Principal Component Analysis for Nonstationary Series

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Abstract

This paper develops a procedure for uncovering the common cyclical factors that drive a mix of stationary and nonstationary variables. The method does not require knowing which variables are nonstationary or the nature of the nonstationarity. An application to the FRED-MD macroeconomic dataset demonstrates that the approach offers similar benefits to those of traditional principal component analysis with some added advantages.

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1 Introduction

Principal component analysis (PCA) has become a key tool for building dynamic models of vector time series with a large cross-sectional dimension. A typical application first subtracts the sample mean from each variable and divides the demeaned variables by their sample standard deviations. PCA finds linear combinations of the standardized variables that have maximum sample variance. These principal components are then used to build dynamic models for each individual series. For surveys of PCA and its usefulness in economics see Bai and Ng (2008) and Stock and Watson (2016).

One difficulty with PCA is that many of the time series encountered in economics and finance are nonstationary. For a nonstationary variable, the population mean is undefined and the sample standard deviation diverges to infinity as the number of time-series observations gets large. Onatski and Wang (2021) detailed some of the problems that can arise from trying to apply PCA to nonstationary data. The typical solution to this problem is for researchers to examine each series individually by hand to determine the transformation of that series that needs to be made before calculating principal components of the set of variables.

This approach has two shortcomings. First, while for some variables it may be fairly clear what transformation is necessary to achieve stationarity, for others it is far from obvious. For example, interest rates have a strong downward trend since 1980. Should yields be treated as stationary? If nonstationary, should we take their first differences or deviations from a time trend before performing PCA? Many finance applications take principal components of interest rates without any transformation; see for example Piazzesi (2010) or Joslin et al. (2011). McCracken and Ng (2016) used either first differences or spreads between interest rates as the first step before including interest rates in PCA. Crump and Gospodinov (2022) recommended using either bond returns or the first differences of bond returns in place of yields themselves to reduce the persistence of these data. Some authors work with first differences of inflation and the unemployment rate while others leave those variables as is. Many decisions like these have to be made before applying PCA to large data sets. In this paper, we propose an automatic procedure that allows the researcher to avoid these judgment calls.

A second fundamental problem is the appropriateness of the methodology itself. Suppose we somehow overcame the first problem and knew for certain the true nature of the trend in each individual series. Suppose for illustration we knew correctly that the first variable y_{1t} is a stationary $AR(1)$ process with autoregressive coefficient $\rho = 0.99$ while the second variable y_{2t} is a random walk. The currently prescribed procedure would instruct the researcher to use the first variable as is and the second variable in the form of first differences. But while the levels of y_{1t} and y_{2t} exhibit very similar properties, the level of y_{1t} and first difference of y_{2t} are radically different. Should we expect that there is some linear combination of y_{1t} and Δy_{2t} that can summarize the common economic drivers behind the two variables? If differencing is the appropriate transformation for a random walk, it seems we should be using some similar transformation for an $AR(1)$ process whose root is close but not quite equal to unity.

This paper proposes an approach to PCA that solves these problems. We first note as in Hamilton (2018) that it is possible to use OLS to estimate an h-period-ahead forecast of the level of any variable as a linear function of its own lags without knowing the nature of the nonstationarity. Moreover, the errors from these linear forecasts are stationary for a broad class of underlying nonstationary processes. Our proposal is to use PCA to identify common factors behind the forecast errors. Specifically, we estimate an OLS regression of y_{it} on $\{1, y_{i,t-h}, y_{i,t-h-1}, \ldots, y_{i,t-h-p+1}\}$, where $i = 1, \ldots, N$ represents the index of the variable, h is the forecasting horizon, and p is the number of lags used for the forecast. We then calculate principal components of the residuals. This approach solves both of the problems identified above. The procedure is fully automatic and treats every variable in the same way. The transformation of y_{it} is a continuous function of the estimated autoregressive coefficients and implies no discontinuity in the way that persistent stationary series are treated as the largest autoregressive root tends toward unity.

Hamilton (2018) suggested that a two-year-ahead forecast error could be interpreted as the stationary cyclical component of the series. Our empirical application follows this suggestion and uses $h = 24$ for monthly data to uncover the common cyclical factors behind a range of economic and financial indicators. We demonstrate that using $h = 24$ also offers a practical solution to the problem of outliers which has posed a severe challenge to using PCA on macroeconomic observations in 2020.

Bai and Ng (2004) and Barigozzi et al. (2021) developed methods that are suitable when the observed data are an unknown mix of $I(0)$ and $I(1)$ variables. Their approaches start by applying

PCA to first-differences of all of the variables whether stationary or not. Bai and Ng (2004) estimated factors for the original data by accumulating factors estimated from the differenced data. Barigozzi et al. (2021) estimated factors by applying the loadings estimated from differenced data to the detrended levels of the original data. Barigozzi et al. (2021) showed that their procedure can estimate the set of factors that account for common stochastic trends in the original data. Bai and Ng (2004) allowed for some of the common factors to be $I(0)$ and others to be $I(1)$, but their equations (1)-(3) do not allow the possibility that a single common factor is influencing the level of a stationary variable y_{1t} and the growth rate of a nonstationary variable y_{2t}. In contrast to these approaches, the goal of our analysis is to uncover the cyclical components that are driving both the stationary and nonstationary variables.

Our results are also useful for estimation of dynamic factor models. One common approach to estimating dynamic factor models is to apply a transformation that is believed to make the individual series stationary, calculate principal components of the transformed data, and then fit a vector autoregression to the estimated factors (Stock and Watson (2002, 2016); Bai and Ng (2002, 2006)). Alternatively, the factors and their dynamics can be estimated jointly by maximum likelihood or quasi-maximum likelihood (Bańbura and Modugno (2014); Doz et al. (2012)). For MLE or QMLE, researchers again typically transform each observed variable y_{it} individually to achieve stationarity (e.g., Forni et al. (2009); D'Agostino et al. (2016)). Antolin-Diaz et al. (2017) allowed the growth rate of a known subset of the variables to follow a random walk as part of the hypothesized state-space framework.[1] In contrast, our approach allows the researcher to implement the initial transformations without the need for subjective judgments about each individual series.

Section 2 discusses the use of forecasting regressions to isolate a stationary component of a possibly nonstationary time series. Section 3 describes how PCA could be used to uncover the common factors behind the forecast errors if we somehow could know the true process followed by each variable, and uses standard results to establish the consistency of PCA in that hypothetical setting. Section 4 analyzes the case when PCA is applied to the residuals from estimated OLS regressions, and establishes consistency of the method in that more realistic setting. Simulations in Section 5 investigate what happens when we apply our procedure to a variety of possible

[1] Stock and Watson (2016) recommended a transformation to render the original raw data stationary that allows for a slowly moving growth rate or trend.

data-generating processes. We find that using two-year-ahead forecast errors works well when we have 50 years of data and that $h = 1$ performs quite well even in much shorter samples. Section 6 illustrates the promise of our approach in an empirical analysis using the FRED-MD large macroeconomics data set.

2 Isolating a stationary component from a nonstationary series

Let y_{it} denote the level of a possibly nonstationary variable at time t. Collect its p most recent values as of date $t-h$ along with a constant term in a vector $\tilde{z}_{i,t-h} = (1, y_{t-h}, y_{t-h-1}, ..., y_{t-h-p+1})'$. The population linear projection of y_{it} on $\tilde{z}_{i,t-h}$ is given by $\mathbb{P}(y_{it}|\tilde{z}_{i,t-h}) = \alpha_{i0}'\tilde{z}_{i,t-h}$ where the coefficient α_{i0} is defined as the vector that minimizes the expected squared error of a forecast of y_{it} based on a linear function of $\tilde{z}_{i,t-h}$:

$$\alpha_{i0} = \arg\min_{\alpha} E(y_{it} - \alpha'\tilde{z}_{i,t-h})^2. \tag{1}$$

For example, if y_{it} is covariance stationary with $E(\tilde{z}_{i,t-h}\tilde{z}_{i,t-h}')$ nonsingular, the population linear projection coefficient is given by

$$\alpha_{i0} = \left[E(\tilde{z}_{i,t-h}\tilde{z}_{i,t-h}')\right]^{-1} E(\tilde{z}_{i,t-h}y_{it}). \tag{2}$$

If y_{it} is ergodic for second moments, the population parameter α_{i0} can be consistently estimated by an OLS regression of y_{it} on $\tilde{z}_{i,t-h}$:

$$\hat{\alpha}_i = \left[\textstyle\sum_{t=1}^T \tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'\right]^{-1} \left[\textstyle\sum_{t=1}^T \tilde{z}_{i,t-h}y_{it}\right] = \left[T^{-1}\textstyle\sum_{t=1}^T \tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'\right]^{-1} \left[T^{-1}\textstyle\sum_{t=1}^T \tilde{z}_{i,t-h}y_{it}\right] \xrightarrow{p} \alpha_{i0}.$$

Note that there is no assumption in the definition (1) nor in its stationary solution (2) that the process is linear or that it can be characterized by an ARMA representation of known order. There may be better forecasts of y_{it} that could be obtained using a nonlinear function or more lags like $y_{i,t-h-p}, y_{i,t-h-p-1}, ...$ But there is an optimal forecast within the class of linear forecasts that use only p lags. For a covariance-stationary process, the optimal forecast within that class is characterized by (2).

A solution to (1) also exists if y_{it} is nonstationary but its first difference Δy_{it} is covariance stationary. This can be seen from the accounting identity that holds for any time series:

$$y_{it} = y_{i,t-h} + \Delta y_{i,t-h+1} + \Delta y_{i,t-h+2} + \cdots + \Delta y_{it}. \tag{3}$$

For an $I(1)$ process,

$$\mathbb{P}(y_{it}|\tilde{z}_{i,t-h}) = y_{i,t-h} + \mathbb{P}(\Delta y_{i,t-h+1}|\tilde{z}^{\dagger}_{i,t-h}) + \mathbb{P}(\Delta y_{i,t-h+2}|\tilde{z}^{\dagger}_{i,t-h}) + \cdots + \mathbb{P}(\Delta y_{it}|\tilde{z}^{\dagger}_{i,t-h}) \tag{4}$$

for $\tilde{z}_{i,t-h} = (1, y_{t-h}, y_{t-h-1}, ..., y_{t-h-p+1})'$, $\tilde{z}^{\dagger}_{t-h} = (1, \Delta y_{t-h}, \Delta y_{t-h-1}, ..., \Delta y_{t-h-p+2})'$ and $\mathbb{P}(\Delta y_{i,t-s}|z^{\dagger}_{i,t-h}) = \left[E(\Delta y_{i,t-s}\tilde{z}^{\dagger\prime}_{i,t-h})\right]\left[E(\tilde{z}^{\dagger}_{i,t-h}\tilde{z}^{\dagger\prime}_{i,t-h})\right]^{-1}\tilde{z}^{\dagger}_{i,t-h}$. Moreover, the population linear projection coefficient α_{i0} can again be consistently estimated by an OLS regression of the level of y_{it} on $(1, y_{i,t-h}, y_{i,t-h-1}, ..., y_{i,t-h-p+1})'$. The intuition for this is that an OLS levels regression minimizes the sample analog to the population minimization in (1):

$$\hat{\alpha}_i = \arg\min_{\alpha} T^{-1} \textstyle\sum_{t=1}^{T} (y_{it} - \alpha' \tilde{z}_{i,t-h})^2. \tag{5}$$

If the OLS estimate $\hat{\alpha}_i$ picks up the unit root, the value of (5) converges to a finite number as T gets large. If it does not, the value of (5) would diverge to infinity. Thus in large samples, OLS estimation is dominated by the incentive inherent in the objective function of OLS to remove any nonstationary features of the data. More formally, the fitted values from the regression (5) are numerically identical to the fitted values from an OLS regression of y_{it} on $(y_{i,t-h}, \tilde{z}^{\dagger'}_{i,t-h})$. In the latter regression, the OLS coefficient on the lagged level $y_{i,t-h}$ will converge in probability to one whenever the dependent variable is an $I(1)$ process. In fact, the OLS estimate of this parameter is superconsistent, converging at rate T rather than at rate $\sqrt{T}$, and the asymptotic distribution of the other coefficients is identical to what would result if we regressed Δy_{it} on $\tilde{z}^{\dagger}_{i,t-h}$; see Hamilton (2018).

Hamilton (2018) showed that these results generalize to any nonstationary process that is covariance stationary around a deterministic polynomial in time t provided that the order of the polynomial is no greater than the number of lags p, or if the process is $I(d)$ with the dth difference of y_{it} covariance stationary for some d no greater than p. One does not need to know the order of the polynomial or the value of d in order to consistently estimate the population linear projection coefficient α_{i0} using a regression of the level of y_{it} on a constant and its p most recent levels as of date $t-h$.

Hamilton (2018) also showed that the residual from the population linear projection, $c_{it} = y_{it} - \alpha_{i0}'\tilde{z}_{i,t-h}$, is covariance stationary for any of the above nonstationary processes. For example, for an $I(1)$ process, we see from (3) and (4) that

$$\begin{aligned} c_{it} &= y_{it} - \mathbb{P}(y_{it}|\tilde{z}_{i,t-h}) \\ &= \left[\Delta y_{i,t-h+1} - \mathbb{P}(\Delta y_{i,t-h+1}|\tilde{z}^{\dagger}_{i,t-h})\right] + \left[\Delta y_{i,t-h+2} - \mathbb{P}(\Delta y_{i,t-h+2}|\tilde{z}^{\dagger}_{i,t-h})\right] + \\ &\quad \cdots + \left[\Delta y_{it} - \mathbb{P}(\Delta y_{it}|\tilde{z}^{\dagger}_{i,t-h})\right] \end{aligned}$$

is covariance stationary for any finite h. It is thus possible to isolate a stationary component of a large range of nonstationary processes from the residuals of an OLS regression of y_{it} on $(1, y_{i,t-h}, y_{i,t-h-1}, ..., y_{i,t-h-p+1})'$ without knowing whether the variable y_{it} is stationary or non-stationary.[2]

It is instructive to compare this approach with other common ways of thinking about the trend in a nonstationary series. If the ith observed variable is a deterministic function of time plus a zero-mean stationary process, $y_{it} = \delta_i^{\text{det}}(t) + c_{it}^{\text{det}}$, we could describe the deterministic time trend as the limit of a forecast that would have been made in the arbitrarily distant past:

$$\delta_i^{\text{det}}(t) = \lim_{h\to\infty}\lim_{p\to\infty} E(y_{it}|y_{i,t-h}, y_{i,t-h-1}, ..., y_{i,t-h-p+1}).$$

By contrast, if the first difference of the ith variable is a zero-mean stationary process, Beveridge

[2] Specifically, if either: (i) y_{it} is stationary around a deterministic polynomial function of time of order $d_i \leq p$ satisfying

$$T^{-1/2}\sum_{s=1}^{[Tr]}(y_{it} - \delta_{i0} - \delta_{i1}t - \delta_{i2}t^2 - \cdots - \delta_i t^{d_i}) \Longrightarrow \omega_i W_i(r)$$

where $[Tr]$ denotes the largest integer no greater than Tr, $W_i(r)$ denotes standard Brownian motion, and $\Rightarrow$ denotes weak convergence of associated probability measures; or alternatively if (ii) d_i differences of y_{it} are stationary for some $d_i \leq p$ satisfying

$$T^{-1/2}\sum_{s=1}^{[Tr]}(\Delta^{d_i} y_{it} - \mu_i) \Longrightarrow \omega_i W_i(r);$$

then Hamilton (2018) showed that c_{it} is stationary and that the estimated coefficient $\hat{\alpha}_i$ from an OLS regression gives a consistent estimate of the population parameter α_{i0}.

and Nelson (1981) suggested that we think of the trend as the forecast of the variable in the arbitrarily distant future. They decomposed $y_{it} = \delta_{it}^{BN} + c_{it}^{BN}$ where

$$\delta_{it}^{BN} = \lim_{h\to\infty}\lim_{p\to\infty} E(y_{i,t+h}|y_{it}, y_{i,t-1}, ..., y_{i,t-p+1}).$$

While these concepts of trend have some appeal, they have the significant practical drawback that both are based on the properties of forecasts at infinite horizons. They thus depend on conjectures of what happens at infinity, conjectures that are impossible to verify on the basis of a finite sample of observations.

