Principal Component Analysis for Nonstationary Series

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Approaches to large data sets

• Sparsity
  – assumption: most variables not useful
  – examples: LASSO, random forest

• Shrinkage
  – assumption: all variables used but each gets small weight
  – Principal components, ridge regression, Bayesian inference

• Problem: how use these methods when some variables may be nonstationary?
• Principal components: subtract sample mean from each variable and divide by standard deviation
• Calculate eigenvectors of correlation matrix associated with largest eigenvalues
• Use eigenvectors associated with largest eigenvalues to calculate linear combinations of variables
• Problem: if a variable is nonstationary, sample mean and standard deviation do not converge to any population parameter
• PCA when some variables are nonstationary can give very misleading results
  – Onatski and Wang, Econometrica 2021
• Usual approach: determine transformation necessary to make each individual variable stationary
Problem 1: necessary transformation can be unclear

Treasury yields for different maturities, 1982:1 to 2022:3

- GS3M
- GS6M
- GS1
- GS2
- GS5
- GS10
• Many finance applications apply PCA to yields themselves
• McCracken and Ng (JBES 2016) use first-differences of yields or yield spreads
• Crump and Gospodinov (Econometrica 2022) use excess returns or first-differences of returns
Problem 2: reproducibility

• Need to communicate decision used for every variable in the study
• Another researcher who did not use same transformations could get different answers
Problem 3: appropriateness of the method

- Suppose we knew for certain that variable 1 is random walk and variable 2 is AR(1) with coefficient 0.99
- Current approach would say use differences of variable 1 and levels of variable 2
- But these have very different properties
Levels and first-differences of yields
Hamilton (REStat, 2018)

- The error in predicting a variable 2 years from now as a linear function of recent values:
  - is a stationary population magnitude for a broad class of nonstationary processes such as ARIMA\((p,d,q)\) or processes stationary around \(d\)th-order polynomial time trends
  - could be described as cyclical component of the series
  - can be consistently estimated by OLS regression without knowing \(d\)
Example: suppose $\Delta y_{it}$ is stationary ($d = 1$).

Accounting identity:

$$y_{it} = y_{i,t-h} + \sum_{j=0}^{h-1} \Delta y_{i,t-j}$$

$y_{it}$ can be written as linear function of $y_{i,t-h}$ plus something stationary.
Error predicting $y_{it}$ from $y_{i,t-h}, y_{i,t-h-1}, \ldots, y_{i,t-h+p-1}$ is stationary. OLS minimizes sample squared forecast errors and consistently estimates this population object.
Suppose $\Delta^2 y_{it}$ is stationary 
($d = 2$).

Accounting identity:

$$y_{it} = y_{i,t-h} + h\Delta y_{i,t-h} + \sum_{j=0}^{h-1} (j + 1)\Delta^2 y_{i,t-j}$$

$y_{it}$ can be written as linear function of $y_{i,t-h}, y_{i,t-h-1}$ plus something stationary.
\( y_{it} = \text{observation on variable } i \text{ in period } t \)

\[
y_{it} = \alpha_{i0} + \alpha_{i1} y_{i,t-h} + \alpha_{i2} y_{i,t-h-1} + \cdots + \alpha_{ip} y_{i,t-h-p+1} + c_{it}
\]

\( c_{it} = \text{population magnitude (exists for large} \) class of possible data-generating processes for } y_{it} \)

