Supply, Demand, and Specialized Production

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Abstract

This paper develops a growth model characterized by equilibrium unemployment and sustained monopoly power. The level of demand is a key factor in deviations from the steady-state growth path with a Keynesian-type spending multiplier despite the absence of any nominal rigidities. The key friction in the model is the technological requirement that production of certain goods requires a dedicated team of workers that takes time to train and assemble.

1 Introduction.

Many of us are persuaded that fluctuations in demand are a key driver of business cycles. Production of automobiles and construction of new homes appear to fall in a recession not because the items become more difficult to build, but instead because fewer people seem willing to buy them. Evidence supporting this conclusion comes from Mian and Sufi (2014), Michaillat and Saez (2015), and Auerbach, Gorodnichenko, and Murphy (2020), among many others.

A common understanding of the mechanism whereby a decrease in demand leads to lower output is based on a failure of prices to adjust sufficiently quickly to the change in demand. This paper contributes to an alternative literature that maintains that demand shocks can cause real fluctuations even when prices are perfectly flexible. Examples include Hamilton (1988), Murphy (2017), Angeletos (2018), Angeletos and Lian (2020), Auerbach, Gorodnichenko, and Murphy (2020), and Ilut and Saijo (2021). This paper is most closely related to the models in Murphy (2017) and Auerbach, Gorodnichenko, and Murphy (2020) who emphasize the role of excess capacity and near-zero marginal production costs. In their models, capacity is exogenous, whereas here, capacity and marginal cost are endogenously determined in a general equilibrium growth model.

The basic technological friction in this model that replaces the nominal friction in Keynesian models is the requirement that resources must be committed in advance in order to produce certain goods. In this paper, labor is the only factor of production, and production of some goods is only possible if a dedicated team of workers is assembled and trained in advance to make that particular good. Training a new team is costly, but if it is successful, the unit has a monopoly in producing the good, and chooses quantity and price to maximize profits subject to a maximum capacity that the team is capable of producing. If demand falls below capacity, profit-maximization calls for lowering both quantity and price. Prices do not fall more than this because it would mean lower profits. Since marginal production costs are zero, there is no market force to bid costs down further. And although the triggering event was a change in relative demand, there is no offsetting gain from higher relative demand for other goods. The reason is that the underutilized specialized workers cannot costlessly shift to producing something else.

A key state variable in this model is n_{1t}, which is the fraction of the population without a high-skill, high-paying job. The value of n_{1t} is determined endogenously as individuals evaluate the costs and benefits of trying to develop a new skill, but it is predetermined at date t as a result of the training requirement. A sufficiently large drop in demand for good j may induce team j to disband and try to develop a new skill, increasing $n_{1,t+1}$. But $n_{1,t+1}$ is also a factor in the demand for all goods, since unskilled workers have lower income on average than skilled workers. There is thus an effect reminiscent of a Keynesian spending multiplier in this model; lower demand for some goods can have a feedback effect that lowers demand for all goods.

The model is consistent with a number of observed features of business cycles. The first is an asymmetry: a decrease in demand can have a bigger effect on output than an increase of the same magnitude. Empirical evidence of such asymmetry was provided by Weise (1999) and Lo and Piger (2005), with Tobin (1972) and Ball and Mankiw (1994) attributing asymmetry to the mechanics of partial price adjustment. Here the asymmetry arises even with perfectly flexible prices. When demand falls below capacity, the profit-maximizing response is to lower both output and price, whereas an increase in demand above capacity leads only to a price increase. Second, the response of output to a demand shock is often found to be humped-shaped, with the maximum effect observed many months after the initial shock. Empirical support and alternative explanations for this finding were provided by Christiano, Eichenbaum and Evans (2005), Hamilton (2008), and Auclert, Rognlie and Straub (2020). In this model, a humped shape can result when a reduction in demand slows the rate of hiring of new skilled workers. As the number of unskilled increases over time, the demand pressures get amplified, and output will remain below the steady-state level even after the initial shock is completely gone. A third striking observation in the data is that the unemployment rate has been remarkably stable despite a century of economic growth and technological innovations. Martellini and Menzio (2020) noted the challenges in explaining this using standard search and matching models and proposed an alternative explanation. In this model, a stable unemployment rate in the face of long-term economic growth is an equilibrium implication of the fact that the opportunity cost and potential benefits of being unemployed along with the tax base that finances compensation paid to the unemployed all grow with the overall level of productivity.

The paper makes a number of contributions to the literature. It develops a unified model of growth and fluctuations in which demand and other variables contribute to short-run fluctuations while long-run growth is determined solely by increases in population and productivity. It shows how monopoly power can be sustained in a growing economy even as new goods are being introduced and some old goods are being discontinued every period. The model allows for considerable heterogeneity, yet both individual and aggregate outcomes can be calculated using only a handful of equations.

2 Demand for goods.

At time t the population consists of a continuum of individuals of measure N_t who each consume a discrete set $j \in \mathcal{J}_t$ of different goods. Goods are nonstorable, and there are no capital or financial markets, so that the budget constraint for individual i is

$$\sum_{j \in \mathcal{J}_t} P_{jt} q_{ijt} \leq y_{it} \tag{1}$$

where P_{jt} is the nominal price of good j, q_{ijt} is the quantity of good j consumed by individual i, and y_{it} the individual's nominal income.