By contrast, forecasts at a two-year horizon are something that can be reasonably investigated without auxiliary assumptions in samples of typical size. If we choose h to correspond to a two-year horizon, Hamilton (2018) argued that the decomposition

$$y_{it} = \mathbb{P}(y_{it}|1, y_{i,t-h}, y_{i,t-h-1}, ..., y_{i,t-h-p+1}) + c_{it} \tag{6}$$

can be viewed as a way to implement traditional approaches to trend-cycle decomposition that is practical and robust. Moreover, the primary reason we would go wrong in making a two-year-ahead forecast of most economic time series is due to unforeseen cyclical changes. For example, the value of y_{it} will be significantly below $\mathbb{P}(y_{it}|\tilde{z}_{i,t-h})$ if the economy goes into a recession after $t-h$ and significantly above the forecast if recovery from a downturn is more robust than expected. For this reason, Hamilton (2018) proposed to use h corresponding to a two-year horizon as a way to isolate the the stationary cyclical component of the ith variable, for example, $h = 24$ for monthly data.

Another important benefit of setting $h = 24$ is that it offers a solution to the huge practical problem of dealing with outliers. To see why this is the case, suppose for illustration that the ith variable is characterized by a random walk: $y_{it} = y_{i,t-1} + \varepsilon_{it}$. For a monthly random walk, the residual from the two-year population linear projection is given by $c_{it} = \sum_{s=0}^{23} \varepsilon_{i,t-s}$. From the Central Limit Theorem, c_{it} exhibits much less kurtosis than the one-period-ahead forecast error ε_{it}. We find in our empirical application in Section 6 that this feature is extremely helpful in dealing with outliers such as those seen during the COVID-19 pandemic in 2020.

Notwithstanding, these benefits do not come without a cost. The larger the value of h, the more serial correlation there will be in c_{it}, and the more prone PCA can be in small samples to the spurious factor problem identified by Onatski and Wang (2021). Our simulations in Section 5 suggest that if we have 50 years of data, using two-year-ahead forecast errors works pretty well. For shorter samples, the spurious factor problem can be more serious, and our recommendation for those applications is to look for common factors in the one-year-ahead ($h = 12$) or one-month-ahead ($h = 1$) forecasts. In the latter case, additional correction for outliers is likely needed.

To summarize, our procedure is to estimate the same regression for every variable y_{it}, regardless of whether we think it is stationary and without making any conjecture about the nature of any nonstationarity. To allow for persistent seasonal components in y_{it}, we recommend choosing p to be the number of observations in a year. Our procedure estimates the following regression by OLS for every variable,

$$y_{it} = k_i + \alpha_{i1} y_{i,t-h} + \alpha_{i2} y_{i,t-h-1} + \cdots + \alpha_{ip} y_{i,t-h-p+1} + c_{it}, \tag{7}$$

with $h = 8$ and $p = 4$ for quarterly data and $h = 24$ and $p = 12$ for monthly data. We will refer to the residual from the estimated regression $\hat{c}_{it}$ as the OLS residual for variable y_{it} and the residual from the population linear projection c_{it} as the true cyclical component. The value $\hat{c}_{it}$ is a consistent estimate of c_{it}, and the true value c_{it} is stationary as long as any nonstationarity in y_{it} is characterized by either a polynomial time trend of order d_i or an $I(d_i)$ process with $d_i \leq p$. Our procedure is to perform PCA on the regression residuals $\{\hat{c}_{1t}, ..., \hat{c}_{Nt}\}$.

One practical decision is whether a nonlinear transformation of the raw data is necessary for $\Delta^{d_i} y_{it}$ to be stationary for some d_i. If taking first differences of the log is the correct way to produce a stationary series, then taking the change in the level would not produce a stationary series. We recommend using logs for variables like output or prices which are usually described in terms of growth rates. For such variables we use the log of the level of the variable, $y_{it} = \log Y_{it}$ as the variable in the regression (7). For variables like interest rates or the unemployment rate that are already quoted in percentage terms, we use the raw data $y_{it} = Y_{it}$ in the regression.

The true cyclical component c_{it} has mean zero and is stationary for a wide range of processes. However, the population value of α_{i0} is not known but must be estimated by regression. In Section

3 we characterize our assumptions about the factor structure that we hypothesize describes the true values of c_{it}, and use standard results to establish that these population factors could be consistently estimated if the true values of c_{it} were observed without error. Section 4 considers the case when we do not know the value of d_i for each series, do not know whether it is stationary or characterized by a deterministic time trend or an $I(d_i)$ process, and the c_{it} are not observed. In that section we analyze the consequences of performing PCA on the estimated OLS residuals $\hat{c}_{it}$.

3 Principal component analysis when the true cyclical component is observed

In the previous section we defined the true cyclical component c_{it} to be the residual from a population linear projection of y_{it} on $(1, y_{i,t-h}, y_{i,t-h-1}, ..., y_{i,t-h-p+1})'$, and noted that c_{it} is stationary for a broad class of possible processes. In this section we provide sufficient conditions under which the true cyclical components for a collection of N different variables would have a factor structure that could be consistently estimated using PCA if we observed the true value of c_{it} for each variable. The set-up and results in this section closely follow Stock and Watson (2002).

3.1 Assumed factor structure of the true cyclical components

Collect the true cyclical components for the N different series at time t in an $(N \times 1)$ vector $C_t = (c_{1t}, ..., c_{Nt})'$. We postulate that these are characterized by a factor structure of the form

$$\underset{(N\times 1)}{C_t} = \underset{(N\times r)}{\Lambda}\underset{(r\times 1)}{F_t} + \underset{(N\times 1)}{e_t}. \tag{8}$$

The number of latent factors r is much less than the number of variables N, but the r factors are assumed to account for most of the variance of C_t in a sense made formal below. Since the factors are unobserved, $C_t = \Lambda H^{-1} H F_t + e_t$ would imply the identical observable model as (8). Thus some normalizations are necessary in order to talk about consistently estimating the jth factor f_{jt}. In empirical estimation, practitioners typically resolve this ambiguity by estimating the jth column of Λ by the eigenvector associated with the jth largest eigenvalue of $T^{-1}\sum_{t=1}^{T} C_t C_t'$. Note that such a procedure implies a normalization in which the columns of Λ are orthogonal to each other and the elements of F_t are uncorrelated with each other and ordered by the size of their variance. We follow Stock and Watson (2002) in how to characterize these conventions as the cross-section dimension N and time-series dimension T get large.[3]

Assumption 1 (factor structure)**.**

(i) $(\Lambda'\Lambda/N) \to I_r$.

(ii) $E[F_tF_t'] = \Omega_{FF}$, *where* Ω_{FF} *is a diagonal matrix with* $\omega_{ii} > \omega_{jj} > 0$ *for* $i < j$.

(iii) $|\lambda_{ij}| \leq \bar{\lambda} < \infty$.

(iv) $T^{-1}\sum_t F_tF_t' \xrightarrow{p} \Omega_{FF}$.

In addition to implementing the property that eigenvectors of a symmetric matrix are orthogonal, Assumption 1(*i*) requires that each factor makes a nonnegligible contribution to the average variance of c_{it} across i. That is, if we were to imagine adding more variables (increasing N) with $\lambda_{ij} = 0$ for all i greater than some fixed N_0, then Assumption 1(*i*) could not hold. Likewise 1(*ii*) and 1(*iv*) require that each factor continues to matter as the number of time-series observations T grows. These conditions are consistent with serial dependence of the factors, but rely on the fact that C_t is stationary.

Let γ denote an $(N \times 1)$ vector and $\Gamma = \{\gamma : \gamma'\gamma/N = 1\}$. Note that if γ were the jth column of Λ, the scalar $\gamma'\Lambda F_t/N$ would converge to f_{jt} and $(N^2T)^{-1}\sum_{t=1}^{T}\gamma'\Lambda F_tF_t'\Lambda'\gamma \xrightarrow{p} \omega_{jj}$. The assumption that the idiosyncratic elements e_t do not have a factor structure requires that there is no value of γ for which the analogous operation applied to e_t would lead to anything other than zero: $\sup\limits_{\gamma\in\Gamma}(N^2T)^{-1}\sum_{t=1}^{T}\gamma'e_te_t'\gamma \xrightarrow{p} 0$. Stock and Watson (2002) used the following assumptions to guarantee the absence of a factor structure in e_t.

Assumption 2 (moments of the errors)**.**

(i) $\lim\limits_{N\to\infty} \sup_t \sum_{s=-\infty}^{\infty} |E[e_t'e_{t+s}/N]| < \infty$.

(ii) $\lim\limits_{N\to\infty} \sup_t N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} |E[e_{it}e_{jt}]| < \infty$, *where* e_{it} *denotes the ith element of* e_t.

(iii) $\lim\limits_{N\to\infty} \sup_{t,s} N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} |cov[e_{is}e_{it}, e_{js}e_{jt}]| < \infty$.

[3] See Bai and Ng (2013) and Stock and Watson (2016) for discussion of alternative normalizations.

Some might be concerned that we have simply postulated that the true cyclical components are characterized by Assumptions 1 and 2. But something very similar is done in traditional applications that assume conditions like these characterize specified stationary transformations of the original data. Indeed, insofar as the cyclical components have a common primitive definition in terms of h-period-ahead forecast errors, we would argue that these assumptions are easier to defend in our application than in many others.

3.2 Consequences of applying PCA to the true cyclical components

Recall that the $(N \times 1)$ vector of true cyclical components C_t is stationary and has population mean zero. If C_t was observed directly, its estimated sample variance matrix would be $S = T^{-1} \sum_{t=1}^{T} C_t C_t'$ and a linear combination $\gamma' C_t$ for any $(N \times 1)$ vector γ would have sample variance $\gamma' S \gamma$. If C_t was observed, the first estimated principal component (denoted $\tilde{f}_{1t} = N^{-1} \tilde{\lambda}_1' C_t$) would be defined as the linear combination that has maximum sample variance subject to a normalization condition such as $\gamma \in \Gamma = \{\gamma : \gamma'\gamma/N = 1\}$:

$$\tilde{\lambda}_1 = \arg \sup_{\gamma \in \Gamma} \tilde{R}(\gamma) \tag{9}$$

$$\tilde{R}(\gamma) = (N^2 T)^{-1} \gamma' \textstyle\sum_{t=1}^{T} C_t C_t' \gamma. \tag{10}$$

Note we are normalizing $\tilde{\lambda}_1' \tilde{\lambda}_1 / N = 1$ as we did asymptotically for the columns of Λ in Assumption 1(i). We also divide the sample variance of $\gamma' C_t$ by N^2 in anticipation of the result that $\tilde{R}(\tilde{\lambda}_1)$ will converge to a fixed constant as N and T grow. The solution to (9) is obtained by setting $\tilde{\lambda}_1$ proportional to the eigenvector of $S = T^{-1} \sum_{t=1}^{T} C_t C_t'$ associated with the largest eigenvalue. For example, if we calculated eigenvectors of this matrix using code that normalizes eigenvectors to have unit length and orders eigenvalues by decreasing size, $\tilde{\lambda}_1$ would be $\sqrt{N}$ times the first eigenvector. The largest eigenvalue of S is equal to $T^{-1} \sum_{t=1}^{T} \tilde{f}_{1t}^2$, the sample variance of the first principal component. The jth principal component $N^{-1} \tilde{\lambda}_j' C_t$ is found by maximizing $\tilde{R}(\gamma)$ subject to the constraint that γ is orthogonal to $\tilde{\lambda}_1, ..., \tilde{\lambda}_{j-1}$. The solution for $\tilde{\lambda}_j$ is proportional to the eigenvector of S associated with the jth largest eigenvalue.

Alternatively, if we observed the true factors F_t and loadings Λ, we could calculate the component of the variance of $\gamma' C_t$ that is attributable to the r factors alone:

$$R^*(\gamma) = (N^2 T)^{-1} \gamma' \textstyle\sum_{t=1}^{T} \Lambda F_t F_t' \Lambda' \gamma. \tag{11}$$

Stock and Watson (2002) showed that under Assumptions 1 and 2, the maximum value for (10) (which is given by the largest eigenvalue of S) and the supremum of (11) over all $\gamma \in \Gamma$ converge in probability to the same number ω_{11}, which is the population variance of the first factor, and that $\tilde{\lambda}_1' C_t / N$ gives a consistent estimate of f_{1t} up to a sign. If we were to estimate $k > r$ principal components, the first r would consistently estimate f_{jt} up to a sign normalization and the last $k - r$ would asymptotically have zero variance. We restate their results in the following theorem.

Theorem 1. *(Stock and Watson, 2002). Suppose that Assumptions 1 and 2 hold. Let $R^*(\gamma)$ be the function in (11) and let $\tilde{f}_{1t}, ..., \tilde{f}_{kt}$ denote the first k estimated principal components of C_t ($\tilde{f}_{jt} = \tilde{\lambda}_j' C_t / N$) with $k \geq r$. Let $\underset{(r\times 1)}{\tilde{F}_t} = (\tilde{f}_{1t}, ..., \tilde{f}_{rt})'$ and $\underset{(N\times r)}{\tilde{\Lambda}} = \begin{bmatrix} \tilde{\lambda}_1 & \cdots & \tilde{\lambda}_r \end{bmatrix}$. Then as N and T go to infinity:*

- *(i)* $\sup_{\gamma\in\Gamma} R^*(\gamma) \xrightarrow{p} \omega_{11}$;
- *(ii) If* $\lambda_1^* = \arg\sup_{\gamma\in\Gamma} R^*(\gamma)$ *and* $\lambda_j^* = \arg \sup_{\gamma\in\Gamma, \gamma'\lambda_1^* = \cdots = \gamma'\lambda_{j-1}^* = 0} R^*(\gamma)$, *then* $R^*(\lambda_j^*) \xrightarrow{p} \omega_{jj}$ *for* $j = 1, ..., r$;
- *(iii)* $T^{-1} \sum_{t=1}^{T} \tilde{f}_{jt}^2 = \tilde{R}(\tilde{\lambda}_j) \xrightarrow{p} \omega_{jj}$ *for* $j = 1, ..., r$;
- *(iv)* $T^{-1} \sum_{t=1}^{T} \tilde{f}_{jt}^2 \xrightarrow{p} 0$ *for* $j = r+1, ..., k$;
- *(v)* $\tilde{S}\tilde{\Lambda}'\Lambda / N \xrightarrow{p} I_r$ *where* $\tilde{S}$ *is a diagonal matrix whose row j column j element is* $+1$ *if* $\tilde{\lambda}_j'\lambda_j > 0$ *and* -1 *if* $\tilde{\lambda}_j'\lambda_j < 0$.
- *(vi)* $\tilde{S}\tilde{F}_t - F_t \xrightarrow{p} 0$.

4 Principal component analysis when the cyclical component must be estimated

In this section we assume that we do not observe the true cyclical component c_{it} of series i but have an estimate $\hat{c}_{it} = c_{it} + v_{it}$. Let $\hat{C}_t = (\hat{c}_{1t}, ..., \hat{c}_{Nt})'$ and $V_t = (v_{1t}, ..., v_{Nt})'$. We investigate the properties of principal components calculated from the estimated cyclical components:

$$\hat{f}_{jt} = N^{-1}\hat{\lambda}_j'\hat{C}_t \tag{12}$$

$$\hat{\lambda}_j = \arg \sup_{\{\gamma\in\Gamma,\gamma'\hat{\lambda}_1=\cdots=\gamma'\hat{\lambda}_{j-1}=0\}} \hat{R}(\gamma)$$

$$\hat{R}(\gamma) = (N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} \hat{C}_t\hat{C}_t'\gamma.$$

We first note a high-level sufficient condition under which PCA applied to the estimated cyclical components $\hat{C}_t$ gives consistent estimates of the true factors F_t. Let $v_{it} = \hat{c}_{it} - c_{it}$ denote the difference between the estimated and true cyclical component of series i at date t. The condition is that v_{it} converges in mean square to zero uniformly in i and t as T goes to infinity.

Assumption 3 (high-level conditions on v_{it})**.** *For $\forall\delta > 0$, $\exists T_\delta : E(v_{it}^2) < \delta\ \forall T > T_\delta$ and $\forall i, t$.*

The following result establishes that if the error in estimating the cyclical component satisfies Assumption 3, the results $(\hat{f}_{jt}, \hat{\lambda}_j, \hat{R}(\gamma))$ of applying PCA to the estimated cyclical components $\hat{C}_t$ give consistent estimates of the magnitudes that characterize the true cyclical components C_t.