\( \hat{c}_{it} = \text{OLS residual} \)
Proposal: estimate by OLS separately for each $i = 1, \ldots, N$

$$y_{it} = z_{it}' \alpha_i + c_{it}$$

$$z_{it}' = (1, y_{i,t-h}, y_{i,t-h-1}, \ldots, y_{i,t-h-p+1})'$$

Perform PCA on regression residuals $\hat{c}_{it}$. 
In principle, would work for any finite $h$. $h = 1$ would correspond to principal component of 1-month-ahead forecast errors which is not usual object of interest. For $h$ too large, $c_{it}$ has lots of persistence and very large sample needed to estimate. We recommend $h = 24$ and $p = 12$ for monthly data.
Suppose true cyclical components are characterized by an approximate factor structure as in Stock and Watson (JASA 2002):

\[ C_t = \Lambda F_t + e_t \]

\[ \begin{align*}
N \times 1 & \quad (N \times r)(r \times 1) & \quad (N \times 1) \\
\lim \sup_{t} \sum_{s=-\infty}^{\infty} \left| \mathbb{E}[e_t'e_{t+s}/N] \right| & < \infty \\
\lim \sup_{t} N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \mathbb{E}[e_{i t}e_{j t}] \right| & < \infty \\
\lim \sup_{t,s} N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \left| \text{cov}[e_{i s}e_{i t}, e_{j s}e_{j t}] \right| & < \infty
\end{align*} \]
\( \nu_{it} = \hat{\nu}_{it} - C_{it} \)

If \( \nu_{it} \overset{m.s.}{\to} 0 \) uniformly in \( i \) and \( t \), then subject to normalization conditions,

\[
\hat{f}_{jt} \overset{p}{\to} f_{jt} \quad \forall j, t
\]

\[
T^{-1} \sum_{t=1}^{T} \hat{f}_{jt}^2 \overset{p}{\to} E(f_{jt}^2) \quad \text{for} \ j \leq r
\]

\[
T^{-1} \sum_{t=1}^{T} \hat{f}_{jt}^2 \overset{p}{\to} 0 \quad \text{for} \ j > r
\]
Should we expect that $E(v^2_{it}) \rightarrow 0$?

$$\sum_{t=1}^{T} v^2_{it} = (\alpha_i - \hat{\alpha}_i)' \sum_{t=1}^{T} z_{it}z_{it}'(\alpha_i - \hat{\alpha}_i)$$

This is proportional to OLS Wald test of the (correct) null hypothesis that $\alpha_i$ is the true value.
\[ \sum_{t=1}^{T} v_{it}^2 \] converges in distribution to some variable in a variety of stationary and nonstationary settings.

\[ T^{-1} \sum_{t=1}^{T} v_{it}^2 \xrightarrow{p} 0 \]
Application 1: Describing the yield curve
Let $\tilde{\lambda}_j = \text{eigenvector of correlation matrix of raw yields associated with } j\text{th largest eigenvalue.}$

Consider plot of weights of $\tilde{\lambda}_j$ as a function of maturity of yield $i$. 
Factor loadings for first 3 PC of raw yields as a function of maturity in months
First PC of raw yields as a function of time
\( \hat{c}_{it} \) = residual from OLS regression of \( y_{it} \) on \( (1, y_{i,t-24}, y_{i,t-25}, \ldots, y_{i,t-35}) \).

\( \hat{\lambda}_j \) = eigenvector of correlation matrix of \( \hat{c}_{it} \) associated with \( j \)th largest eigenvalue.

Now plot elements of \( \hat{\lambda}_j \) as a function of maturity of yield \( i \).
Factor loadings for first 3 PC of cyclical components of yields
First principal component of raw yields and cyclical component of yields
• For this application, PCA on levels works fine because all variables share the same trend component.

• Principal components capture both level and trend.

• If we mix U.S. nominal interest rates with other variables that have different trends, nonstationarity is bigger concern.
Application 2. Large macroeconomic data set

- Stock and Watson (JME 1999) found that first PC of a set of 85 different measures of real economic activity was best way to use big data set to predict inflation.
- This evolved into the Chicago Fed National Activity Index (CFNAI).
• McCracken and Ng (JBES 2016) developed FRED-MD data set
  – output and income; labor market; housing; consumption, orders, and inventories; money and credit; interest and exchange rates; prices; and stock market
  – 134 variables in 2015:4 vintage
  – continually updated
  – McCracken and Ng selected a transformation to make each variable stationary
Plant managers index

