objective function does not change the feasible set.

On the other form I changed the order of the questions.