Theorem 2. *Suppose that C_t and e_t in equation (8) satisfy Assumptions 1 and 2. Let $\hat{C}_t = C_t + V_t$ for $V_t = (v_{1t}, ..., v_{Nt})'$ where v_{it} satisfy Assumption 3. Let $\hat{f}_{1t}, ..., \hat{f}_{kt}$ denote the first k estimated principal components of $\hat{C}_t$ ($\hat{f}_{jt} = \hat{\lambda}_j'\hat{C}_t/N$) with $k \geq r$ and let $\underset{(r\times 1)}{\hat{F}_t} = (\hat{f}_{1t}, ..., \hat{f}_{rt})'$ and $\underset{(N\times r)}{\hat{\Lambda}} = \begin{bmatrix} \hat{\lambda}_1 & \cdots & \hat{\lambda}_r \end{bmatrix}$. Then as N and T go to infinity:*

(i) $T^{-1}\sum_{t=1}^{T}\hat{f}_{jt}^2 = \hat{R}(\hat{\lambda}_j) \xrightarrow{p} \omega_{jj}$ *for* $j = 1, ..., r$;

(ii) $T^{-1}\sum_{t=1}^{T}\hat{f}_{jt}^2 \xrightarrow{p} 0$ *for* $j = r+1, ..., k$;

(iii) $\hat{S}\hat{\Lambda}'\Lambda/N \xrightarrow{p} I_r$ *where $\hat{S}$ is a diagonal matrix whose row j column j element is $+1$ if $\hat{\lambda}_j'\lambda_j > 0$ and -1 if $\hat{\lambda}_j'\lambda_j < 0$;*

(iv) $\hat{S}\hat{F}_t - F_t \xrightarrow{p} 0$.

Under what conditions can we expect Assumption 3 to hold? For $\tilde{z}_{i,t-h} = (1, y_{i,t-h}, y_{i,t-h-1}, ..., y_{i,t-h-p+1})'$, the true cyclical component c_{it} is the residual from a population linear projection of y_{it} on $\tilde{z}_{i,t-h}$ and $\hat{c}_{it}$ is the residual from the corresponding estimated regression:

$$c_{it} = y_{it} - \alpha_{i0}'\tilde{z}_{i,t-h}$$

$$\hat{c}_{it} = y_{it} - \hat{\alpha}_i'\tilde{z}_{i,t-h}$$

$$v_{it} = \hat{c}_{it} - c_{it} = (\alpha_{i0} - \hat{\alpha}_i)'\tilde{z}_{i,t-h}$$

$$(\alpha_{i0} - \hat{\alpha}_i) = -\left(\sum_{t=1}^{T} \tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'\right)^{-1} \left(\sum_{t=1}^{T} \tilde{z}_{i,t-h}c_{it}\right) \tag{13}$$

$$v_{it}^2 = (\alpha_{i0} - \hat{\alpha}_i)'\tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'(\alpha_{i0} - \hat{\alpha}_i). \tag{14}$$

Consider first the case of a single stationary regressor $(p = 1, d_i = 0)$ and $v_{it}^2 = (\alpha_{i0} - \hat{\alpha}_i)^2 y_{i,t-h}^2$. If y_{it} is stationary we expect that $\sqrt{T}(\alpha_{i0} - \hat{\alpha}_i) \xrightarrow{d} N(0, V_i)$ and $E(y_{i,t-h}^2) < \infty$, in which case v_{it}^2 should converge to zero as T gets large. Next consider the case of a single $I(1)$ regressor $(p = 1, d_i = 1)$. Take for illustration the case of a random walk: $y_{it} = y_{i,t-1} + \varepsilon_{it} = \varepsilon_{it} + \varepsilon_{i,t-1} + \cdots + \varepsilon_{i1}$ and $c_{it} = \varepsilon_{it} + \varepsilon_{i,t-1} + \cdots + \varepsilon_{i,t-h+1}$. Then

$$\begin{aligned} \hat{\alpha}_i - \alpha_{i0} &= \left(\sum_{t=1}^{T} y_{i,t-h}^2\right)^{-1} \left(\sum_{t=1}^{T} y_{i,t-h}c_{it}\right) \\ &= \left(\sum_{t=1}^{T} y_{i,t-h}^2\right)^{-1} \left(\sum_{t=1}^{T} y_{i,t-h}(\varepsilon_{i,t-h+1} + \varepsilon_{i,t-h+2} + \cdots + \varepsilon_{it})\right) \end{aligned}$$

$$\begin{aligned} T(\hat{\alpha}_i - \alpha_{i0}) &= \left[T^{-2}\sum_{t=1}^{T} y_{i,t-h}^2\right]^{-1} \Big[T^{-1}\sum_{t=1}^{T}(y_{i,t-h}\varepsilon_{i,t-h+1}) + T^{-1}\sum_{t=1}^{T}(y_{i,t-h+1} - \varepsilon_{i,t-h+1})\varepsilon_{i,t-h+2} \\ &\quad + \cdots + T^{-1}\sum_{t=1}^{T}(y_{i,t-1} - \varepsilon_{i,t-h-1} - \varepsilon_{i,t+h-2} - \cdots \varepsilon_{i,t-h+1})\varepsilon_{it}\Big] \\ &\xrightarrow{d} \left[\sigma_i^2 \int_0^1 [W_i(r)]^2 dr\right]^{-1} \left[h\sigma_i^2 \int_0^1 W_i(r)dW_i(r)\right] \end{aligned}$$

which is h times the Dickey-Fuller distribution.[4] Also for $t - h = [rT]$ for $[rT]$ the largest integer less than or equal to rT and $r \in (0, 1)$,

$$T^{-1}y_{i,[rT]}^2 \underset{T\to\infty}{\xrightarrow{d}} \sigma_i^2[W_i(r)]^2.$$

If we assume that $E(T^{-1}y_{i,t-h}^2)$ is bounded for all t, then $v_{it}^2 = T^{-1}[T(\alpha_{i0} - \hat{\alpha}_i)]^2[T^{-1}y_{i,t-h}^2]$ should

[4]See Hamilton (1994, Proposition 17.3) for similar derviations.

again converge to zero for each t as $T \to \infty$.

Returning to the general case, we have from (14) and (13) that

$$\begin{aligned}
\textstyle\sum_{t=1}^{T} v_{it}^2 &= (\alpha_{i0} - \hat{\alpha}_i)' \left(\textstyle\sum_{t=1}^{T} \tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'\right) (\alpha_{i0} - \hat{\alpha}_i) \\
&= \left(\textstyle\sum_{t=1}^{T} c_{it}\tilde{z}_{i,t-h}'\right) \left(\textstyle\sum_{t=1}^{T} \tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'\right)^{-1} \left(\textstyle\sum_{t=1}^{T} \tilde{z}_{i,t-h}c_{it}\right). \qquad (15)
\end{aligned}$$

This will be recognized as the OLS Wald statistic for testing the true null hypothesis $H_0 : \alpha_i = \alpha_{i0}$ multiplied by $\hat{\sigma}_i^2 = (T-k)^{-1} \sum_{t=1}^{T} \hat{c}_{it}^2$, the average squared regression residual. To find the asymptotic distribution of $\sum_{t=1}^{T} v_{it}^2$ we can make use of the insight of Sims et al. (1990) that the residuals from a regression of y_{it} on $\tilde{z}_{i,t-h}$ are numerically identical to the residuals from a regression of y_{it} on $z_{i,t-h} = R_i\tilde{z}_{i,t-h}$ for R_i any nonsingular matrix. For a particular form of nonstationarity, there is a particular value for R_i that makes the asymptotic properties of the residuals easiest to analyze. Note that we do not need to know the form of the nonstationarity since the observed residual $\hat{c}_{it} = y_{it} - \hat{\alpha}_i'\tilde{z}_{i,t-h}$ is numerically identical to the analyzed residual $y_{it} - \hat{\beta}_i' z_{i,t-h}$ and thus $v_{it} = \hat{c}_{it} - c_{it}$ is identical whichever way one chooses to describe the regression. For example, for $d_i = 2$ and $p = 4$, we would conveniently characterize the regressors as $z_{i,t-h} = (\Delta^2 y_{i,t-h}, \Delta^2 y_{i,t-h-1}, 1, \Delta y_{i,t-h}, y_{i,t-h})'$. The estimated residuals from a regression of y_{it} on $z_{i,t-h}$ are numerically identical to the residuals from a regression on $\tilde{z}_{i,t-h}$. We do not need to know the value of d_i to implement the first regression, and therefore do not need to know the value of d_i to find the residuals from the second regression:

$$\begin{aligned}
\textstyle\sum_{t=1}^{T} v_{it}^2 &= \left(\textstyle\sum_{t=1}^{T} c_{it}\tilde{z}_{i,t-h}'\right) \left(\textstyle\sum_{t=1}^{T} \tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'\right)^{-1} \left(\textstyle\sum_{t=1}^{T} \tilde{z}_{i,t-h}c_{it}\right) \\
&= \left(\textstyle\sum_{t=1}^{T} c_{it}z_{i,t-h}'(R_i')^{-1}\right) \left(\textstyle\sum_{t=1}^{T} R_i^{-1}z_{i,t-h}z_{i,t-h}'R_i'^{-1}\right)^{-1} \left(\textstyle\sum_{t=1}^{T} R_i^{-1}z_{i,t-h}c_{it}\right) \\
&= \left(\textstyle\sum_{t=1}^{T} c_{it}z_{i,t-h}'\right) \left(\textstyle\sum_{t=1}^{T} z_{i,t-h}z_{i,t-h}'\right)^{-1} \left(\textstyle\sum_{t=1}^{T} z_{i,t-h}c_{it}\right). \qquad (16)
\end{aligned}$$

For any given unknown true value of d_i, there is also an unknown true value of a diagonal scaling matrix Υ_{iT} that facilitates calculation of the asymptotic distribution:

$$\textstyle\sum_{t=1}^{T} v_{it}^2 = \left(\textstyle\sum_{t=1}^{T} c_{it}z_{i,t-h}'\Upsilon_{iT}^{-1}\right) \left(\Upsilon_{iT}^{-1}\textstyle\sum_{t=1}^{T} z_{i,t-h}z_{i,t-h}'\Upsilon_{iT}^{-1}\right)^{-1} \left(\Upsilon_{iT}^{-1}\textstyle\sum_{t=1}^{T} z_{i,t-h}c_{it}\right). \qquad (17)$$

Again we do not need to know the value of Y_{iT} in order to know that (17) characterizes the fitted residuals. The results in Hamilton (2018) establish that for a broad class of stationary and nonstationary processes,

$$\left(\sum_{t=1}^{T} c_{it} z'_{i,t-h} Y_{iT}^{-1}\right)\left(Y_{iT}^{-1}\sum_{t=1}^{T} z_{i,t-h} z'_{i,t-h} Y_{iT}^{-1}\right)^{-1}\left(Y_{iT}^{-1}\sum_{t=1}^{T} z_{i,t-h} c_{it}\right) \xrightarrow{d} q_i' Q_i q_i'.$$

Note the existence of limiting variables q_i and Q_i does not depend on any assumption that the c_{it} are serially uncorrelated. The result that $\sum_{t=1}^{T} v_{it}^2 \sim O_p(1)$ implies $T^{-1}\sum_{t=1}^{T} v_{it}^2 \sim o_p(1)$ when the regressors are stationary or nonstationary of any unknown order $d_i < p$. For example, if $d_i = 2$, $p = 4$ and $\Delta^2 y_{it}$ is a mean-zero $I(0)$ process, then

$$Y_{iT} = \begin{bmatrix} T^{1/2} & 0 & 0 & 0 & 0 \\ 0 & T^{1/2} & 0 & 0 & 0 \\ 0 & 0 & T^{1/2} & 0 & 0 \\ 0 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & T^2 \end{bmatrix} \tag{18}$$

$$Q_i = \begin{bmatrix} \gamma_{i0} & \gamma_{i1} & 0 & 0 & 0 \\ \gamma_{i1} & \gamma_{i0} & 0 & 0 & 0 \\ 0 & 0 & 1 & \omega_i \int_0^1 W_i(r)dr & \omega_i \int W_i^{(2)}(r)dr \\ 0 & 0 & \omega_i \int_0^1 W_i(r)dr & \omega_i^2 \int_0^1 \left[W_i(r)\right]^2 dr & \omega_i^2 \int W_i(r)W_i^{(2)}(r)dr \\ 0 & 0 & \omega_i \int W_i^{(2)}(r)dr & \omega_i^2 \int W_i(r)W_i^{(2)}(r)dr & \omega_i^2 \int \left[W_i^{(2)}(r)\right]^2 dr \end{bmatrix} \tag{19}$$

for $\gamma_{ij} = E(\Delta^2 y_{it} \Delta^2 y_{i,t-j})$, $\omega_i^2 = \sum_{j=-\infty}^{\infty} \gamma_{ij}$, $W_i(r)$ standard Brownian motion, and $W_i^{(2)}(r) = \int_0^r W_i(s)ds$. Thus $T^{-1}\sum_{t=1}^{T} v_{it}^2$ should converge to zero for a broad class of processes.

Consider next the convergence of v_{it}^2 for each individual t. In the case of a single stationary regressor, $v_{it}^2 = (\alpha_{i0} - \hat{\alpha}_i)^2 y_{i,t-1}^2$ and $\sum_{t=1}^{T} v_{it}^2 = (\alpha_{i0} - \hat{\alpha}_i)^2 \sum_{t=1}^{T} y_{i,t-1}^2$ so

$$\frac{v_{it}^2}{T^{-1}\sum_{t=1}^{T} v_{it}^2} = \frac{y_{i,t-1}^2}{T^{-1}\sum_{t=1}^{T} y_{i,t-1}^2} \tag{20}$$

$$v_{it}^2 = (A_{it}/B_{iT}) T^{-1} \sum_{t=1}^{T} v_{it}^2 \tag{21}$$

for $A_{it} = y_{i,t-1}^2$ and $B_{iT} = T^{-1}\sum_{t=1}^T y_{i,t-1}^2$. As $T \to \infty$, $B_{iT} \xrightarrow{p} E(y_{i,t-1}^2)$ meaning that if $T^{-1}\sum_{t=1}^T v_{it}^2 \xrightarrow{p} 0$, then also $v_{it}^2 \xrightarrow{p} 0$. A sufficient condition to ensure that $E(v_{it}^2) < \delta$ is that $E(A_{it}/B_{iT})^2 < \kappa_4$ is uniformly bounded.

For the random-walk example, expression (20) again holds identically and we can again rewrite it in the form of (21) now defining $A_{it} = T^{-1}y_{i,t-1}^2$ and $B_{iT} = T^{-2}\sum_{t=1}^T y_{i,t-1}^2$. Then $T^{-1/2}y_{i,[Ts]} \underset{T\to\infty}{\xrightarrow{d}} \sigma_i W_i(s)$ and

$$\frac{A_{it}}{B_{iT}} = \frac{T^{-1}y_{i,[Ts]}^2}{\int_0^1 T^{-1}y_{i,[Ts]}^2 ds} \xrightarrow{d} \frac{\left[W_i(s)\right]^2}{\int_0^1 [W_i(s)]^2 ds}.$$

This limiting distribution again has finite variance. We can ensure that Condition 3 holds for finite T and each t as before by assuming that $E(A_{it}/B_{iT})^2 < \kappa_4$.

For the general case,

$$\begin{aligned}
v_{it}^2 &= (\alpha_{i0} - \hat{\alpha}_i)'\tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'(\alpha_{i0} - \hat{\alpha}_i) \\
&= (\alpha_{i0} - \hat{\alpha}_i)'R_i^{-1}\Upsilon_{iT}\Upsilon_{iT}^{-1}R_i\tilde{z}_{i,t-h}\tilde{z}_{i,t-h}'R_i'\Upsilon_{iT}^{-1}\Upsilon_{iT}(R_i')^{-1}(\alpha_{i0} - \hat{\alpha}_i) \\
&= \bar{\alpha}_i'(\Upsilon_{iT}^{-1}z_{i,t-h}z_{i,t-h}'\Upsilon_{iT}^{-1})\bar{\alpha}_i \\
&= T^{-1}\bar{\alpha}_i'A_{it}\bar{\alpha}_i
\end{aligned}$$

for $\bar{\alpha}_i = \Upsilon_{iT}(R_i')^{-1}(\alpha_{i0} - \hat{\alpha}_i)$ and $A_{it} = T\Upsilon_{iT}^{-1}z_{i,t-h}z_{i,t-h}'\Upsilon_{iT}^{-1}$. Likewise $\sum_{t=1}^T v_{it}^2 = \bar{\alpha}_i'B_{iT}\bar{\alpha}_i$ for $B_{iT} = \Upsilon_{iT}^{-1}\left(\sum_{t=1}^T z_{i,t-h}z_{i,t-h}'\right)\Upsilon_{iT}^{-1}$. Thus

$$\frac{v_{it}^2}{T^{-1}\sum_{t=1}^T v_{it}^2} = \frac{\bar{\alpha}_i'A_{it}\bar{\alpha}_i}{\bar{\alpha}_i'B_{iT}\bar{\alpha}_i}. \tag{22}$$

In general, the ratio in (22) converges in distribution as $T \to \infty$ to a variable with finite variance, and condition 3 will hold if the ratio has finite variance for each t as well.