Individual preferences. The objective of consumer i is to maximize

$$U_{it} = \sum_{j \in \mathcal{J}_t} \frac{-\gamma_{ijt}}{2} (\bar{q}_{ijt} - q_{ijt})^2 \tag{2}$$

subject to (1). Quadratic preferences have some advantages and some disadvantages relative to the more common assumption of isoelastic preferences. As emphasized by Murphy (2017), quadratic preferences imply that the elasticity of demand changes as we move along the demand curve, which is important for understanding how decisions of monopolist producers respond to changing conditions. A disadvantage of quadratic preferences is that $\bar{q}_{ijt}$ is a bliss point, a level of consumption that consumer i would never want to exceed. This would not be sensible in an economy in which q_{ijt} is growing but $\bar{q}_{ijt}$ is constant. Our approach is to model the preference parameters γ_{ijt} and $\bar{q}_{ijt}$ as changing along a steady-state growth path. We motivate this in terms of a second-order approximation to log preferences at point $\{q^0_{ijt},\ j \in \mathcal{J}_t\}$ along a steady-state growth path,

$$\begin{aligned} \sum_{j \in \mathcal{J}_t} \alpha_{jt} \log q_{ijt} &\simeq \sum_{j \in \mathcal{J}_t} \alpha_{jt} \left[\log q^0_{ijt} + \frac{1}{q^0_{ijt}} (q_{ijt} - q^0_{ijt}) - \frac{1}{2\left(q^0_{ijt}\right)^2} (q_{ijt} - q^0_{ijt})^2 \right] \\ &= \sum_{j \in \mathcal{J}_t} \left[\delta^0_{ijt} - \frac{\gamma_{ijt}}{2} (\bar{q}_{ijt} - q_{ijt})^2 \right] \end{aligned} \tag{3}$$

with $\sum_{j \in \mathcal{J}_t} \alpha_{jt} = 1$ and

$$\bar{q}_{ijt} = 2q^0_{ijt} \tag{4}$$

$$\gamma_{ijt} = \frac{\alpha_{jt}}{\left(q^0_{ijt}\right)^2}. \tag{5}$$

See the top panel of Figure 1.

The advantage of taking (2) rather than the left side of (3) to be the specification of preferences is that (2) allows the possibility that producers of j could be driven out of business if productivity or demand is too low. If the price P_{jt} becomes too high, a consumer with preferences (2) will choose $q_{ijt} = 0$, whereas the left side of (3) would imply that consumers always buy every good in equilibrium, willing to pay $P_{jt} \to \infty$ as $q_{ijt} \longrightarrow 0$. Allowing $\bar{q}_{ijt}$ and γ^{-1}_{ijt} to grow along the steady-state growth path as specified in (4)-(5) allows us to adapt the quadratic preferences in (2) into a growth economy that shares some of the convenient long-run properties that would be implied by log preferences.

Individual demand curves. The first-order conditions for an interior solution are

$$\gamma_{ijt}(\bar{q}_{ijt} - q_{ijt}) = \lambda_{it} P_{jt} \quad j \in \mathcal{J}_t \tag{6}$$

for λ_{it} the marginal utility of income. The individual demand curve (6) is plotted in the bottom panel of Figure 1. Demand is zero at a price above $\gamma_{ijt}\bar{q}_{ijt}/\lambda_{it}$. If we ignore the implications of a change in the individual price P_{jt} on λ_{it}, the elasticity of demand is

$$\varepsilon_{ijt} = \left| \frac{\partial q_{ijt}}{\partial P_{jt}} \frac{\gamma_{ijt}(\bar{q}_{ijt} - q_{ijt})/\lambda_{it}}{q_{ijt}} \right| = \frac{\bar{q}_{ijt} - q_{ijt}}{q_{ijt}}. \tag{7}$$

The elasticity equals 1 if $q_{ijt} = \bar{q}_{ijt}/2$. It is greater than 1 if $q_{ijt} < \bar{q}_{ijt}/2$ and less than 1 if $q_{ijt} > \bar{q}_{ijt}/2$. This is unlike log utility, for which elasticity would always equal 1.

Market-wide demand curves. Summing across all consumers i gives the market demand curve shown in the top panel of Figure 2:

$$P_{jt} = A_{jt} - B_{jt} Q_{jt}$$

Note we will be following the notational convention of using lower-case letters like q_{ijt} to refer to magnitudes for individual consumers i and upper case like Q_{jt} to refer to magnitudes for individual goods j. Here $A_{jt} = \bar{Q}_{jt}/\Lambda_{jt}$, $B_{jt} = 1/\Lambda_{jt}$, $\Lambda_{jt} = \int_0^{N_t} (\lambda_{it}/\gamma_{ijt}) di$ and

$$\bar{Q}_{jt} = \int_0^{N_t} \bar{q}_{ijt} di. \tag{8}$$

The marginal revenue for producers of good j is $MR_{jt} = A_{jt} - 2B_{jt}Q_{jt}$. The good-level elasticity has the same properties as the demand curves for individual consumers.

Expenditure shares. If the preference parameters are characterized by expressions (4)-(5) and q^0_{ijt} is the quantity of good j that individual i would consume along the steady-state growth path, then the magnitude α_{jt} in (5) turns out to be the expenditure share along the steady-state growth path, as shown in the following proposition.

Proposition 1. *Suppose that there is a set consisting of R_{kt} different individuals at date t (denoted $\mathcal{M}_{kt}$) who are all on the same steady-state income path:*

$$q^0_{ijt} = q^0_{kjt} \quad i \in \mathcal{M}_{kt}, \ j \in \mathcal{J}_t.$$

Suppose that in period t the average quantity of good j purchased by members of the group is equal to this common steady-state value:

$$\frac{\int_{i \in \mathcal{M}_{kt}} q_{ijt} di}{R_{kt}} = q^0_{kjt} \quad j \in \mathcal{J}_t. \tag{9}$$

Then in period t the group as a whole spends a fraction a_{jt} of its income on good j:

$$\int_{i\in\mathcal{M}_{kt}} P_{jt}q_{ijt}di = \alpha_{jt}\int_{i\in\mathcal{M}_{kt}} y_{it}di. \tag{10}$$