PMI (level)

PMI (transformed)

PMI (cyclical)
Log of industrial production index
Unemployment rate

Unemployment (level)  Unemployment (transformed)  Unemployment (cyclical)


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Series as transformed by McCracken and Ng

PMI (transformed)  IP (transformed)  Unemployment (transformed)
PC1 of transformed data and CFNAI
Cyclical components as identified by regressions

PMI (cyclical) | IP (cyclical) | Unemployment (cyclical)

PC1 of transformed data and of cyclical components
Dealing with outliers

- Traditional approach to outliers:
  - Calculate interquartile range of transformed data
  - If observation exceeds $k$ times the interquartile range, treat as missing
  - CFNAI historically used $k = 6$
  - McCracken-Ng used $k = 10$ and found 79 outliers in 22 different variables in 1960-2014 data set
How identify outliers if don’t know form of nonstationarity?

If we observed true $c_{it}$, could compare it with its interquartile range. Can estimate $\hat{c}_{it}$, but outliers will unduly influence regression.
Consider regression that does not use $y_{it}$ as dependent variable. Use these coefficients to predict $y_{it}$ and form “leave-one-out” residual $\tilde{c}_{it}$. Compare $\tilde{c}_{it}$ with its interquartile range. Leave-one-out regression with $h = 1$ identifies similar but not identical outliers as McCracken-Ng. 86 outliers in 26 different variables in 1960-2014 data set.
<table>
<thead>
<tr>
<th>variable</th>
<th>id</th>
<th>description</th>
<th>McKracken-Ng</th>
<th>Regression (h=1)</th>
<th>Regression (h=24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PPIITM</td>
<td>108</td>
<td>PPI intermediate materials</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>PPICRM</td>
<td>109</td>
<td>PPI crude materials</td>
<td>1</td>
<td>2001:2</td>
<td>0</td>
</tr>
<tr>
<td>OILPRICEEx</td>
<td>110</td>
<td>crude oil price</td>
<td>2</td>
<td>1974:1,1974:2</td>
<td>0</td>
</tr>
<tr>
<td>CPITRNSL</td>
<td>115</td>
<td>CPI transportation</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>CUSR0000SAC</td>
<td>117</td>
<td>CPI commodities</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>CUSR0000SAS</td>
<td>119</td>
<td>CPI services</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>DSERRG3M086SBEA</td>
<td>126</td>
<td>PCE consumption</td>
<td>1</td>
<td>2001:10</td>
<td>0</td>
</tr>
<tr>
<td>MZMSL</td>
<td>131</td>
<td>MZM money stock</td>
<td>1</td>
<td>1983:1</td>
<td>2</td>
</tr>
<tr>
<td>DTCOLN-VHFN M</td>
<td>132</td>
<td>motor vehicle loans</td>
<td>3</td>
<td>1977:12,2010:3, 2010:4</td>
<td>0</td>
</tr>
<tr>
<td>DTCTHFN M</td>
<td>133</td>
<td>consumer loans</td>
<td>2</td>
<td>2010:12,2011:1</td>
<td>2</td>
</tr>
<tr>
<td>total</td>
<td></td>
<td></td>
<td>79</td>
<td>86</td>
<td>45</td>
</tr>
</tbody>
</table>
But regressions with $h = 24$ have far fewer outliers.

If $y_{it}$ is random walk, then $c_{it}$ is sum of 24 individual innovations.

By CLT, $c_{it}$ has a distribution much closer to Normal distribution.

In 1960-2014, outliers detected in only two variables (nonborrowed and total reserves) essentially all in the Great Recession.
Our recommended procedure makes no corrections for outliers.
• When dataset is expanded to include recent data, McCracken-Ng identifies 40 outliers in 2020:4 observations alone
• CFNAI modified their treatment of outliers to accommodate COVID observations