The stationary $p = 1$ example is a special case of this general formulation with $z_{i,t-h} = y_{i,t-h}$, $\Upsilon_{iT} = \sqrt{T}$, $A_{it} = y_{i,t-h}^2$, and $B_{iT} = T^{-1}\sum_{t=1}^T y_{i,t-h}^2$. For the random-walk example, $z_{i,t-h} = y_{i,t-h}$, $\Upsilon_{iT} = T$, $A_{it} = T^{-1}y_{i,t-h}^2$, and $B_{iT} = T^{-2}\sum_{t=1}^T y_{i,t-h}^2$. For the $p = 4, d_i = 2$ example, Υ_{iT} is given by

(18) and $B_{iT} \xrightarrow{d} Q_i$ given in (19), while the lower right (3×3) block of A_{it} is characterized by

$$T \begin{bmatrix} T^{-1/2} & 0 & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & T^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ \Delta y_{i,t-h} \\ y_{i,t-h} \end{bmatrix} \begin{bmatrix} 1 & \Delta y_{i,t-h} & y_{i,t-h} \end{bmatrix} \begin{bmatrix} T^{-1/2} & 0 & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & T^{-2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & T^{-1/2}\Delta y_{i,t-h} & T^{-3/2} y_{i,t-h} \\ T^{-1/2}\Delta y_{i,t-h} & T^{-1}(\Delta y_{i,t-h})^2 & T^{-2} y_{i,t-h}\Delta y_{i,t-h} \\ T^{-3/2} y_{i,t-h} & T^{-2} y_{i,t-h}\Delta y_{i,t-h} & T^{-3} y_{i,t-h}^2 \end{bmatrix}$$

$$\xrightarrow{d} \begin{bmatrix} 1 & \omega_i W_i(s) & \omega_i W_i^{(2)}(s) \\ \omega_i W_i(s) & \omega_i^2 [W_i(s)]^2 & \omega_i^2 W_i(s) W_i^{(2)}(s) \\ \omega_i W_i^{(2)}(s) & \omega_i^2 W_i(s) W_i^{(2)}(s) & \omega_i^2 \left[W_i^{(2)}(s) \right]^2 \end{bmatrix}$$

for $[sT] = t - h$ and $\omega_i^2 = \sum_{j=-\infty}^{\infty} E(\Delta^2 y_{it} \Delta^2 y_{i,t-j})$.

We now formally state sufficient conditions that guarantee that Assumption 3 holds.

Assumption 4 (sufficient conditions for Assumption 3)**.**

(i) The true cyclical component c_{it} is uniformly bounded, that is, there exists $\kappa_1 < \infty : \forall i\ c_{it}^2 < \kappa_1$.

(ii) $\sum_{t=1}^{T} v_{it}^2 \xrightarrow{d} U_i = q_i' Q_i^{-1} q_i$ with $E(U_i) < \kappa_2\ \forall i$.

(iii) The convergence is uniform in i, that is, for all $\kappa_3, \varepsilon_3 > 0$, $\exists T_3(\kappa_3, \varepsilon_3) : \forall T > T_3(\kappa_3, \varepsilon_3)$ and $\forall i$,

$$\left| Prob\left(\sum_{t=1}^{T} v_{it}^2 > \kappa_3 \right) - \ Prob(U_i > \kappa_3) \right| < \varepsilon_3.$$

(iv) Let $\lambda^{\max}(A_{it})$ denote the largest eigenvalue of $A_{it} = T \Upsilon_{iT}^{-1} z_{i,t-h} z_{i,t-h}' \Upsilon_{iT}^{-1}$ and $\lambda^{\min}(B_{iT})$ the smallest eigenvalue of $B_{iT} = \Upsilon_{iT}^{-1} \left(\sum_{t=1}^{T} z_{i,t-h} z_{i,t-h}' \right) \Upsilon_{iT}^{-1}$. There exists a $\kappa_4, T_4 < \infty$ such that $E\left[\lambda^{\max}(A_{it}) / \lambda^{\min}(B_{iT}) \right]^2 < \kappa_4$ for all $T > T_4$ and all i and t.

Presumably it is possible to replace condition 4(i) with restrictions on the tail behavior of c_{it}, though we have not attempted that here. Sufficient conditions for 4(ii) are analyzed in Hamilton (2018). The following result establishes that Assumption 4 can replace Assumption 3.

Theorem 3. *Assumption 4 implies Assumption 3.*

5 Results from simulations

In this section we report results from applying our method in a variety of different settings. In these simulations we take the number of cross-section variables to be $N = 100$ and vary the number of time-series observations T from 100 to 1000. For each generated sample, we calculate the $(N \times N)$ correlation matrices of: (1) the raw data; (2) the OLS regression residuals for that data set; and (3) the true cyclical components for that data set implied by the particular process that was used to generate that sample. For each of these correlation matrices, let $\hat{\xi}_j$ denote the jth largest eigenvalue of the correlation matrix. The fraction of the variance of the sample explained by the jth principal component is

$$R_j^2 = \hat{\xi}_j / N. \tag{23}$$

We also calculated the number of factors that would be selected for that sample based on the IC_{p2} criterion of Bai and Ng (2002) recommended by Stock and Watson (2016, p. 436),

$$r^* = \arg \left\{ \min_{r \in \{0,1,\ldots,r_0\}} \log \left(1 - \frac{\sum_{j=0}^{r} \hat{\xi}_j}{N} \right) + r \left(\frac{N+T}{NT} \right) \min\{N,T\} \right\}, \tag{24}$$

with $\hat{\xi}_0$ defined to be 0. We took $r_0 = 10$ and for each of the different cases generated 100 different samples.

5.1 Mix of unrelated stationary and nonstationary variables

For our first example, half the variables are random walks and the other half are white noise,

$$y_{it} = \begin{cases} y_{it-1} + \varepsilon_{it} = \varepsilon_{it} + \varepsilon_{i,t-1} + \cdots + \varepsilon_{i1} & \text{for } i = 1, 2, \ldots, N/2 \\ \varepsilon_{it} & \text{for } i = (N/2)+1, \ldots, N \end{cases}$$

for $t = 1, \ldots, T$. The innovations $\varepsilon_{it} \sim N(0,1)$ are independent across all i and t. Thus each of the N variables is completely independent of the others and there is no factor structure in the true data-generating process.

The first two columns of Table 1 report the results from calculating principal components of the raw data. In a sample of $T = 100$ observations, the first three principal components seem to account for 38% of the variance of the full set of $N = 100$ variables. This result is entirely spurious, and illustrates the cautions raised by Onatski and Wang (2021) about using PCA when some of the variables are nonstationary. The criterion (24) would always lead us incorrectly to conclude that there is more than one factor in a sample of size $T = 100$. This problem in the apparent number of factors gets even worse when the sample size increases. The latter is the expected result, since (24) is in the class of criteria for which Onatski and Wang (2021, p. 602) demonstrated that the number of factors selected diverges as N and T go to infinity.

The next two columns of Table 1 report what the results would be if we somehow knew the true cyclical component of each variable. For this case, we applied PCA to a sample of $T - h$ observations for which the ith observed variable for $t = h+1, h+2, ..., T$ is given by

$$c_{it} = \begin{cases} y_{it} - y_{i,t-h} = \varepsilon_{it} + \varepsilon_{i,t-1} + \cdots + \varepsilon_{i,t-h+1} & \text{for } i = 1, 2, ..., N/2 \\ \varepsilon_{it} & \text{for } i = (N/2)+1, ..., N \end{cases} . \tag{25}$$

Note that the cyclical components in (25) can be serially correlated, but this autocorrelation vanishes for observations separated by more than h periods. The cyclical components c_{it} thus satisfy by construction the conditions under which Bai and Ng (2002) demonstrated that (24) would asymptotically select the correct number of factors. We find in our simulations that (24) does indeed correctly conclude there is no factor structure for these data sets provided the number of time-series observations is 600 or larger. For smaller T it is less reliable. The reason is that there is a small-sample version of the Onatski and Wang (2021) spurious factor problem that arises from the serial correlation of some of the variables that is induced by the definition of the true cyclical component in equation (25). If T is large enough, this problem goes away, but for smaller T it can make a difference.

Columns (5) and (6) examine the case where the analysis is based on the residuals from running an OLS regression on the raw data Y_{it} for all variables $i = 1, ..., N$ without making any judgments about which variables are stationary and which are not. For large samples, the results are similar to those that we would obtain if we somehow knew the exact correct transformation to use for every variable.

The small-sample problem in columns (3)-(6) results from the serial correlation that is a conse-

Table 1: Mixture of independent random walks and white noise

j	Raw data		c_t (h=24)		$\hat{c}_t$ (h=24)		$\hat{c}_t$ (h=12)		$\hat{c}_t$ (h=8)		$\hat{c}_t$ (h=1)	
	R^2	r^*	R^2	r^*	R^2	r^*	R^2	r^*	R^2	r^*	R^2	r^*
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
$T = 100$												
0	—	0	—	0	—	1	—	13	—	96	—	100
1	22.6	0	16.0	0	11.9	26	10.3	48	8.4	4	4.0	0
2	9.9	67	12.0	6	9.4	54	8.5	35	7.1	0	3.8	0
3	5.6	33	8.3	94	7.3	19	6.9	4	6.1	0	3.6	0
$T = 200$												
0	—	0	—	0	—	0	—	78	—	100	—	100
1	22.7	0	9.4	0	10.2	0	6.6	22	5.2	0	2.9	0
2	9.4	13	7.8	3	8.2	8	5.7	0	4.6	0	2.7	0
3	5.3	87	6.6	97	6.5	92	4.9	0	4.2	0	2.6	0
$T = 400$												
0	—	0	—	23	—	3	—	100	—	100	—	100
1	22.5	0	5.9	51	6.5	34	4.3	0	3.5	0	2.2	0
2	9.4	0	5.2	22	5.5	47	3.8	0	3.2	0	2.1	0
3	5.1	100	4.5	4	4.7	16	3.5	0	3.0	0	2.1	0
$T = 600$												
0	—	0	—	99	—	79	—	100	—	100	—	100
1	22.4	0	4.6	1	5.0	21	3.4	0	2.9	0	1.9	0
2	9.3	0	4.1	0	4.4	0	3.1	0	2.7	0	1.9	0
3	5.2	100	3.7	0	3.9	0	2.9	0	2.5	0	1.8	0
$T = 800$												
0	—	0	—	100	—	100	—	100	—	100	—	100
1	22.7	0	4.0	0	4.2	0	3.0	0	2.6	0	1.8	0
2	9.3	0	3.6	0	3.7	0	2.7	0	2.4	0	1.7	0
3	5.0	100	3.3	0	3.4	0	2.6	0	2.3	0	1.7	0
$T = 1000$												
0	—	0	—	100	—	100	—	100	—	100	—	100
1	22.6	0	3.5	0	3.7	0	2.7	0	2.4	0	1.7	0
2	9.3	1	3.2	0	3.3	0	2.5	0	2.2	0	1.7	0
3	5.1	99	3.0	0	3.0	0	2.4	0	2.1	0	1.6	0

Notes to Table 1. R^2 indicates the percentage of total variance accounted for by the jth principal component for $j = 1, 2$ or 3. r^* indicates the percentage of samples for which the criterion (24) selects the number of factors to be $j = 0, 1, 2,$ or ≥ 3. In every case, the true number of factors is $r = 0$ and the cross-section dimension is $N = 100$.

quence defining the cyclical component to be the error from a 24-period-ahead forecast. Columns (7) and (8) report results if we instead were to look for common factors in the 12-period-ahead forecast errors. This typically would reach the correct conclusion even in a sample of only $T = 200$ observations. Columns (9) and (10) consider 8-period-ahead forecast errors, such as our suggested cyclical calculation would use for quarterly data. The results indicate that if we have more than 50 years of data ($T = 600$ for monthly data or $T = 200$ for quarterly data), conducting PCA on the two-year-ahead OLS forecast residuals is reasonably reliable.

The last two columns of Table 1 examine looking for common factors in the one-period-ahead forecast errors. In these simulations, this reaches the correct conclusion 100% of the time that there are zero factors in these data sets even for a sample of $T = 100$ observations. Thus our proposed method appears to be quite reliable if the interest is in identifying common factors behind one-period-ahead forecast errors. However, one-period-ahead forecast errors are more sensitive to outliers. This is an important consideration, as will be demonstrated in our analysis of actual data in Section 6.

5.2 Mix of unrelated stationary variables with differing persistence

In our second example, for $i = 1, 2, ..., N/2$ the variables are generated by a stationary but persistent AR(1) process:

$$y_{i1} \sim N(0, 1/(1-\rho^2))$$

$$y_{it} = \rho y_{i,t-1} + \varepsilon_{it} \quad \text{for } t = 2, 3, ..., T.$$

The remaining $N/2$ variables are white noise ($y_{it} = \varepsilon_{it}$ for $i = (N/2) + 1, ..., N$). Our example uses $\rho = 0.99$, so all the variables are stationary but half of them are highly persistent. The innovations $\varepsilon_{it} \sim N(0, 1)$ are independent across all i and t, so there is no factor structure in the true data-generating processes.

Columns (1) and (2) of Table 2 report the results from applying PCA to the raw data. Note that for this example, the raw data themselves satisfy the Bai and Ng (2002) conditions for asymptotic validity of PCA. Nevertheless, even in a sample of size $T = 1000$, the first principal component alone appears to explain a third of the data, and the criterion in (24) would always conclude incorrectly that there is at least one factor. This is a small-sample manifestation of the Onatski and

Wang (2021) spurious factor phenomenon. In a sufficiently large sample, this problem would go away. But $T = 1000$ is not large enough for persistence characterized by $\rho = 0.99$.

In columns (3)-(4) we apply PCA to the residuals from a 24-period-ahead forecasting regression. Again we estimated the same regression for all variables, whether persistent or not. And again using regression residuals solves the problem pretty reliably in samples larger than $T = 600$. If we look for a factor structure in the one-period-ahead regression residuals as in columns (5)-(6), the problem is solved 100% of the time even in a sample of $T = 100$.

5.3 Cointegration

In our next example, the nonstationary variables are cointegrated and there is no factor structure for the stationary variables. The single common factor for the nonstationary variables follows a random walk:

$$F_t = F_{t-1} + v_t \quad t = 1, ..., T;\ F_0 = 0$$

$$y_{it} = F_t + \varepsilon_{it} \quad i = 1, 2, ..., N/2$$

$$y_{it} = \varepsilon_{it} \quad i = (N/2) + 1, ..., N$$

with $v_t \sim N(0, 1)$ and $\varepsilon_{it} \sim N(0, 1)$ independent for all i and t. Notice that the first $(N/2)$ variables are cointegrated with $(N/2) - 1$ linearly independent cointegrating relations.[5] Thus the true number of common factors in this example is $r = 1$.

Columns (7) and (8) of Table 2 report the results from applying PCA to the raw data. Note that even though half the variables are nonstationary, PCA always correctly concludes that there is a single common factor in these data. This is an illustration of the well-known result that PCA on levels data can correctly identify cointegrating relations; for more discussion see Harris (1997) and Onatski and Wang (2018).

Columns (9) and (10) report results from applying PCA to the residuals from 24-period-ahead forecasting regressions. Again the same regression is estimated in the same way for stationary and nonstationary observations. And again PCA on the regression residuals results in the correct answer 100% of the time, even if the sample size is as small as $T = 100$.

[5] Linearly independent cointegrating relations are given by $y_{it} - y_{1t} \sim I(0)$ for $i = 2, 3, ..., (N/2)$.

Table 2: Stationary and cointegrated variables

	$\rho = 0.99$						Cointegrated			
	Raw data		$\hat{c}_t$ (h=24)		$\hat{c}_t$ (h=1)		Raw data		$\hat{c}_t$ (h=24)	
j	R^2 (1)	r^* (2)	R^2 (3)	r^* (4)	R^2 (5)	r^* (6)	R^2 (7)	r^* (8)	R^2 (9)	r^* (10)
$T = 100$										
0	—	0	—	0	—	100	—	0	—	0
1	43.3	98	16.0	1	4.1	0	46.1	100	40.3	100
2	3.6	2	10.6	99	3.8	0	2.8	0	3.4	0
3	2.8	0	5.5	0	3.6	0	2.5	0	3.1	0
$T = 200$										
0	—	0	—	0	—	100	—	0	—	0
1	42.5	82	10.9	0	2.9	0	48.1	100	45.7	100
2	3.3	18	8.7	2	2.7	0	2.1	0	2.3	0
3	2.3	0	6.9	98	2.6	0	2.0	0	2.2	0
$T = 400$										
0	—	0	—	7	—	100	—	0	—	0
1	40.5	54	6.3	26	2.2	0	48.8	100	46.5	100
2	3.4	46	5.5	54	2.1	0	1.8	0	1.8	0
3	2.2	0	4.8	13	2.1	0	1.7	0	1.7	0
$T = 600$										
0	—	0	—	96	—	100	—	0	—	0
1	38.5	40	4.8	4	1.9	0	49.3	100	47.0	100
2	3.5	57	4.3	0	1.9	0	1.6	0	1.6	0
3	2.3	3	3.9	0	1.8	0	1.5	0	1.6	0
$T = 800$										
0	—	0	—	100	—	100	—	0	—	0
1	36.9	32	4.1	0	1.8	0	49.4	100	47.0	100
2	3.4	60	3.7	0	1.7	0	1.5	0	1.5	0
3	2.4	8	3.3	0	1.7	0	1.5	0	1.5	0
$T = 1000$										
0	—	0	—	100	—	100	—	0	—	0
1	35.4	33	3.6	0	1.7	0	49.5	100	47.1	100
2	3.5	60	3.3	0	1.7	0	1.5	0	1.5	0
3	2.5	7	3.0	0	1.6	0	1.4	0	1.4	0

Notes to Table 2. R^2 indicates the percentage of total variance accounted for by the jth principal component for $j = 1, 2$ or 3. r^* indicates the percentage of samples for which the criterion (24) selects the number of factors to be $j = 0, 1, 2,$ or ≥ 3. In columns (1)-(6), the true number of factors is $r = 0$. In columns (7)-(10), the true number of factors is $r = 1$. In every case, the cross-section dimension is $N = 100$.