3 Production of specialized goods.

Good $j = 1$ can be produced by anyone without any training or coordination with others. By contrast, goods $j > 1$ are specialized in the sense that their production requires a dedicated team who work together to produce the good. Once the workers who form a team are assembled, they enjoy a monopoly in producing good j and base their production and pricing decisions on that monopoly power. Team j consists of a measure of N_{jt} workers and has total production capacity $X_{jt}N_{jt}$ where productivity per worker X_{jt} for the team evolves according to an exogenous process. At the time that its production and pricing decisions for period t are made, unit j takes X_{jt} and N_{jt} as given and chooses P_{jt} and Q_{jt} to maximize total profits $P_{jt}Q_{jt}$ subject to $P_{jt} = A_{jt} - B_{jt}Q_{jt}$, $P_{jt} \in [0, A_{jt}]$, and $Q_{jt} \leq X_{jt}N_{jt}$. The number of specialized goods is sufficiently large that unit j ignores the effect of its decisions on the price and output of other units, so that maximizing nominal profits is the same decision as maximizing real profits. The team's profit-maximizing strategy is to produce up to the point where marginal revenue equals zero if there is sufficient production capacity and to produce at production capacity if not:

$$Q_{jt} = \begin{cases} \bar{Q}_{jt}/2 & \text{if } X_{jt}N_{jt} \geq \bar{Q}_{jt}/2 \quad \text{[demand constrained]} \\ X_{jt}N_{jt} & \text{if } X_{jt}N_{jt} < \bar{Q}_{jt}/2 \quad \text{[supply constrained]} \end{cases}. \tag{11}$$

We will describe production of good j as demand constrained in the first instance and supply constrained in the second.

New hiring. In period t, unit j takes its total capacity $N_{jt}X_{jt}$ as given. We assume that the hiring decision for $N_{j,t+1}$ is based on the goal of maximizing expected revenue of the unit. This is not the same as maximizing expected revenue per current worker. We think of an observed firm as a collection of a large number of separate producing units, and the objective of the firm is to maximize total profit subject to the constraint that individuals are available to do the work at the offered terms. If instead we took the objective to be to maximize expected income of existing team members, that would add an additional friction to hiring in the model.

Let $N^*_{j,t+1}$ denote the level of employment that maximizes expected revenue:

$$N^*_{j,t+1}E_t(X_{j,t+1}) = E_t(\bar{Q}_{j,t+1}/2). \tag{12}$$

We assume that the technology does not allow the team to be productive if any current member leaves, so workers are not laid off even if $N^*_{j,t+1} < N_{jt}$. The number of positions offered to new employees who would begin working in $t+1$ is thus

$$O_{jt} = \max\{N^*_{j,t+1} - N_{jt}, 0\}. \tag{13}$$

4 Unskilled workers.

We will refer to an individual who is not part of a specialized team at time t as "unskilled." Unskilled workers can choose between 3 options.

Option 1: seek to join an existing specialized unit. To pursue this option, an individual trains and applies in period t for a position to produce good j beginning in period $t+1$. With probability π_{jt} the individual will be successful. Each individual takes π_{jt} as given, though in equilibrium π_{jt} will be determined by the number of people applying for the job and the number of openings available. An individual who pursues this option will receive nominal compensation C_t while unemployed, financed through a proportional tax levied on the income of specialized workers during period t.

Option 2: seek to create a new good. An individual who is trying to join a team that creates a new good also receives unemployment compensation C_t during period t and has a probability k_π of being successful. There is also a utility cost k_U of making an effort to create a new good. The parameters k_π and k_U are exogenously fixed parameters that summarize the importance of frictions in creating new goods. If $k_\pi \to 1$ and $k_U \to 0$, the monopoly power of specialized teams would not be sustained along the steady-state growth path.

Option 3: produce good 1. Good 1 is assumed to be produced in a nonspecialized sector in which anyone could work with no training or coordination with others. If individual i works in sector 1, s/he could produce x_{it} units of good 1. The productivity parameter x_{it} is distributed independently across workers and across time. A favorable productivity x_{it} for individual i at time t has no implications for that same individual's productivity at $t+1$. The nominal income of individual i during period t is given by

$$y_{it} = \begin{cases} P_{1t}x_{it} & \text{if produces good 1} \\ C_t & \text{if looks for a job} \end{cases}.$$

Objective of unskilled workers. Unskilled workers choose between the above three options, seeking to maximize

$$v_{it} = E_t \sum_{s=1}^{\infty} \beta^s \log y_{i,t+s} \tag{14}$$

where E_t denotes an expectation conditional on information available at date t and $0 < \beta < 1$ is a discount rate[1].

[1]This objective function can be movitated as follows. Maximization of a pure log utility function (the left-hand side of (3)) subject to the budget constraint (1) has the solution $P_{jt}q_{ijt} = \alpha_{jt}y_{it}$ which results in a

Let Y_{jt} be the after-tax nominal income of each individual who is part of specialized team j at date t,

$$Y_{jt} = (1-\tau)P_{jt}Q_{jt}/N_{jt},$$

for τ the tax rate. Let V_{jt} denotes the value of (14) for such an individual:

$$V_{jt} = \log Y_{jt} + \beta(1-k_{jt})E_tV_{j,t+1} + \beta k_{jt}E_tV_{1,t+1}. \tag{15}$$

Here k_{jt} is the probability that unit j will still be disbanded in $t+1$. If the good is discontinued, next period those individuals will be unskilled. Since productivity x_{it} is drawn independently over time, the expected lifetime utility in the event that the team is disbanded is $E_tV_{1,t+1}$, the same for all individuals.