These results confirm the asymptotic theory that applying PCA to OLS regression residuals is a promising approach to handling both nonstationarity and stationary persistence of unknown form in reasonably large samples.

6 Characterizing a large macroeconomic data set

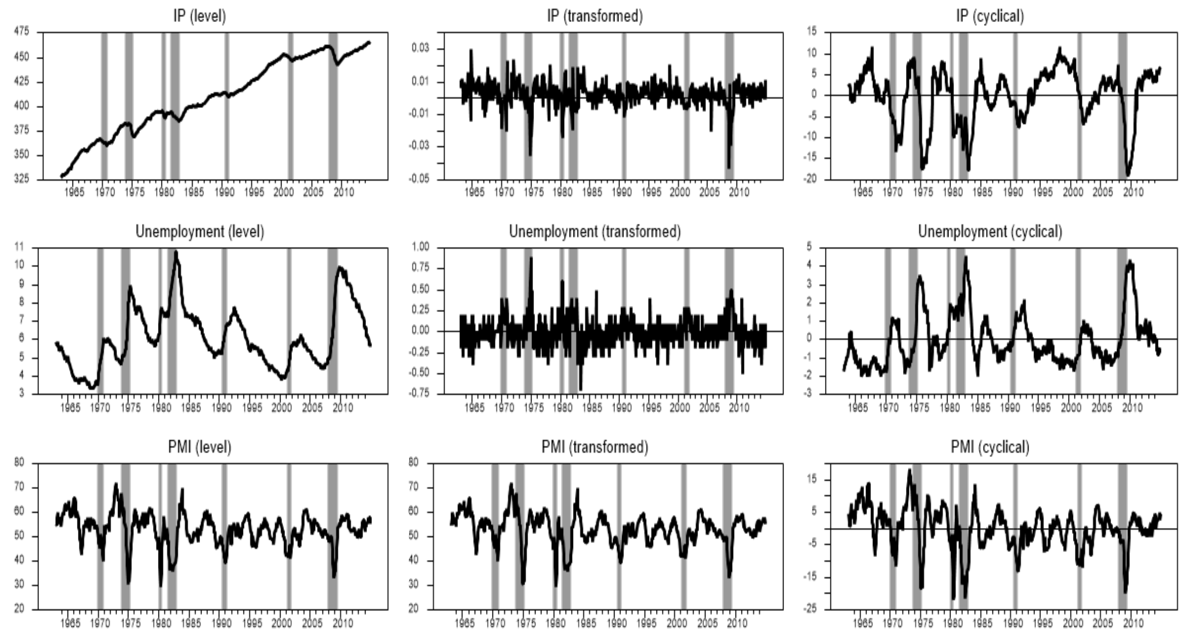
The use of large macroeconomic data sets was pioneered by Stock and Watson (1999), whose goal was to use the information of 168 different macroeconomic variables to produce better forecasts of inflation. They found that the first principal component of macroeconomic variables that are related to the level of real economic activity produced the best inflation forecasts over the period 1959:1 to 1997:9. Their findings led to the development of the Chicago Fed National Activity Index, which is the first principal component of a subset of 85 different measures of economic activity.[6]

McCracken and Ng (2016) reviewed subsequent uses of large macroeconomic sets and developed the FRED-MD database whose 2015:4 vintage covered 134 macroeconomic variables. These variables include monthly measures divided into 8 broad categories: (1) output and income; (2) labor market; (3) housing; (4) consumption, orders, and inventories; (5) money and credit; (6) interest and exchange rates; (7) prices; and (8) stock market. This data set offers benefits of continuity and continuous updating and is the basis for our analysis in this paper.

In previous applications of PCA to large macroeconomic data sets, each of the variables needed to be transformed using a detrending method that was selected individually for each series. For details of how this has been done for the CFNAI see Federal Reserve Bank of Chicago (2021) and for FRED-MD see the data appendix to McCracken and Ng (2016). Figure 1 illustrates these transformations for three important macroeconomic indicators. The first column plots the raw data, while the second column plots the data as transformed by McCracken and Ng (2016) in order to ensure stationarity, using the same data set as in their original paper. Everyone agrees that industrial production (row 1) is nonstationary, and all previous researchers have used first differences of the log of industrial production shown in panel (1,2). While there is little doubt that this is a good way to generate a stationary series for this variable, monthly growth rates of

[6]The variables used to calculate the Chicago Fed National Activity Index (CFNAI) fall into four broad groups: (1) production and income; (2) employment, unemployment, and hours; (3) personal consumption and housing; and (4) sales, orders, and inventories

Figure 1: Level, transformed value, and cyclical component of industrial production, unemployment, and PMI Composite, 1962:3 to 2014:12



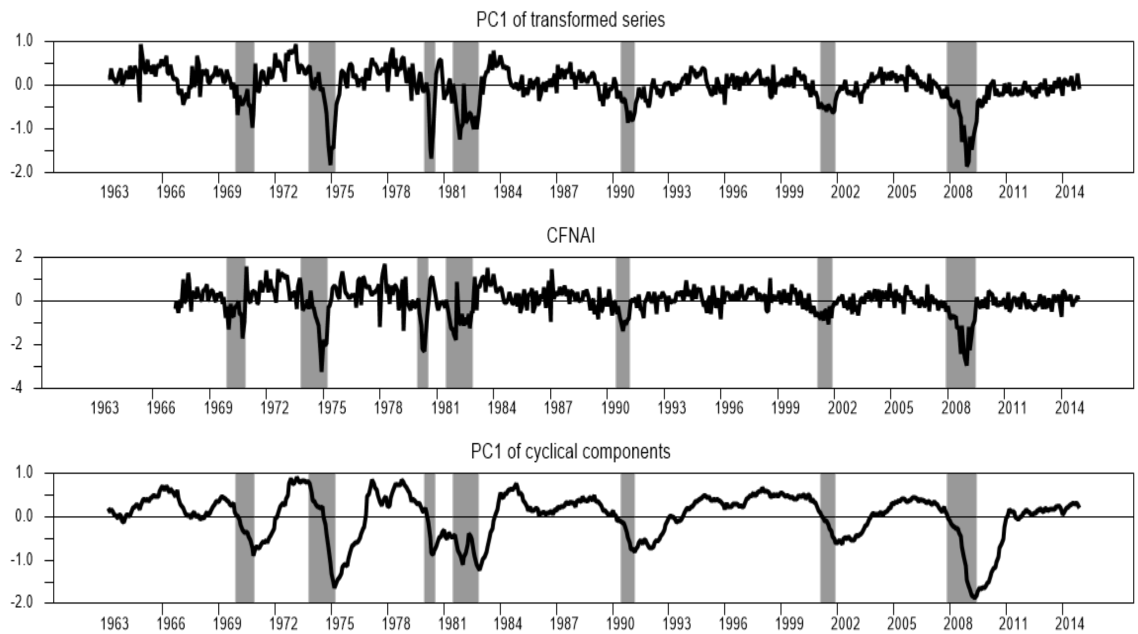
industrial production exhibit a lot of high-frequency fluctuations around the dominant cyclical patterns. For the unemployment rate (row 2), it is less clear whether the series should be regarded as stationary. McCracken and Ng (2016) used first differences of unemployment, which behave quite differently from the level. The purchasing managers composite index from the Institute of Supply Management (row 3) appears to be stationary, and McCracken and Ng (2016) entered this series directly into PCA without any transformation.

The top panel of Figure 2 plots the first principal component of the transformed series arrived at by McCracken and Ng (2016).[7] This inherits some of the high-frequency fluctuations seen in the (1,2) and (2,2) panels of Figure 1. Indeed, McCracken and Ng (2016) regarded this series as too volatile to reliably identify business cycles and turning points, and instead plotted in their Figure 3 the accumulation of this series. The CFNAI (shown in panel 2 of Figure 2) is very similar to the first principal component of the FRED-MD macro data set.

The third column of Figure 1 plots the cyclical components of industrial production, unem-

[7] We generated this figure using the exact data and code posted at https://research.stlouisfed.org/econ/mccracken/fred-databases/. Note that we have multiplied the series by -1 in order to give it the property that the factor declines in recessions, and that their Figure 3 plots the accumulations $(s_{1t} = \sum_{j=1}^{t} \tilde{f}_{1t})$ whereas our graph shows $\tilde{f}_{1t}$ itself.

Figure 2: First PC of FRED-MD variables as transformed by McCracken and Ng (2016), the Chicago Fed National Activity Index, and first PC of cyclical components of FRED-MD variables, 1962:3 to 2014:12



ployment, and PMI as estimated by the residuals of the OLS regression (7) with $h = 24$ and $p = 12$.[8] PMI is almost impossible to predict two years in advance, and our cyclical component is almost identical to the original series. Thus both our method and the traditional approach use this variable essentially as is. There is some but not much predictability of the unemployment rate at the two-year horizon, so for this variable our transformation much more closely resembles the original series than it does the first-difference transformation. For industrial production, our approach takes out the broad trend while retaining the essential cyclical behavior observed in the raw data. The three variables in the third column, unlike those in the second column, all share a common characterization of what is happening over the business cycle. Consistent with a long tradition in business cycle research, when plotted this way PMI appears as a leading indica-

[8]For those series that McCracken and Ng (2016) transformed using logs, first differences of logs, or second differences of logs (their transformations 4-6), we simply took the log of the variable before performing the regresssion. Thus for example the series plotted in the upper left panel of Figure 1 is 100 times the natural logarithm of the industrial production index. For those series that they used as is, as first differences, or second differences (their transformations 1-3), we simply used the variable as is. They employed a special transformation (7) for nonborrowed reserves. One would have expected to take logs of a variable like this, but the variable took on negative values in 2008. For this series their transformation was $y_{it} = \Delta(x_t/x_{t-1} - 1.0)$ and we used $y_{it} = x_t/x_{t-1}$.

tor, industrial production as a coincident indicator, and unemployment as a coincident or lagging indicator, with all three clearly following the same cycle.

The first principal component of the estimated cyclical components of the variables in the data set is plotted in the bottom panel of Figure 2. Unlike the CFNAI, this provides a very clean summary of historical business cycles. There is another interesting difference between the third panel and the first two. The NBER defines the business-cycle trough (the end point of the shaded regions) as the low point in the *level* of overall economic activity. For example, in the first month of a new expansion, the unemployment rate is still very high, but it has started to come down. Our series in the bottom panel of Figure 2 captures this feature very well, reaching a trough at exactly the point identified by the NBER Business Cycle Dating Committee. By contrast, the low point in the series plotted in the first two panels typically comes more towards the middle of the recession. This is because the *rate of decline* of real output (a common raw input in the variables as usually transformed) starts to ease well before the recession has ended. Both in terms of the cleanness of the series and its timing, we would suggest that our approach offers a better characterization of the state of the U.S. business cycle over this sample period.

6.1 Outliers

Previous users of large macro data sets devoted a lot of attention to outliers and implemented procedures to mitigate their influence. Prior to the COVID-19 recession of 2020, the CFNAI discarded observations that were more than six times the interquartile range, as did Stock and Watson (1999) in some of their analysis. McCracken and Ng (2016) discarded observations that were more than ten times the interquartile range. This criterion identifies 79 different observations on 22 different variables as outliers in the 1960:3 to 2014:12 data set; for details see Table B2.

This approach to identifying outliers requires one to know the form of the transformation that is needed to render each variable stationary. Can one accomplish the same task without knowing which variables are nonstationary or the form of the nonstationarity? If we knew the true cyclical component c_{it} we could easily identify outliers of c_{it} relative to its interquartile range. But outliers also exert undue influence on the estimated regression coefficients, tilting the coefficients so as to tend to make $\hat{c}_{it}$ closer to zero than the true value c_{it}. We therefore calculated residuals using leave-one-out regressions.[9] Outliers that exceed ten times the interquartile range for $h = 1$-month-ahead forecast errors are reported in the middle columns of Table B2. There is quite a bit of overlap between these outliers and those identified by conventional methods, though there are also some differences. For example, interest rate spreads in 1980 and some price measures in November 2008 register as outliers to the regression but not in the transformed series. Overall, the regression identifies 98 outliers in this data set compared to 79 identified by McCracken and Ng. If one were to use $h = 1$-month-ahead forecasting regressions to extract a stationary component, then we would recommend removing or downweighting outliers as identified using leave-one-out interquartile ranges.

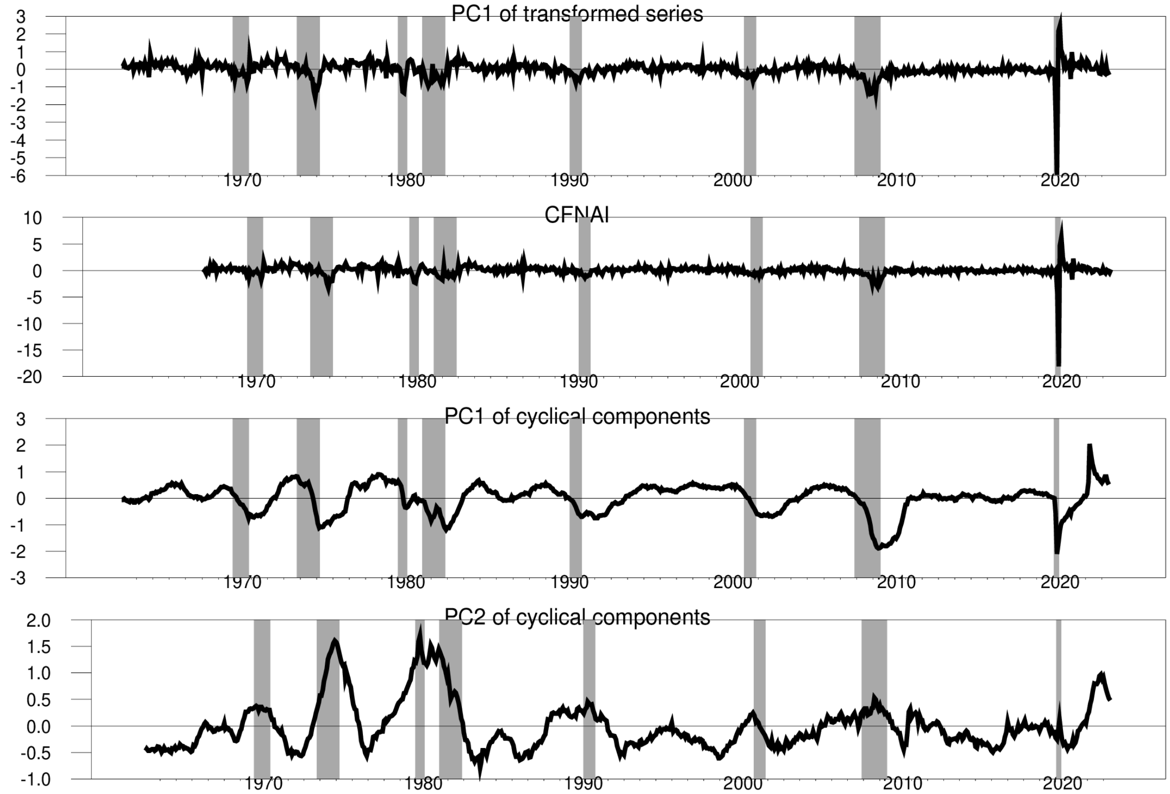
We noted in Section 2 that two-year-ahead forecast errors should be much less influenced by outliers than is the case for typical transformations like taking the monthly growth rate. In fact, we found outliers in the two-year-ahead regression residuals in only 2 of the 134 series, as reported in the last columns of Table B2. The behavior of total and nonborrowed reserves was certainly anomalous during the Federal Reserve's response to the Great Recession, but nothing else in this sample is a clear outlier by this criterion. Our recommended procedure is to use two-year-ahead regression residuals and make no corrections for outliers. The series that we have plotted in the bottom panel of Figure 2 is the unadjusted first principal component of the full set of OLS residuals $\hat{c}_{it}$.

Outliers are an even bigger issue when data for 2020 are included. For the 2023:11 vintage of FRED-MD, the McCracken-Ng procedure would identify 40 of the 127 variables as all being outliers in the single month of 2020:4. Despite dropping all of these 40 observations, the first principal component calculated using their algorithm shows an enormous decline in this month. Indeed, in order to include the 2020 observations in the top panel of Figure 3, the scale must be so large that it makes all the previous cyclical fluctuations barely noticeable. The CFNAI modified its procedure for dealing with anomalous observations to handle these observations. Even so the CFNAI still displays an unprecedented drop in 2020, as seen in the second panel.

By contrast, only two variables are identified as outliers for 2020:4 for purposes of our approach, these being new claims for unemployment insurance and the number unemployed for

[9] That is, we calculated $\tilde{c}_{it} = y_{it} - \tilde{\alpha}_{i.t}\tilde{z}_{i,t-h}$ with $\tilde{\alpha}_{i.t} = \left(\sum_{s=1,s\neq t}^{T} \tilde{z}_{i,s-h}\tilde{z}'_{i,s-h}\right)^{-1}\left(\sum_{s=1,s\neq t}^{T} \tilde{z}_{i,s-h}y_{is}\right)$ estimated separately for each i and t and then divided $\tilde{c}_{i.t}$ by its observed interquartile range.

Figure 3: First PC of FRED-MD variables as transformed by McCracken and Ng (2016), the Chicago Fed National Activity Index, and first and second PC of cyclical components of FRED-MD variables, 1962:3 to 2023:6



less than 5 weeks. The result of applying our procedure to the FRED-MD database up through June of 2023 with no corrections for outliers is displayed in the third panel of Figure 3. Note that, unlike the top two panels, our series describes the downturn in 2020 on a comparable scale as earlier recessions, although our series indicates that the speed of the downturn was unprecedented, as was the growth in the first two months of the recovery. Also in contrast to the first two panels, our series indicates (correctly, in our view) that the economy did not fully recover from the COVID-19 shock until September of 2021. The sharp spike up in our series in April of 2022 reflects the fact that most macro variables were substantially higher in April 2022 than one would have predicted based on observations in April 2020. This conclusion is also consistent with the aggressive actions of policy makers in the spring of 2020. Our series further indicates that economic activity remained unusually strong through the summer of 2023.

6.2 Missing observations

Another issue with large data sets comes from discontinued, newly added, or missing variables. McCracken and Ng (2016) adapted the Stock and Watson (2002) algorithm for unbalanced panels, though they found in their original data set that the results are essentially identical if one simply drops variables as needed to create a balanced panel. For our application, we have simply calculated principal components of $\hat{c}_{it}$ on a balanced panel, though there is no obstacle to applying the Stock and Watson (2002) algorithm to an unbalanced panel of $\hat{c}_{it}$.[10]

6.3 Uses of macroeconomic cyclical factors

A key use of PCA is to summarize the statistical information in a large cross section of indicators; for illustrations see Bernanke et al. (2005), Bai and Ng (2008), Forni et al. (2009), Bai and Ng (2010), and Stock and Watson (2016). The movement in variable i that is captured by the jth factor alone is given by $\hat{\lambda}_{ij}\hat{f}_{jt}$. Since $\hat{c}_{it}$ is normalized to have unit variance, the fraction of the variance of the stationary component of variable i that is explained by the jth macro factor is given by $\hat{\lambda}_{ij}^2 T^{-1}\sum_{t=1}^{T}\hat{f}_{jt}^2$.

Table B1 reports the R^2 explained by the first macroeconomic cyclical factor for each of the 120 variables used in our analysis of the 2023 vintage data set. The first factor alone accounts for almost 2/3 of the variance of typical indicators of real output or income and more than half of the variance of typical indicators of labor-market conditions. The first cyclical factor is far less successful at describing financial indicators and nominal prices. It is interesting that when we add the second cyclical factor, the R^2 for the median price indicator rises to 66%. The first factor thus seems mainly to capture real economic conditions and the second characterizes nominal prices and interest rates.

The fourth panel in Figure 3 plots the second cyclical factor. This by construction is orthogonal to the first, and often continues to fall even as the recovery in real economic activity is beginning.

[10] A balanced panel was created from the 127 variables in the 2023:11 dataset by: using only data over 1960:1-2023:6; dropping the Michigan Survey of Consumer Sentiment (UMCSENT), trade-weighted exchange rate (TWEXAFEGSMTH), and new orders for consumer goods (ACOGNO) and nondefense capital goods (ANDENO), which are the same four series dropped by McCracken and Ng to create a balanced panel from the 2015:4 dataset; dropping the VIX (VIXCLS), which was not included in the 2015:4 dataset and whose first value is July 1962; and dropping the financial commercial paper rate (CP3M) and the commercial paper-fed funds spread (COMPAPFF) which were not reported for April 2020. The particular variables used in our analysis of the 2023 vintage dataset are described in Table B1.

This is consistent with the view that nominal variables may respond sluggishly to business-cycle developments. It describes events beginning in 2022 as a third big U.S. inflation wave, though less dramatic than the big inflations of 1973-74 and 1979-81.

We next explore the use of the cyclical factors in forecasting. Stock and Watson (1999) demonstrated that the first principal component of a large data set of real macroeconomic variables could be very helpful for forecasting inflation. Their finding gave rise to the Chicago Fed National Activity Index (CFNAI), a PCA-based indicator that is still widely used today. We compare the usefulness for forecasting of the CFNAI (denoted $\hat{f}_t^{CF}$), the first or second principal component calculated using the algorithm and data set of McCracken and Ng (2016) (denoted $\hat{f}_t^{MN1}$ and $\hat{f}_t^{MN2}$, respectively), or the first or second principal component of the forecasting residuals (denoted $\hat{f}_t^{HX1}$ or $\hat{f}_t^{HX2}$) calculated from the McCracken-Ng data set using equation (12). Our approach to comparing different forecasts is similar to that used by Stock and Watson (1999) and McCracken and Ng (2016).

A particular model m uses a set of variables $\mathbf{x}_t^m$ that are observed at date t to try to forecast the value of a variable of interest y_{t+h}^h that will not be observed until $t + h$:

$$y_{t+h}^h = \boldsymbol{\pi}^{m\prime}\mathbf{x}_t^m + u_{t+h}^{m,h}. \tag{26}$$

We estimated the value of $\boldsymbol{\pi}^m$ by OLS regression on the subsample $t = T_0, T_0 + 1, ..., T_1 - h - 1$ and used these coefficients to forecast $y_{T_1}^h$. We then augmented the sample by one observation, estimating the regression over $t = T_0, T_0 + 1, ..., T_1 - h$ and using those coefficients to forecast $y_{T_1+1}^h$. We repeated this for an evaluation period T_1 to T_2 and calculated the average squared forecast error over this evaluation period. We performed this analysis using three different evaluation periods. The first evaluation period is specified by $T_1 = 1970{:}1$ to $T_2 = 1996{:}12$, which was the evaluation period in the original study by Stock and Watson (1999). The second evaluation period is $T_1 = 1997{:}1$ to $T_2 = 2014{:}12$, which corresponds to the new data used by McCracken and Ng (2016) that were not available to Stock and Watson (1999). The third evaluation period is $T_1 = 2015{:}1$ to $T_2 = 2023{:}6$, which is the new data available since publication of McCracken and Ng

(2016). The models we considered were a pure autoregressive model,

$$\mathbf{x}_t^{AR} = (1, y_t^1, y_{t-1}^1, ..., y_{t-5}^1)',$$

and models that add to the autoregressive model six lags of one of the principal components estimates. For example,

$$\mathbf{x}_t^{CF} = (\mathbf{x}_t^{AR\prime}, \hat{f}_t^{CF}, \hat{f}_{t-1}^{CF}, ..., \hat{f}_{t-5}^{CF})'.$$

This differs a little from the forecast evaluations performed by Stock and Watson (1999) and McCracken and Ng (2016) in that these authors used BIC to select different lag lengths for the autoregressive and principal components and for each subsample, whereas we set the lag length to six for every evaluation. Also, since $\hat{f}_t^{CF}$ is only available beginning in 1967:3, we used $T_0 = 1967{:}9$ as the first date for estimation of all models.

In every case, for each T we re-estimated the coefficients $\hat{\boldsymbol{\pi}}^m$ for the forecasting regression (26) using an expanding data set ending h periods before the variable being forecast. A separate question is the data set used to estimate the factors $\hat{f}_t$ themselves. Insofar as the factors are only identified up to sign, the meaning of coefficients multiplying $\hat{f}_t$ could change across expanding samples. To construct $\hat{f}_t^{MN}$ for the first two evaluation samples we followed McCracken and Ng (2016) in calculating the factor $\hat{f}_t^{MN}$ using the full historical vintage of the FRED-MD database available as of 2015:4.[11] For the third evaluation sample, we re-estimated $\hat{f}_t^{MN}$ using the full database available as of 2023:11. For the predicting variable $\hat{f}_t^{CF}$ we used the value of the CFNAI as it is currently reported for all historical dates.[12] For $\hat{f}_t^{HX}$, for all three samples we first detrended each individual variable using regressions estimated using the full 2015:4 vintage data set. For the first two evaluation samples, we calculated the principal component of $\hat{c}_{it}$ using the full 2015:4 vintage. For the third evaluation sample, we calculated the principal component of $\hat{c}_{it}$ using the full 2023:11 vintage.[13] Our approach for the third sample thus is not affected by the potential

[11]Our series for $\hat{f}_t^{MN1}$ for this subsample is almost (but not quite) identical to the series analyzed by McCracken and Ng (2016). We have been unable to identify the source of the small discrepancies.

[12]We downloaded $\hat{f}_t^{CF}$ on December 27, 2023 from the FRED database at https://fred.stlouisfed.org/series/CFNAI.

[13]That is, we estimated the coefficients for the forecasting regression (7) using the 2015:4 vintage data set. We used those coefficients to obtain $\hat{c}_{it}$ for all dates through 2023:6. We calculated $\hat{f}_t^{HX1}$ for the first two subsamples from the first principal component of $\hat{c}_{it}$ for $t = 1962{:}3$ to 2014:12. We calculated $\hat{f}_t^{HX1}$ for the third subsample from the first principal component of $\hat{c}_{it}$ for $t = 1962{:}3$ to 2023:6. We obtained similar results (not reported here) when the detrending regressions used to calculate $\hat{c}_{it}$ for the third subsample were estimated using the 2023:11 vintage data set.

concern that the detrending regression is using information that was unavailable at the time the forecast was made.

For our first set of evaluations we set y^h_{t+h} to be the average inflation rate between month t and $t+h$, quoted at an annual rate,

$$y^h_{t+h} = (1200/h)\log(CPI_{t+h}/CPI_t),$$

where CPI_t denotes the level of the consumer price index in month t.[14] The column labeled AR in Table 3 reports the simulated out-of-sample mean squared error of a purely autoregressive model for each of the three evaluation samples and for forecast horizons of $h = 1$, 6, or 12 months.[15] The first panel reproduces the finding of Stock and Watson (1999) that an index like CFNAI significantly improves forecasts for longer horizons over the 1970-1996 period. The alternative measures $\hat{f}^{MN1}_t$ or $\hat{f}^{HX1}_t$ offer similar improvements. The indexes offer little or no improvement for one-month-ahead forecasts over this period, but again are similar to each other. All three indexes are outperformed by simple autoregressive forecasts over either of the later two evaluation periods. The observation that inflation has become much harder to forecast in data since 1996 has been reported by a number of other researchers, including Atkeson and Ohanian (2001), Fisher et al. (2002), Stock and Watson (2007), and Stock and Watson (2008).[16] Interestingly, the second cyclical factor $\hat{f}^{HX2}_t$ does better than any of the other four indexes at forecasting inflation at the one-month horizon for the 1970-1996 sample and at the 6-12 month horizons for the 2015-2023 sample.

Table 3 also reports forecasts of industrial production, setting

$$y^h_{t+h} = (1200/h)\log(IP_{t+h}/IP_t),$$

for IP_t the level of the industrial production index in month t. The indexes $\hat{f}^{CF}_t$, $\hat{f}^{MN1}_t$, and $\hat{f}^{HX1}_t$

[14] Here again our evaluation design differs slightly from that in Stock and Watson (1999) and McCracken and Ng (2016) in that those authors took the object of interest to be to forecast the *change* in the inflation rate as a function of lagged changes:

$$y^h_{t+h} = (1200/h)\log(CPI_{t+h}/CPI_t) - 1200\log(CPI_t/CPI_{t-1}).$$

[15] Stock and Watson (1999) reported results for $h = 12$ months whereas McCracken and Ng (2016) reported results for $h = 1$, 6, and 12.

[16] The result is also revealed in the numbers reported in Table 5 of McCracken and Ng (2016) , though the authors did not comment on it.

Table 3: Mean squared forecast errors for different models

sample	horizon	Consumer Price Index					
		AR	CF	MN1	HX1	MN2	HX2
1970-1996	h=1	7.91	1.00	0.99	1.03	0.98	0.90
	h=6	4.26	0.81	0.77	0.80	0.95	0.88
	h=12	5.32	0.70	0.62	0.74	1.02	1.33
1997-2014	h=1	12.26	1.03	1.04	1.02	0.97	1.09
	h=6	6.08	1.23	1.23	1.23	0.96	1.11
	h=12	4.21	1.22	1.22	1.28	0.95	1.17
2015-2023	h=1	8.31	1.58	1.30	1.19	0.96	1.05
	h=6	3.98	2.03	1.52	1.81	1.04	1.02
	h=12	3.84	1.79	1.31	1.51	1.06	0.98

sample	horizon	Industrial Production					
		AR	CF	MN1	HX1	MN2	HX2
1970-1996	h=1	76.73	0.97	0.94	0.96	1.01	1.02
	h=6	38.66	0.92	0.93	0.83	0.71	0.79
	h=12	27.19	1.01	1.06	1.21	0.49	0.87
1997-2014	h=1	58.90	0.85	0.83	0.98	1.04	1.00
	h=6	22.61	0.93	0.94	1.05	1.24	1.12
	h=12	20.11	0.96	1.01	1.06	1.26	1.11
2015-2023	h=1	584.33	1.76	0.92	1.11	1.02	1.00
	h=6	100.90	2.33	1.16	1.04	1.06	0.94
	h=12	43.97	2.38	1.15	0.85	1.04	0.91

Notes to Table 3. AR column reports simulated out-of-sample mean squared forecast error for purely autoregressive model evaluated over three different out-of-sample periods. CF column reports the relative MSE when lags of the Chicago Fed National Activity Index are added to the autoregression, with a value less than one indicating the variable is useful for forecasting. MN1 column reports the relative MSE when lags of the first principal component calculated using the procedures in McCracken and Ng (2016) are used in place of the CFNAI. HX1 reports relative MSE when lags of the first principal component of the estimated cyclical components are used in place of CFNAI. MN2 and HX2 report results when the second principal component is used instead of the first.

all help forecast industrial production over near horizons in the first two evaluation periods. The CFNAI does particularly poorly at forecasting either inflation or industrial production at any horizon for 2015-2023. Both $\hat{f}_t^{HX1}$ and $\hat{f}_t^{HX2}$ do significantly better than CFNAI in every case over this period.

We conclude that our approach offers similar benefits to conventional PCA when evaluated in terms of simulated out-of-sample forecasts, and does much better than measures like the CFNAI for recent data. We share the conclusion of the earlier literature that the usefulness of any principal-component-based measure for purposes of forecasting depends on the variable, evaluation period, and horizon of the forecast.

7 Conclusion

Calculating principal components of medium-horizon forecast errors is a viable approach to identifying the common cyclical factors that drive a large collection of potentially nonstationary economic indicators. This avoids the need to decide how to detrend each individual series and is much more promising than approaches such as the Chicago Fed National Activity Index for handling data that include the large outliers of 2020.

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A Proofs

The proof of Theorem 2 makes use of the following lemma.

Lemma 1. *Let S be any symmetric positive semidefinite $(N \times N)$ matrix with diagonal elements $s_{ii}, i = 1, ..., N$. Then*

$$\sup_{\gamma \in \Gamma} \frac{\gamma' S \gamma}{\gamma' \gamma} \leq \textstyle\sum_{i=1}^{N} s_{ii}. \tag{A-1}$$

Proof of Lemma 1.

Notice that the left side of (A-1) is equal to the largest eigenvalue of S. Since S is positive semidefinite, all eigenvalues are nonnegative so the largest eigenvalue is less than or equal to the sum of all the eigenvalues. But the sum of all the eigenvalues is equal to the trace of S, which is defined as the sum of its diagonal elements. Thus the left side of (A-1) must be less than or equal to the right side.