If an unskilled individual successfully creates a new good, the expected lifetime utility is $E_tV^\sharp_{t+1}$, whose value will be described below. Thus the value of (14) for an unskilled individual at time t is

$$v_{it} = \begin{cases} \log(P_{1t}x_{it}) + \beta E_tV_{1,t+1} & \text{if produces good 1} \\ \log C_t + \beta\pi_{jt}E_tV_{j,t+1} + \beta(1-\pi_{jt})E_tV_{1,t+1} & \text{if applies to join existing unit } j \\ \log C_t - k_U + \beta k_\pi E_tV^\sharp_{t+1} + \beta(1-k_\pi)E_tV_{1,t+1} & \text{if tries to create a new good} \end{cases}. \tag{16}$$

Decisions of unskilled workers. Individual i chooses the most favorable of the options in (16). The optimal decision is characterized by a productivity threshold X^*_{1t} such that individual i chooses to produce good 1 if $x_{it} \geq X^*_{1t}$ and looks for something better otherwise. If some individuals choose to produce good 1 and others try to create new goods, then X^*_{1t} would be the level of productivity at which the marginal unskilled individual is indifferent between working or trying to create a new good:

$$\log(P_{1t}X^*_{1t}) + \beta E_tV_{1,t+1} = \log C_t - k_U + \beta k_\pi E_tV^\sharp_{t+1} + \beta(1-k_\pi)E_tV_{1,t+1}. \tag{17}$$

Expression (17) can equivalently be written

$$\log(P_{1t}X^*_{1t}) - \log C_t = -k_U + \beta k_\pi E_t\tilde{V}^\sharp_{t+1} \tag{18}$$

where $\tilde{V}^\sharp_t = V^\sharp_t - V_{1t}$ is the expected lifetime advantage of specializing in a newly created good relative to being nonspecialized. Alternatively, when there is an incentive to try to specialize

level of utility given by

$$\sum_{j\in\mathcal{J}_t}\alpha_{jt}\log\left(\frac{\alpha_{jt}y_{it}}{P_{jt}}\right) = \log y_{it} + \sum_{j\in\mathcal{J}_t}\alpha_{jt}\log\alpha_{jt} - \sum_{j\in\mathcal{J}_t}\alpha_{jt}\log P_{jt}.$$

in continuing good j, (16) would require

$$\log(P_{1t}X_{1t}^*) + \beta E_t V_{1,t+1} = \log C_t + \beta\pi_{jt}E_t V_{j,t+1} + \beta(1-\pi_{jt})E_t V_{1,t+1} \tag{19}$$

$$\log(P_{1t}X_{1t}^*) - \log C_t = \beta\pi_{jt}E_t\tilde{V}_{j,t+1} \tag{20}$$

for $\tilde{V}_{jt} = V_{jt} - V_{1t}$ the lifetime advantage of specializing in j. In a typical equilibrium in which some individuals try to create a new good while others seek to join existing unit j, both conditions (18) and (20) hold, requiring that in equilibrium π_{jt} must satisfy

$$\beta\pi_{jt}E_t\tilde{V}_{j,t+1} = -k_U + \beta k_\pi E_t\tilde{V}_{t+1}^\sharp. \tag{21}$$

From equations (16) and (17) it follows that the lifetime income of nonspecialized individual i is characterized by

$$v_{it} = \begin{cases} \log(P_{1t}x_{it}) + \beta E_t V_{1,t+1} & \text{if } x_{it} \geq X_{1t}^* \\ \log(P_{1t}X_{1t}^*) + \beta E_t V_{1,t+1} & \text{if } x_{it} < X_{1t}^* \end{cases}. \tag{22}$$

The expression $E_tV_{1,t+1}$ is the expected value for $v_{i,t+1}$ across individuals i. Since x_{it} is distributed independently across time, we can find the current value of V_{1t} by taking the expected value of (22) across all unskilled individuals i at time t:

$$V_{1t} = \log(P_{1t}\tilde{X}_{1t}) + \beta E_t V_{1,t+1} \tag{23}$$

$$\log\tilde{X}_{1t} = P(x_{it} \geq X_{1t}^*)E[\log(x_{it})|x_{it} \geq X_{1t}^*] + P(x_{it} < X_{1t}^*)\log X_{1t}^*. \tag{24}$$

Another object of interest is $\hat{X}_{1t}$, the average output of unskilled individuals:

$$\hat{X}_{1t} = E\left(x_{it}|x_{it} \geq X_{1t}^*\right)P(x_{it} \geq X_{1t}^*). \tag{25}$$

Note that this definition of $\hat{X}_{1t}$ means that if N_{1t} denotes the total number of unskilled individuals (including both those working and those unemployed), the total amount of good 1 that is produced is given by

$$Q_{1t} = N_{1t}\hat{X}_{1t}. \tag{26}$$

Distribution of productivity across unskilled workers. We will be using a parametric distribution for x_{it} for which simple closed-form expressions for the key magnitudes are easily obtained. The assumption is that $\log x_{it} \sim U(R_t, S_t)$ where the bounds (R_t, S_t) on the uniform distribution will grow over time at the same rate as productivity of specialized goods. Implications of the uniform distribution are summarized in the following proposition.

Proposition 2. *Suppose that the log of potential productivity for producing good 1 is distributed independently across individuals as* $\log x_{it} \sim U(R_t, S_t)$ *and let* $\log X_{1t}^* \in [R_t, S_t]$ *be* *the threshold level of productivity above which unskilled individuals choose to produce good 1 (that is, X^*_{1t} satisfies (17) or (19)). Then:*

(a) the fraction of unskilled individuals who are employed is

$$h_{1t} = P(x_{it} \geq X^*_{1t}) = \frac{S_t - \log X^*_{1t}}{S_t - R_t}; \tag{27}$$

(b) the expected flow-equivalent productivity of unskilled individuals (value of (24)) is

$$\log \tilde{X}_{1t} = \frac{S_t^2 - 2R_t \log X^*_{1t} + (\log X^*_{1t})^2}{2(S_t - R_t)} \tag{28}$$

which is monotonically increasing in X^*_{1t};

(c) the average output of unskilled individuals (expression (25)) is

$$\hat{X}_{1t} = \frac{\exp(S_t) - X^*_{1t}}{S_t - R_t}. \tag{29}$$

Moreover, if $R_{t+1} = R_t + g$, $S_{t+1} = S_t + g$, and $\log X^*_{1,t+1} = \log X^*_{1t} + g$, *then:*

(d) $h_{1,t+1} = h_{1t}$;

(e) $\log \tilde{X}_{1,t+1} = \log \tilde{X}_{1t} + g$;

(f) $\log \hat{X}_{1,t+1} = \log \hat{X}_{1t} + g$.