Proof of Theorem 2(i)-(ii).

Notice from $\hat{C}_t = C_t + V_t$ that

$$\begin{aligned} & (N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} \hat{C}_t \hat{C}_t' \gamma \\ = \quad & (N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} C_t C_t' \gamma + (N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} V_t V_t' \gamma + 2(N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} C_t V_t' \gamma. \end{aligned} \tag{A-2}$$

We first show that the second and third terms on the right side of (A-2) converge in probability to 0 for all $\gamma \in \Gamma$. To show this for the second term, notice from Lemma 1 that

$$\begin{aligned} \sup_{\gamma \in \Gamma} (N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} V_t V_t' \gamma \quad &= \quad \sup_{\gamma \in \Gamma} (NT)^{-1} \frac{\gamma' \sum_{t=1}^{T} V_t V_t' \gamma}{\gamma' \gamma} \\ &\leq \quad (NT)^{-1} \textstyle\sum_{i=1}^{N} \sum_{t=1}^{T} v_{it}^2. \end{aligned} \tag{A-3}$$

Since $Ev_{it}^2 < \delta$ for all i and t, it follows that $E\left[(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} v_{it}^2\right] < \delta$ and thus $\sup_{\gamma \in \Gamma}(N^2T)^{-1}\gamma' \sum_{t=1}^{T} V_t V_t' \gamma \xrightarrow{p} 0$ by Markov's Inequality.

For the last term in (A-2),

$$\left|(N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} C_t V_t' \gamma\right| \leq \left[(N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} C_t C_t' \gamma\right]^{1/2} \left[(N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} V_t V_t' \gamma\right]^{1/2}.$$

The first term converges in probability to a number no larger than $\omega_{11}^{1/2}$ from Theorem 1, and the second converges in probability to 0 for all γ from (A-3). Thus for all $\gamma \in \Gamma$, $(N^2T)^{-1}\gamma' \sum_{t=1}^{T} C_t V_t' \gamma \xrightarrow{p} 0$. We thus conclude from (A-2) that

$$(N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} \hat{C}_t\hat{C}_t'\gamma - (N^2T)^{-1}\gamma' \sum_{t=1}^{T} C_t C_t'\gamma \xrightarrow{p} 0 \qquad \text{(A-4)}$$

for all $\gamma \in \Gamma$. Since

$$T^{-1}\textstyle\sum_{t=1}^{T} \hat{f}_{1t}^2 - T^{-1}\sum_{t=1}^{T} \tilde{f}_{it}^2 = \sup\limits_{\gamma\in\Gamma}(N^2T)^{-1}\gamma' \sum_{t=1}^{T} \hat{C}_t\hat{C}_t'\gamma - \sup\limits_{\gamma\in\Gamma}(N^2T)^{-1}\gamma' \sum_{t=1}^{T} C_t C_t\gamma,$$

it follows that this difference converges in probability to zero, establishing result (i) of Theorem 2 for $j = 1$. Analogous calculations establish results (i) and (ii) for $j = 2, ..., k$.

Proof of Theorem 2(iii).

Notice that $(N^2T)^{-1}\hat{\Lambda}' \sum_{t=1}^{T} \hat{C}_t\hat{C}_t'\hat{\Lambda}$ is a diagonal matrix for all N and T by the definition of $\hat{\Lambda}$ with diagonal elements converging in probability to ω_{jj} by result (i):

$$(N^2T)^{-1}\hat{\Lambda}' \textstyle\sum_{t=1}^{T} \hat{C}_t\hat{C}_t'\hat{\Lambda} \xrightarrow{p} \Omega_{FF}. \qquad \text{(A-5)}$$

Equation (A-4) then establishes that $(N^2T)^{-1}\hat{\Lambda}' \sum_{t=1}^{T} C_t C_t'\hat{\Lambda} \xrightarrow{p} \Omega_{FF}$. We also know from results (R2)-(R6) in Stock and Watson (2002) that

$$(N^2T)^{-1}\gamma' \textstyle\sum_{t=1}^{T} C_t C_t'\gamma - (N^2T)^{-1}\gamma' \sum_{t=1}^{T} \Lambda' F_t F_t' \Lambda\gamma \xrightarrow{p} 0$$

for all $\gamma \in \Gamma$, meaning

$$\begin{aligned}(N^2T)^{-1}\hat{\Lambda}' \textstyle\sum_{t=1}^{T} C_t C_t'\hat{\Lambda} \xrightarrow{p} \quad & (N^2T)^{-1}\hat{\Lambda}' \textstyle\sum_{t=1}^{T} \Lambda F_t F_t' \Lambda'\hat{\Lambda} \\ = \quad & (N^{-1}\hat{\Lambda}'\Lambda)\left(T^{-1}\textstyle\sum_{t=1}^{T} F_t F_t'\right)(N^{-1}\Lambda'\hat{\Lambda}) \\ & \xrightarrow{p} H\Omega_{FF}H' \end{aligned} \qquad \text{(A-6)}$$

for $H = \text{plim } (N^{-1}\hat{\Lambda}'\Lambda)$. Combining results (A-4)-(A-6),

$$\Omega_{FF} = H\Omega_{FF}H'. \tag{A-7}$$

Let $\hat{h}_j'$ denote the jth row of $\hat{\Lambda}'\Lambda/N$,

$$\underset{(1\times r)}{\hat{h}_j'} = \underset{(1\times N)}{\hat{\lambda}_j'} \underset{(N\times r)}{\Lambda} /N,$$

for $\hat{\lambda}_j'$ the jth row of $\hat{\Lambda}'$. Then

$$\hat{h}_j'\hat{h}_j = \frac{\hat{\lambda}_j'}{\sqrt{N}} \frac{\Lambda\Lambda'}{N} \frac{\hat{\lambda}_j}{\sqrt{N}}.$$

This is less than or equal to the largest eigenvalue of $\Lambda\Lambda'/N$, which converges to 1. Letting $h_j' = (h_{j1}, h_{j2}, ..., h_{jr})'$ denote the jth row of H, we thus have

$$\hat{h}_j'\hat{h}_j \xrightarrow{p} h_{j1}^2 + h_{j2}^2 + \cdots h_{jr}^2 \leq 1.$$

The (1,1) element of (A-7) states

$$h_1'\Omega_{FF}h_1 = h_{11}^2\omega_{11} + h_{12}^2\omega_{22} + \cdots + h_{1r}^2\omega_{rr} = \omega_{11}.$$

Since $\omega_{11} > \omega_{22} > \cdots > \omega_{rr} > 0$, this requires $h_{11}^2 = 1$ and $h_{12} = \cdots = h_{1r} = 0$. Thus the (1,1) element of $\hat{\Lambda}'\Lambda/N$ converges in probability to ± 1 and other elements of the first row converge to zero.

The (2,2) element of (A-7) states

$$h_{21}^2\omega_{11} + h_{22}^2\omega_{22} + \cdots + h_{2r}^2\omega_{rr} = \omega_{22} \tag{A-8}$$

where $h_{21} = \text{plim } \hat{\lambda}_2\lambda_1/N$. Regress λ_1 on $\hat{\lambda}_1$ with residual q_1:

$$\lambda_1 = \hat{k}_1\hat{\lambda}_1 + q_1 \tag{A-9}$$

$$\hat{k}_1 = (\hat{\lambda}_1'\hat{\lambda}_1/N)^{-1}(\hat{\lambda}_1'\lambda_1/N)$$

$$q_1'\hat{\lambda}_1 = 0$$

$$\lambda_1'\lambda_1/N = \hat{k}_1^2(\hat{\lambda}_1'\hat{\lambda}_1/N) + q_1'q_1/N.$$

We saw above that $\hat{k}_1^2 \xrightarrow{p} 1$, which along with $\lambda_1'\lambda_1/N \to 1$ and $\hat{\lambda}_1'\hat{\lambda}_1/N = 1$ establishes $q_1'q_1/N \xrightarrow{p} 0$. Premultiply (A-9) by $\hat{\lambda}_2'/N$:

$$\hat{\lambda}_2'\lambda_1/N = \hat{k}_1\hat{\lambda}_2'\hat{\lambda}_1/N + \hat{\lambda}_2'q_1/N = \hat{\lambda}_2'q_1/N.$$

But from Cauchy-Schwarz

$$(\hat{\lambda}_2'q_1/N)^2 \leq (\hat{\lambda}_2'\hat{\lambda}_2/N)(q_1'q_1/N) \xrightarrow{p} 0.$$

Thus $\hat{\lambda}_2'\lambda_1/N \xrightarrow{p} h_{21} = 0$ and (A-8) becomes

$$h_{22}^2\omega_{22} + h_{23}^2\omega_{33} + \cdots + h_{2r}^2\omega_{rr} = \omega_{22}.$$

Since $\omega_{22} > \omega_{33} > \cdots > \omega_{rr}$ and $h_{22}^2 + h_{23}^2 + \cdots + h_{2r}^2 \leq 1$, this requires $h_{22}^2 = 1$ and all other elements of the second row of H to be zero, establishing the second row of the claim in Theorem 2(*iii*). Proceeding iteratively through rows 3,4,..,r establishes the rest of the result in (*iii*).

Proof of Theorem 2(iv).

Write

$$\begin{aligned}
\hat{S}\hat{F}_t - F_t &= N^{-1}\hat{S}\hat{\Lambda}'\hat{C}_t - F_t \\
&= N^{-1}\hat{S}\hat{\Lambda}'(\Lambda F_t + e_t + V_t) - F_t \\
&= (N^{-1}\hat{S}\hat{\Lambda}'\Lambda - I_r)F_t + N^{-1}\hat{S}\hat{\Lambda}'e_t + N^{-1}\hat{S}\hat{\Lambda}'V_t. \qquad \text{(A-10)}
\end{aligned}$$

The task is to show that all three terms in (A-10) have plim 0. That $(N^{-1}\hat{S}\hat{\Lambda}'\Lambda - I_r)F_t \xrightarrow{p} 0$ follows immediately from result (*iii*). For the second term,

$$N^{-1}\hat{S}\hat{\Lambda}'e_t = N^{-1}(\hat{S}\hat{\Lambda}' - \Lambda')e_t + N^{-1}\hat{S}\Lambda'e_t. \qquad \text{(A-11)}$$

Consider the square of the jth element of the first term in (A-11):

$$\left[\frac{(\hat{s}_j\hat{\lambda}_j' - \lambda_j')e_t}{N}\right]^2 \leq \left[\frac{(\hat{s}_j\hat{\lambda}_j' - \lambda_j')(\hat{s}_j\hat{\lambda}_j - \lambda_j)}{N}\right]\left[\frac{e_t'e_t}{N}\right]. \tag{A-12}$$

The first term in (A-12) is

$$\frac{(\hat{s}_j\hat{\lambda}_j' - \lambda_j')(\hat{s}_j\hat{\lambda}_j - \lambda_j)}{N} = \frac{\hat{s}_j^2\hat{\lambda}_j'\hat{\lambda}_j}{N} - \frac{\lambda_j'\hat{s}_j\hat{\lambda}_j}{N} - \frac{\hat{s}_j\hat{\lambda}_j'\lambda_j}{N} + \frac{\lambda_j'\lambda_j}{N},$$

which converges in probability to zero by Theorem 2(*iii*). The second term in (A-12) is $O_p(1)$, by result (R1) in Stock and Watson (2002), meaning the plim of (A-12) is zero. The second term in (A-11) also converges in probability to zero as in Stock and Watson (2002) Result (R15). Hence $N^{-1}\hat{S}\hat{\Lambda}'e_t \xrightarrow{p} 0$.

For the third term in (A-10), $N^{-1}\hat{S}\hat{\Lambda}'V_t$, note that the jth element is $N^{-1}\hat{s}_j\hat{\lambda}_j'V_t$ whose square is

$$N^{-2}\hat{\lambda}_j'V_tV_t'\hat{\lambda}_j \leq N^{-1}\textstyle\sum_{i=1}^N v_{it}^2 \xrightarrow{p} 0 \tag{A-13}$$

with the inequality following from Lemma 1 and the convergence in probability from Assumption 3 and Markov's Inequality.

Proof of Theorem 3.

We first demonstrate that Assumptions 4(i)-(iii) imply $E\left(T^{-1}\sum_{t=1}^T v_{it}^2\right) < \delta$ for all $T > T_\delta$. Note from (16) $U_{iT} = \sum_{t=1}^T v_{it}^2$ can be written as the sum of squares of the fitted values from a regression of c_{it} on $z_{i,t-h}$, which by construction it must be smaller than the sum of squares of c_{it} itself:

$$U_{iT} < \textstyle\sum_{t=1}^T c_{it}^2 < T\kappa_1.$$

We therefore know

$$\begin{aligned} E[T^{-1}U_{iT}] &= E[T^{-1}U_{iT}\mathbf{1}\{T^{-1}U_{iT} \leq \delta/2\}] + E[T^{-1}U_{iT}\mathbf{1}\{\delta/2 \leq T^{-1}U_{iT} \leq \kappa_1\}] \\ &< (\delta/2) + \kappa_1 \text{Prob}(T^{-1}U_{iT} \geq \delta/2). \end{aligned}$$

The condition $E[T^{-1}\sum_{t=1}^T v_{it}^2] < \delta$ will then follow if we can show that for any $\delta > 0$ $\exists T_\delta$ such that $\text{Prob}(T^{-1}U_{iT} \geq \delta/2) \leq \delta/(2\kappa_1)$ whenever $T > T_\delta$. Given any $\delta > 0$, let $\kappa_3 = 4\kappa_1\kappa_2/\delta$ and $\varepsilon_3 = \delta/(4\kappa_1)$. Then from Assumption 4(iii), $\exists T_3(\kappa_3, \varepsilon_3)$ (call this $T_3(\delta)$) such that $\forall T \geq T_3(\delta)$,

$$\text{Prob}(U_{iT} \geq \kappa_3) < \text{Prob}(U_i \geq \kappa_3) + \varepsilon_3 = \text{Prob}(U_i \geq \kappa_3) + \delta/(4\kappa_1).$$

From Markov's Inequality and the definition of κ_3 this means

$$\begin{aligned} \text{Prob}\left(U_{iT} \geq \frac{4\kappa_1\kappa_2}{\delta}\right) &< \frac{E(U_i)}{\kappa_3} + \frac{\delta}{4\kappa_1} \\ &\leq \frac{\kappa_2}{(4\kappa_1\kappa_2/\delta)} + \frac{\delta}{4\kappa_1} \\ &= \frac{\delta}{2\kappa_1} \end{aligned} \tag{A-14}$$

with the second inequality coming from Assumption 4(ii). Let $T_\delta = \max\{T_3(\delta), 8\kappa_1\kappa_2/\delta^2\}$. For all $T \geq T_\delta$ we know from this definition of T_δ that

$$\begin{aligned} \text{Prob}(T^{-1}U_{iT} > \delta/2) &\leq \text{Prob}(T_\delta^{-1}U_{iT} > \delta/2) \\ &= \text{Prob}(U_{iT} > \delta T_\delta/2) \\ &\leq \text{Prob}(U_{iT} > 4\kappa_1\kappa_2/\delta). \end{aligned} \tag{A-15}$$

Putting (A-14) together with (A-15) establishes that for all $T > T_\delta$, $\text{Prob}(T^{-1}U_{iT} \geq \delta/2) \leq \delta/(2\kappa_1)$ which was to be shown.

We next show that $E\left(T^{-1}\sum_{t=1}^T v_{it}^2\right) < \delta$ along with Assumption 4(iv) imply that Assumption 3 holds. Notice that

$$v_{it}^2 = T^{-1}\bar{\alpha}_i' A_{it}\bar{\alpha}_i \leq T^{-1}(\bar{\alpha}_i'\bar{\alpha}_i)\lambda^{\max}(A_{it})$$

$$\textstyle\sum_{t=1}^T v_{it}^2 = \bar{\alpha}_i' B_{iT}\bar{\alpha}_i \geq (\bar{\alpha}_i'\bar{\alpha}_i)\lambda^{\min}(B_{iT})$$

$$\frac{v_{it}^2}{T^{-1}\sum_{t=1}^T v_{it}^2} \leq \frac{\lambda^{\max}(A_{it})}{\lambda^{\min}(B_{iT})}$$

$$
\begin{aligned}
E(v_{it}^2) &\leq E\left[\frac{\lambda^{\max}(A_{it})}{\lambda^{\min}(B_{iT})} T^{-1} \textstyle\sum_{t=1}^{T} v_{it}^2\right] \\
&\leq \left(E\left[\frac{\lambda^{\max}(A_{it})}{\lambda^{\min}(B_{iT})}\right]^2\right)^{1/2} \left(E\left[T^{-1} \textstyle\sum_{t=1}^{T} v_{it}^2\right]^2\right)^{1/2} \qquad \text{(A-16)}
\end{aligned}
$$

by Cauchy-Schwarz. The first term in (A-16) is less than $\sqrt{\kappa_4}$ by Assumption 4(iv). For the second term, $T^{-1} \sum_{t=1}^{T} v_{it}^2 < \kappa_1$ so $E\left[T^{-1} \sum_{t=1}^{T} v_{it}^2\right]^2 < \kappa_1 E\left[T^{-1} \sum_{t=1}^{T} v_{it}^2\right]$. For any $\delta > 0$, let $\delta_4 = \delta/(\kappa_1 \sqrt{\kappa_4})$. There exists a T_4 such that $E(T^{-1} \sum_{t=1}^{T} v_{it}^2) < \delta_4$ for $T > T_4$ and all i, meaning $E(v_{it}^2) < \delta$. Since this is uniform in i, it follows that $E\left[N^{-1} \sum_{1=1}^{N} v_{it}^2\right] < \delta$, which was to be shown.

B Data appendix

Table B1: R^2 for each variable explained by first and second principal components, 1962:3 to 2023:6

Group 1. Output and income

Index	FRED	Description	PC1	PC1&2
1	RPI	Real Personal Income	0.23	0.40
2	W875RX1	Real personal income ex transfer receipts	0.61	0.75
6	INDPRO	IP Index	0.77	0.85
7	IPFPNSS	IP: Final Products and Nonindustrial Supplies	0.81	0.88
8	IPFINAL	IP: Final Products (Market Group)	0.78	0.82
9	IPCONGD	IP: Consumer Goods	0.52	0.83
10	IPDCONGD	IP: Durable Consumer Goods	0.49	0.82
11	IPNCONGD	IP: Nondurable Consumer Goods	0.41	0.55
12	IPBUSEQ	IP: Business Equipment	0.71	0.71
13	IPMAT	IP: Materials	0.65	0.73
14	IPDMAT	IP: Durable Materials	0.64	0.75
15	IPNMAT	IP: Nondurable Materials	0.62	0.69
16	IPMANSICS	IP: Manufacturing (SIC)	0.77	0.87
17	IPB51222S	IP: Residential Utilities	0.02	0.03
18	IPFUELS	IP: Fuels	0.07	0.07
19	CUMFNS	Capacity Utilization: Manufacturing	0.68	0.73
		Median	0.63	0.74

Table B1 (continued)

Group 2. Labor market

Index	FRED	Description	PC1	PC1&2
20	HWI	Help-Wanted Index for United States	0.55	0.56
21	HWIURATIO	Ratio of Help Wanted/No. Unemployed	0.55	0.55
22	CLF16OV	Civilian Labor Force	0.25	0.34
23	CE16OV	Civilian Employment	0.75	0.75
24	UNRATE	Civilian Unemployment Rate	0.69	0.71
25	UEMPMEAN	Average Duration of Unemployment (Weeks)	0.24	0.24
26	UEMPLT5	Civilians Unemployed - Less Than 5 Weeks	0.38	0.46
27	UEMP5TO14	Civilians Unemployed for 5-14 Weeks	0.65	0.68
28	UEMP15OV	Civilians Unemployed - 15 Weeks and Over	0.66	0.66
29	UEMP15T26	Civilians Unemployed for 15-26 Weeks	0.65	0.66
30	UEMP27OV	Civilians Unemployed for 27 Weeks and Over	0.59	0.59
31	CLAIMSx	Initial Claims	0.42	0.45
32	PAYEMS	All Employees: Total nonfarm	0.81	0.81
33	USGOOD	All Employees: Goods-Producing Industries	0.85	0.85
34	CES1021000001	All Employees: Mining and Logging: Mining	0.03	0.41
35	USCONS	All Employees: Construction	0.67	0.74
36	MANEMP	All Employees: Manufacturing	0.74	0.74
37	DMANEMP	All Employees: Durable goods	0.77	0.77
38	NDMANEMP	All Employees: Nondurable goods	0.52	0.53
39	SRVPRD	All Employees: Service-Providing Industries	0.67	0.68
40	USTPU	All Employees: Trade, Transportation and Utilities	0.80	0.80
41	USWTRADE	All Employees: Wholesale Trade	0.74	0.80
42	USTRADE	All Employees: Retail Trade	0.67	0.68
43	USFIRE	All Employees: Financial Activities	0.42	0.43
44	USGOVT	All Employees: Government	0.08	0.09
45	CES0600000007	Avg Weekly Hours : Goods-Producing	0.33	0.48
46	AWOTMAN	Avg Weekly Overtime Hours : Manufacturing	0.38	0.62
47	AWHMAN	Avg Weekly Hours : Manufacturing	0.31	0.53
115	CES0600000008	Avg Hourly Earnings : Goods-Producing	0.04	0.42
116	CES2000000008	Avg Hourly Earnings : Construction	0.00	0.34
117	CES3000000008	Avg Hourly Earnings : Manufacturing	0.02	0.33
		Median	0.55	0.59

Group 3. Housing

Index	FRED	Description	PC1	PC1&2
48	HOUST	Housing Starts: Total New Privately Owned	0.14	0.37
49	HOUSTNE	Housing Starts, Northeast	0.17	0.39
50	HOUSTMW	Housing Starts, Midwest	0.11	0.42
51	HOUSTS	Housing Starts, South	0.12	0.28
52	HOUSTW	Housing Starts, West	0.11	0.26
53	PERMIT	New Private Housing Permits (SAAR)	0.11	0.35
54	PERMITNE	New Private Housing Permits, Northeast (SAAR)	0.13	0.43
55	PERMITMW	New Private Housing Permits, Midwest (SAAR)	0.10	0.47
56	PERMITS	New Private Housing Permits, South (SAAR)	0.08	0.30
57	PERMITW	New Private Housing Permits, West (SAAR)	0.09	0.24
		Median	0.11	0.36

Table B1 (continued)

Group 4. Consumption, orders, and inventories

Index	FRED	Description	PC1	PC1&2
3	DPCERA3M086SBEA	Real personal consumption expenditures	0.54	0.77
4	CMRMTSPLx	Real Manu. and Trade Industries Sales	0.73	0.89
5	RETAILx	Retail and Food Services Sales	0.46	0.50
58	AMDMNOx	New Orders for Durable Goods	0.69	0.69
59	AMDMUOx	Unfilled Orders for Durable Goods	0.26	0.40
60	BUSINVx	Total Business Inventories	0.27	0.75
61	ISRATIOx	Total Business: Inventories to Sales Ratio	0.22	0.27
		Median	0.46	0.69

Group 5. Money and credit

Index	FRED	Description	PC1	PC1&2
62	M1SL	M1 Money Stock	0.02	0.02
63	M2SL	M2 Money Stock	0.03	0.05
64	M2REAL	Real M2 Money Stock	0.03	0.45
65	BOGMBASE	Monetary Base	0.20	0.20
66	TOTRESNS	Total Reserves of Depository Institutions	0.28	0.28
67	NONBORRES	Reserves Of Depository Institutions	0.00	0.00
68	BUSLOANS	Commercial and Industrial Loans	0.10	0.17
69	REALLN	Real Estate Loans at All Commercial Banks	0.23	0.23
70	NONREVSL	Total Nonrevolving Credit	0.29	0.30
71	CONSPI	Nonrevolving consumer credit to Personal Income	0.09	0.17
118	DTCOLNVHFNM	Consumer Motor Vehicle Loans Outstanding	0.03	0.03
119	DTCTHFNM	Total Consumer Loans and Leases Outstanding	0.21	0.22
120	INVEST	Securities in Bank Credit at All Commercial Banks	0.02	0.04
		Median	0.09	0.17

Group 6. Interest and exchange rates

Index	FRED	Description	PC1	PC1&2
76	FEDFUNDS	Effective Federal Funds Rate	0.34	0.68
77	TB3MS	3-Month Treasury Bill:	0.37	0.69
78	TB6MS	6-Month Treasury Bill:	0.38	0.71
79	GS1	1-Year Treasury Rate	0.36	0.71
80	GS5	5-Year Treasury Rate	0.16	0.63
81	GS10	10-Year Treasury Rate	0.08	0.59
82	AAA	Moody’s Seasoned Aaa Corporate Bond Yield	0.02	0.60
83	BAA	Moody’s Seasoned Baa Corporate Bond Yield	0.00	0.61
84	TB3SMFFM	3-Month Treasury C Minus FEDFUNDS	0.06	0.31
85	TB6SMFFM	6-Month Treasury C Minus FEDFUNDS	0.04	0.27
86	T1YFFM	1-Year Treasury C Minus FEDFUNDS	0.00	0.16
87	T5YFFM	5-Year Treasury C Minus FEDFUNDS	0.12	0.34
88	T10YFFM	10-Year Treasury C Minus FEDFUNDS	0.21	0.41
89	AAAFFM	Moody’s Aaa Corporate Bond Minus FEDFUNDS	0.33	0.49
90	BAAFFM	Moody’s Baa Corporate Bond Minus FEDFUNDS	0.40	0.49
91	EXSZUSx	Switzerland / U.S. Foreign Exchange Rate	0.00	0.01
92	EXJPUSx	Japan / U.S. Foreign Exchange Rate	0.00	0.03
93	EXUSUKx	U.S. / U.K. Foreign Exchange Rate	0.07	0.08
94	EXCAUSx	Canada / U.S. Foreign Exchange Rate	0.00	0.04
		Median	0.08	0.49

Table B1 (concluded)

Group 7. Prices

Index	FRED	Description	PC1	PC1&2
95	WPSFD49207	PPI: Finished Goods	0.07	0.74
96	WPSFD49502	PPI: Finished Consumer Goods	0.07	0.72
97	WPSID61	PPI: Intermediate Materials	0.06	0.65
98	WPSID62	PPI: Crude Materials	0.11	0.44
99	OILPRICEx	Crude Oil, spliced WTI and Cushing	0.02	0.50
100	PPICMM	PPI: Metals and metal products:	0.17	0.36
101	CPIAUCSL	CPI : All Items	0.09	0.82
102	CPIAPPSL	CPI : Apparel	0.04	0.40
103	CPITRNSL	CPI : Transportation	0.04	0.56
104	CPIMEDSL	CPI : Medical Care	0.11	0.41
105	CUSR0000SAC	CPI : Commodities	0.08	0.73
106	CUSR0000SAD	CPI : Durables	0.00	0.33
107	CUSR0000SAS	CPI : Services	0.01	0.67
108	CPIULFSL	CPI : All Items Less Food	0.04	0.77
109	CUSR0000SA0L2	CPI : All items less shelter	0.07	0.78
110	CUSR0000SA0L5	CPI : All items less medical care	0.10	0.82
111	PCEPI	Personal Cons. Expend.: Chain Index	0.08	0.76
112	DDURRG3M086SBEA	Personal Cons. Exp: Durable goods	0.00	0.37
113	DNDGRG3M086SBEA	Personal Cons. Exp: Nondurable goods	0.06	0.78
114	DSERRG3M086SBEA	Personal Cons. Exp: Services	0.04	0.66
		Median	0.06	0.66

Group 8. Stock market

Index	FRED	Description	PC1	PC1&2
72	S&P 500	S&P's Common Stock Price Index: Composite	0.22	0.33
73	S&P: indust	S&P's Common Stock Price Index: Industrials	0.18	0.28
74	S&P div yield	S&P's Composite Common Stock: Dividend Yield	0.05	0.40
75	S&P PE ratio	S&P's Composite Common Stock: Price-Earnings Ratio	0.11	0.44
		Median	0.15	0.36
		Overall median	0.19	0.50

Notes to Table B1. Index refers to the index number of the variable in our database. FRED refers to variable name in the FRED database. PC1 is the fraction of the variance of the cyclical component of that variable that is explained by the first principal component. PC1&2 is the fraction of the variance of the cyclical component of that variable that is explained by the first and second principal components combined.

Table B2: Outliers associated with McCracken-Ng algorithm, 1-month-ahead regressions, and 24-month-ahead regressions

			McKracken-Ng		**Regression (h=1)**		**Regression (h =24)**	
variable	id	description	no.	dates	no.	dates	no.	dates
RPI	1	real personal income	1	2013:1	1	2013:1	0	
W875RX1	2	RPI less transfers	1	2013:1	1	2013:1	0	
IPDMAT	14	durables industrial production	1	1959:12	0		0	
CES1021-000001	35	employment: mining	6	1971:10,1971:12, 1977:12,1978:4, 1981:4,1981:6	6	1971:10,1971:12 1977:12,1978:4, 1981:4,1981:6	0	
MANEMP	37	employment: manufacturing	0		2	1970:10,1970:12	0	
DMANEMP	38	employment: durables	0		4	1964:10,1964:11, 1970:10,1970:12	0	
USGOVT	45	employment: government	0		2	1960:3,2010:5	0	
BUSINV	68	total business inventories	1	1982:1	1	1982:1	0	
M1SL	70	M1 money stock	2	2001:10,2009:1	0		0	
AMBSL	73	St. Louis monetary base	9	2008:9,2008:10, 2008:12-2009:3, 2010:2,2010:3, 2011:2	8	2008:9-2008:11, 2009:2,2009:3, 2009:10,2010:3, 2011:2	0	
TOTRESNS	74	total reserves	3	2001:10,2008:9, 2008:11	4	2001:9,2001:10, 2008:9,2008:10,	22	2008:11-2010:8
NONBORRES	75	nonborrowed reserves	14	2001:10, 2008:1-2008:5, 2008:9, 2008:11-2009:5	18	2001:9, 2007:12-2009:4	22	2001:9, 2007:12-2008:4, 2008:9-2009:4, 2010;2,2010:3, 2010:5-2010:8, 2010:10,2010:12
REALLN	77	real estate loans	3	2006:10,2006:11, 2008:11	2	2006:10,2008:10	0	
NONREVSL	78	total nonrevolving credit	4	1977:1,1977:2, 2010:12,2011:1	2	1977:1,2010:12	0	

Table B2 (continued)

			McKracken-Ng		Regression (h=1)		Regression (h =24)	
variable	id	description	no.	dates	no.	dates	no.	dates
CONSPI	79	nonrevolving consumer credit	0		3	1977:1,2010:12, 2013:1	0	
FEDFUNDS	84	fed funds rate	8	1979:10,1980:3, 1980:5,1980:11, 1980:12,1981:2, 1981:5,1982:8	6	1979:10,1980:3, 1980:5,1980:10, 1980:11,1981:5	0	
CP3M	85	3-month commercial paper rate	7	1980:3,1980:5, 1980:11,1980:12, 1981:5,1981:11, 1982:8	4	1980:3,1980:5, 1980:10,1981:5	0	
TB3MS	86	3-month Tbill rate	5	1980:3,1980:5, 1981:5,1981:11, 1982:8	4	1980:4,1980:5, 1981:5,1981:6	0	
TB6MS	87	6-month Tbill rate	2	1980:5,1981:11	2	1980:4,1980:5	0	
GS1	88	1-year Treasury rate	2	1980:5,1981:11	0		0	
COMPAPFF	93	com paper fed funds spread	0		2	1981:1,1981:6	0	
TB3SMFFM	94	3-month Tbill ff spread	0		3	1980:4,1981:2, 1981:6	0	
TB6SMFFM	95	6-month Tbill ff spread	0		4	1980:4,1980:5, 1980:12,1981:2	0	
T1YFFM	96	1-year Tbill ff spread	0		4	1980:4,1980:5, 1980:12,1981:2	0	
T5YFFM	97	5-year Treas ff spread	0		2	1980:5,1981:2	0	
T10YFFM	98	10-year Treas fed funds spread	0		1	1980:5	0	
AAAFFM	99	Aaa corporate fed funds spread	0		3	1980:5,1980:11, 1981:2	0	

Table B2 (concluded)

			McKracken-Ng		**Regression (h=1)**		**Regression (h =24)**	
variable	id	description	no.	dates	no.	dates	no.	dates
BAAFFM	100	Baa corporate fed funds spread	0		2	1980:5,1980:11	0	
PPIITM	108	PPI intermediate materials	0		1	2008:11	0	
PPICRM	109	PPI crude materials	1	2001:2	0		0	
OILPRICE	110	crude oil price	2	1974:1,1974:2	1	1974:1	0	
CPITRNSL	115	CPI transportation	0		1	2008:11	0	
CUSR0000SAS	119	CPI services	0		1	1980:7	0	
DSERRG3M086SBEA	126	PCE consumption	1	2001:10	0		0	
MZMSL	131	MZM money stock	1	1983:1	1	1983:1	0	
DTCOLNVHFNM	132	motor vehicle loans	3	1977:12,2010:3, 2010:4	1	2010:3	0	
DTCTHFNM	133	consumer loans	2	2010:12,2011:1	2	2010:12,2011:1	0	
total			79		98		44	