Results (d)-(f) of Proposition 2 give some invariance properties that will be helpful in establishing that the equilibrium unemployment rate is constant along the steady-state growth path.

5 Entry and exit of specialized goods.

The set of goods produced $\mathcal{J}_t$ is potentially different for each date t. In this section we first specify how the preference parameters adapt to these changes, and then provide further details on what happens upon entry or exit of a particular good.

Preferences in a changing world. Preferences over goods are determined by the parameters γ_{ijt} and $\bar{q}_{ijt}$ in the utility function (2). We specify following equation (4) that $\bar{q}_{ijt} = 2\chi_{jt} q^0_{ijt}$ where q^0_{ijt} is the consumption of good j along the steady-state growth path for individual i and χ_{jt} is a potential additional factor influencing demand for good j at time t, with $\chi_{jt} = 1$ along the steady-state growth path and $\chi_{jt} > 1$ capturing stronger than normal demand for good j. We likewise from (5) specify $\gamma_{ijt} = \xi_{jt}\alpha_{jt}/(q^0_{ijt})^2$ where ξ_{jt} is a slope-demand shock with $\xi_{jt} = 1$ along the steady-state path. The first question we discuss is how to treat the share parameter α_{jt} when new goods are being created and others are discontinued.

We want to maintain the interpretation of α_{jt} as a share parameter by imposing the normalization that $\sum_{j \in \mathcal{J}_t} \alpha_{jt} = 1$ for all t. We focus on an economy for which the share parameter for the nonspecialized good is constant over time: $\alpha_{1t} = \alpha_1$, so that in the absence of demand shifts, the share of income spent on specialized goods $1 - \alpha_1$ will be constant along the steady-state growth path. We accomplish this by specifying that when new specialized goods are created, they carve out some of the demand share that had gone to earlier specialized goods, leaving α_1 unchanged. We now describe the details of how this works.

Demand for newly created goods. Let $\mathcal{J}_{2t}^{\sharp}$ denote the set of goods that were produced for the first time in period t and $\alpha_t^{\sharp} = \sum_{j \in \mathcal{J}_{2t}^{\sharp}} \alpha_{jt}$ denote their combined demand share. Let $\mathcal{J}_{2t}^{\natural}$ denote the set of specialized goods that were produced in both $t-1$ and t and $\alpha_t^{\natural} = \sum_{j \in \mathcal{J}_{2t}^{\natural}} \alpha_{jt}$ their cumulative demand share. We require

$$\alpha_t^{\natural} + \alpha_t^{\sharp} = 1 - \alpha_1 \tag{30}$$

to be constant for all t. Let $n_t^{\sharp} = \sum_{j \in \mathcal{J}_{2t}^{\sharp}} n_{jt}$ denote the fraction of the population at date t who are producing newly created goods and $n_t^{\natural} = \sum_{j \in \mathcal{J}_{2t}^{\natural}} n_{jt}$ the fraction producing continuing specialized goods. Our assumption is that the more people $n_t^{\sharp}$ involved in the process of creating new goods, the more success they have in creating products that consumers want:

$$\alpha_t^{\sharp} = \left(\frac{n_t^{\sharp}}{n_t^{\sharp} + n_t^{\natural}} \right) (1 - \alpha_1) \tag{31}$$

Assumptions (30) and (31) imply that some of the success in creating new goods comes at the expense of demand for continuing goods,

$$\alpha_t^{\natural} = \left(\frac{n_t^{\natural}}{n_t^{\sharp} + n_t^{\natural}} \right) (1 - \alpha_1). \tag{32}$$

If $n_t^{\sharp}$ is higher than normal, continuing goods will see a lower demand share.

Discontinued goods. A good will be discontinued if the expected benefit to workers from retaining that specialization is less than they could anticipate by returning to the pool of unskilled workers:

$$\text{if } E_t V_{j,t+1} < E_t V_{1,t+1}, \text{ then } j \in \mathcal{J}_{2t}^{\flat}. \tag{33}$$

Let k_{Xt} denote the fraction of goods in t that are discontinued in $t+1$. If $J_{2t} = \sum_{j \in \mathcal{J}_{2t}} 1$ is the total number of specialized goods at time t, the fraction that continue into $t+1$ is given by $J_{2t}^{-1} \sum_{j \in \mathcal{J}_{2,t+}^{\natural}} 1 = (1 - k_{Xt})$. Along the steady-state growth path, $k_{Xt} = k_X$ will be constant.

A parametric example. It is sometimes useful also to follow the market for individual goods. For this purpose the following parametric example allows some simple closed-form expressions. Suppose that goods are one of k_J different types in terms of their expenditure share, where the fraction of $n_t^{\sharp}$ who succeed in creating type ℓ is determined by the parameter

a_ℓ with $a_1 + \cdots + a_{k_J} = 1$:

$$n_{jt} = a_{\ell_j} n_t^\sharp \quad \text{for } j \in \mathcal{J}_{2t}^\sharp \text{ and } \ell_j \in \{1, ..., k_J\}.$$

There are k_J different new goods created at each date, one of each type ℓ_j, which acquire demand shares

$$\alpha_{jt} = a_{\ell_j} \alpha_t^\sharp = \left(\frac{n_{jt}}{1 - n_{1t}} \right) (1 - \alpha_1) \quad \text{for } j \in \mathcal{J}_{2t}^\sharp \tag{34}$$

where $1 - n_{1t} = n_t^\sharp + n_t^\natural$ is the fraction of the population who are skilled. The same adjustment to the share parameter for continuing goods,

$$\alpha_{jt} = \left(\frac{n_{jt}}{1 - n_{1t}} \right) (1 - \alpha_1) \quad \text{for } j \in \mathcal{J}_{2t}^\natural, \tag{35}$$

turns out to ensure the aggregate adjustment required by (32). To see this, sum (35) over all $j \in \mathcal{J}_{2t}^\natural$,

$$\alpha_t^\natural = \sum\nolimits_{j \in \mathcal{J}_{2t}^\natural} \alpha_{jt} = \left(\frac{1 - \alpha_1}{1 - n_{1t}} \right) \sum\nolimits_{j \in \mathcal{J}_{2t}^\natural} n_{jt} = \left(\frac{1 - \alpha_1}{1 - n_{1t}} \right) n_t^\natural$$

as required by (32).

The parameter k_J determines the total number of specialized goods produced in any given period along the steady-state growth path. Along the steady-state path, $k_X J_{2t}$ goods are discontinued and k_J new goods are created each period. The number discontinued will equal the number created if

$$k_X J_{2t} = k_J \tag{36}$$

and the constant number of specialized goods along the steady-state growth path is given by $J_2^0 = k_J / k_X$.

6 Equilibrium unemployment and the creation of new goods.

Here we describe unemployment dynamics in the case when some workers create new goods while others apply for existing positions, so that (18) and (21) both hold. For a given value of n_{1t} (the fraction of the population who are unskilled), this involves finding $1 - h_{1t}$ (the fraction of unskilled workers who are unemployed), h_{0t} (the fraction of unemployed who try to create new goods), and h_{jt} (the fraction of unemployed who apply for openings in j), with $h_{0t} = 1 - \sum_{j \in \mathcal{J}_{2t}} h_{jt}$.

An individual worker takes π_{jt} as given. In equilibrium π_{jt} is determined by the number of people who apply for the positions $((1 - h_{1t}) h_{jt} n_{1t} N_t)$ relative to the number of openings

O_{jt} available. For $0 < O_{jt} < h_{jt}(1 - h_{1t})n_{1t}N_t$,[2]

$$\pi_{jt} = \frac{O_{jt}}{(1 - h_{1t})h_{jt}n_{1t}N_t}. \tag{37}$$

Given unemployment compensation C_t and the value of creating a new good $E_t\tilde{V}^{\sharp}_{t+1}$, equation (18) determines the productivity threshold X^*_{1t}, which from (27) tells us $(1 - h_{1t})$. From (21) we also know π_{jt} as a function of $E_t\tilde{V}^{\sharp}_{t+1}$ and $E_t\tilde{V}_{j,t+1}$. Figure 3 illustrates the equilibrium value of π_{jt} in the special case when $E_t\tilde{V}^{\sharp}_{t+1} = E_t\tilde{V}_{j,t+1}$, a special case that in fact turns out to characterize the steady-state growth path. When the advantage to specialization is a large value like $E_t\tilde{V}^{[1]}_{t+1}$, equation (21) requires π_{jt} to be large which from (37) means h_{jt} is small. A higher advantage to specialization induces more workers to create new goods rather than apply for existing positions.

A key determinant of the advantage to specialization is n_1 (see equation (42) below) – higher n_1 means a bigger value of $\tilde{V}$. From Figure 3, higher $\tilde{V}$ induces more workers to become specialized by creating new goods, which would bring future n_1 and $\tilde{V}$ down. The steady-state growth path is characterized by values of n_{1t}, h_{1t}, and $\tilde{V}_t$ that are constant, and thus a constant unemployment rate.

7 Steady-state growth path.

In this section we consider an economy in which productivity grows deterministically at rate g and population grows at rate n. The key endogenous variables whose values we calculate are X^*_{1t}, the level of productivity at which unskilled individuals decide to look for a better job, h_{0t}, the fraction of the unemployed who try to create new goods, and n_{jt}, the equilibrium fraction of the population producing j.

Unemployment compensation. Along the steady-state growth path, a constant fraction of spending goes to good 1:

$$\frac{\sum_{j\in\mathcal{J}_{2t}} P_{jt}Q_{jt}}{P_{1t}Q_{1t}} = \frac{1 - \alpha_1}{\alpha_1}. \tag{38}$$

The numerator on the left side of (38) is the tax base, and from (26), the denominator is $P_{1t}N_{1t}\hat{X}_{1t}$. With a total of $(1 - h_{1t})N_{1t}$ individuals collecting unemployment compensation, the compensation per individual is

$$C_t = \frac{\tau \sum_{j\in\mathcal{J}_{2t}} P_{jt}Q_{jt}}{(1 - h_{1t})N_{1t}} = \left[\frac{\tau(1 - \alpha_1)}{\alpha_1(1 - h_{1t})}\right] P_{1t}\hat{X}_{1t}. \tag{39}$$

[2]This differs from typical search models of labor frictions such as Kaas and Kircher (2015) in that here hiring units can post multiple vacancies at zero cost and the primary uncertainty facing applicants is whether they can successfully complete the training.

Advantage from specialization. Let Y_{st} denote the average after-tax income per person of skilled workers. Result (38) establishes that

$$Y_{st} = \frac{(1-\tau)\sum_{j\in\mathcal{J}_{2t}} P_{jt}Q_{jt}}{(1-n_{1t})N_t} = \left[\frac{(1-\tau)(1-\alpha_1)}{\alpha_1}\right]\left[\frac{P_{1t}N_{1t}\hat{X}_{1t}}{(1-n_{1t})N_t}\right].$$

It turns out that along the steady-state growth path, all skilled workers earn the same income: $Y_{jt} = Y_{st}\ \forall j \in \mathcal{J}_{2t}$. Let $\tilde{Y}_t$ be the ratio of Y_{st} to $P_{1t}\tilde{X}_{1t}$, the flow-equivalent income of unskilled in (24):

$$\tilde{Y}_t = \frac{Y_{st}}{P_{1t}\tilde{X}_{1t}} = \left[\frac{(1-\tau)(1-\alpha_1)n_{1t}}{\alpha_1(1-n_{1t})}\right]\frac{\hat{X}_{1t}}{\tilde{X}_{1t}}. \tag{40}$$

Along the steady-state growth path, n_{1t} and $\hat{X}_{1t}/\tilde{X}_{1t}$ are constant, meaning there is a constant proportional income advantage to specialization.

Setting $k_{jt} = k_X$ and $Y_{jt} = Y_{st}$ in (15) and subtracting (23) from the result gives

$$\tilde{V}_t = \log\tilde{Y}_t + \beta(1-k_X)E_t\tilde{V}_{t+1}. \tag{41}$$

Since $\tilde{Y}_t$ is constant along the steady-state growth path, the discounted life-time advantage is given by

$$\tilde{V} = \left[\frac{1}{1-\beta(1-k_X)}\right]\left\{\log\left[\frac{(1-\tau)(1-\alpha_1)n_{1t}}{\alpha_1(1-n_{1t})}\right] + \log\hat{X}_{1t} - \log\tilde{X}_{1t}\right\} \tag{42}$$

which is also constant along the steady-state growth path.

Creation of new goods. Let h_{Yt} denote the log difference between the income that the marginal unskilled individual could earn from producing good 1 and the income collected from unemployment compensation:

$$h_{Yt} = \log(P_{1t}X^*_{1t}) - \log C_t. \tag{43}$$

From expression (39) this is

$$h_{Yt} = -\log\left[\frac{\tau(1-\alpha_1)}{\alpha_1}\right] + \log(1-h_{1t}) + \log X^*_{1t} - \log\hat{X}_{1t}. \tag{44}$$

We can write the equilibrium condition for creation of new goods (18) as

$$h_{Yt} = -k_U + k_\pi\beta\tilde{V}. \tag{45}$$

Recall from Proposition 2 that h_{1t}, $\tilde{X}_{1t}$, and $\hat{X}_{1t}$ are known functions of X^*_{1t}. Substituting (44) and (42) into (45) gives an expression in two endogenous variables, which are the fraction of the population that is unskilled n_{1t} and the value of X^*_{1t} at which an unskilled individual would be indifferent between producing good 1 and seeking to create a new good. We show in the proof of Proposition 3 below that given any $n_{1t} \in (0, 1)$, there exists a unique $\log X_{1t}^* \in (R_t, S_t)$ for which (45) holds.

New hiring. Along the steady-state growth path, condition (12) holds exactly and all specialized goods will be on the knife edge between being supply- or demand-constrained:

$$Q_{jt} = N_{jt}X_{jt} = \bar{Q}_{jt}/2 \quad j \in \mathcal{J}_{2t}. \tag{46}$$

In each period t, a fraction k_X of producers period learn of a change in demand parameters coming in $t+1$ that leads the workers to disband the unit and return to the pool of unskilled. We will describe details of such a change in Section 9. Surviving units add new workers at the population growth rate n: $N_{j,t+1} - N_{jt} = (e^n - 1)N_{jt}$ for $j \in \mathcal{J}^{\natural}_{2,t+1}$. New openings with continuing units are thus

$$O_t = (1 - k_X)(e^n - 1)(1 - n_{1t})N_t. \tag{47}$$

Since each continuing unit offers the same lifetime advantage, the probability of successfully applying for one of these positions is the same across all continuing goods. With $(1 - h_{1t})(1 - h_{0t})n_{1t}N_t$ individuals applying for these positions, this probability of success from (37) is

$$\pi_t = \frac{(1 - k_X)(e^n - 1)(1 - n_{1t})}{(1 - h_{1t})(1 - h_{0t})n_{1t}}. \tag{48}$$

Individuals are indifferent between applying for existing jobs and trying to create new goods when (21) holds:

$$-k_U + k_\pi \tilde{V} = \pi_t \beta \tilde{V}. \tag{49}$$

Since $\tilde{V}$ and h_{1t} are known functions of X_{1t}^* and n_{1t}, expression (49) gives another restriction on the three endogenous variables X_{1t}^*, n_{1t}, and h_{0t}.

Changes in the number of skilled workers. Let $n_{1t} = N_{1t}/N_t$ denote the fraction of the population that is unskilled. The value of h_{0t} implies that $(1 - h_{1t})h_{0t}k_\pi n_{1t}N_t$ individuals will join newly created units in $t+1$ which would be added to the $(1-k_X)(1-n_{1t})e^n N_t$ workers at continuing units. The total number of unskilled at $t+1$, which could be written as $n_{1,t+1}e^n N_t$, would then consist of the total population at $t+1$ ($e^n N_t$) minus the total number of skilled individuals:

$$n_{1,t+1}e^n N_t = e^n N_t - (1 - h_{1t})h_{0t}k_\pi n_{1t}N_t - (1 - k_X)(1 - n_{1t})e^n N_t$$

$$n_{1,t+1} = n_{1t} + k_X(1 - n_{1t}) - e^{-n}h_{0t}(1 - h_{1t})k_\pi n_{1t}. \tag{50}$$

Thus the fraction of unskilled workers will be constant if

$$k_X(1 - n_{1t}) = e^{-n}h_{0t}k_\pi(1 - h_{1t})n_{1t}. \tag{51}$$

The steady-state growth path is characterized by a value of h_{0t}, n_{1t}, and X^*_{1t} in which the three equations (51), (49) and (45) all hold. The next proposition establishes that there is a unique solution to these three equations.

Proposition 3. *Let $h_{1t}, \tilde{X}_{1t}, \hat{X}_{1t}, h_{Yt}, \tilde{V}$ be the functions of X^*_{1t} and n_{1t} given in (27)-(29), (44), and (42). If $k_\pi, k_X, \alpha_1, \beta, \tau$ are all $\in (0,1)$ and n and k_U are both positive, there exists a unique value of $(X^{*0}_{1t}, n^0_{1t}, h^0_{0t})$ for which ((51), (49) and (45) simultaneously hold. At this solution, $\log X^*_{1t} \in (R_t, S_t)$ and the values of h_{Yt} and $\tilde{V}$ are positive.*

The fact that $\tilde{V}$ is positive means that individuals would prefer to be skilled if they could acquire skills at no cost. The barriers to becoming specialized (a probability less than one of being able to join an existing enterprise and a cost of trying to create a new one) require as compensation that $\tilde{V}$ be positive in equilibrium.

We are now in a position to characterize steady-state growth in this model.

Assumptions behind the steady-state growth path. Population grows at a fixed rate n starting from a value N_{t_0} at an initial date t_0. The initial productivity for producing good j can be any positive value X_{jt_0} and the productivity parameters governing the uniform distribution of productivity among unskilled workers are given by (R_{t_0}, S_{t_0}). Let $(X^{*0}_{1t_0}, n^0_{1t_0}, h^0_{0t_0})$ denote the values described in Proposition 3 and let n^0_1 and h^0_0 denote the latter two values, a notation in anticipation of the fact that they turn out to be constant along the steady-state growth path. In each period $t = t_0 + 1, t_0 + 2, ...$ unskilled individuals maximize (16) resulting in $n_{1t}h_{1t}N_t$ individuals producing good 1, $n_{1t}(1 - h_{1t})h_{0t}N_t$ trying to create new goods, and $n_{1t}(1 - h_{1t})(1 - h_{0t})N_t$ seeking positions with continuing units. Productivity grows at rate g, so $R_{t+1} = g + R_t$, $S_{t+1} = g + S_t$ for all t and $\log X_{j,t+1} = g + \log X_{jt}$ for $j \in \mathcal{J}^\natural_{2t}$, with log productivities for any good newly created at date t drawn from a $N(\log \tilde{X}_{t_0} + g(t - t_0), \sigma^2)$ distribution where $\log \tilde{X}_{t_0} = J^{-1}_{2t_0} \sum_{j \in \mathcal{J}_{2t_0}} \log X_{jt_0}$ is the average initial log productivity and $J_{2t_0} = \sum_{j \in \mathcal{J}_{2t_0}} 1$ is the initial number of specialized goods.

The following proposition characterizes the steady-state growth path for this economy.

Proposition 4. *If at initial date t_0 there are $1/k_X$ specialized goods of each type a_k (so that the initial total number of goods is $J_{2,t_0} = k_J/k_X$), the initial share of unskilled workers is $n_{1t_0} = n^0_1$, and the initial share of the population specialized in good j satisfies*

$$n_{jt_0} = \frac{\alpha_{jt_0}(1 - n^0_1)}{1 - \alpha_1} \qquad j \in \mathcal{J}_{2t_0}, \tag{52}$$

then for all $t \geq t_0$:

(a) the fraction of the population that is unskilled, the fraction of the unskilled who are employed, and fraction of the unskilled who try to create new goods are constant over time,

$$n_{1t} = n^0_1 \quad h_{1t} = h^0_1 \quad h_{0t} = h^0_0$$

(implying a constant population unemployment rate of $n_1^0(1-h_1^0)$) while the values $X_{1t}^*, \hat{X}_{1t}, \tilde{X}_{1t}$ all grow at rate g:

$$\log X_{1,t+1}^* = g + \log X_{1t}^* \quad \log \hat{X}_{1,t+1} = g + \log \hat{X}_{1t} \quad \log \tilde{X}_{1,t+1} = g + \log \tilde{X}_{1t};$$

(b) the number of specialized goods in production is constant: $J_{2t} = k_J/k_X$;

(c) the consumption of good j by every skilled worker at time t is given by

$$q_{sjt}^0 = \frac{(1-\alpha_1)(1-\tau)}{1-n_1^0} n_{jt} X_{jt} \quad j \in \mathcal{J}_t \tag{53}$$

where X_{1t} is defined to be $\hat{X}_{1t}$;

(d) the average consumption of good j by unskilled workers at time t is given by

$$q_{njt}^0 = \frac{[\alpha_1 + \tau(1-\alpha_1)]}{n_1^0} n_{jt} X_{jt} \quad j \in \mathcal{J}_t; \tag{54}$$

(e) the share of the population that produces good j and the demand share parameter α_{jt} remain constant as long as the good remains in production:

$$n_{jt} = n_j^0 \quad j \in \mathcal{J}_{2t}^{\natural}$$

$$\alpha_{jt} = \alpha_j \quad j \in \mathcal{J}_{2t}^{\natural};$$

(f) the relative price of good j at time t is given by

$$\frac{P_{jt}}{P_{1t}} = \frac{\alpha_j n_1^0 \hat{X}_{1t}}{\alpha_1 n_j^0 X_{jt}} \tag{55}$$

which is constant over time as long as the good continues to be produced;

(g) the total demand parameter for good j is given by

$$\frac{\bar{Q}_{jt}^0}{2} = [n_1^0 q_{njt}^0 + (1-n_1^0) q_{sjt}^0] N_t \quad j \in \mathcal{J}_t;$$

(h) the quantity of any good grows at rate $n+g$ for as long as it is produced:

$$\log Q_{j,t+1} = n + g + \log Q_{jt} \quad j \in \{\{1\} \cup \mathcal{J}_{2,t+1}^{\natural}\};$$

(i) at any given date t, all skilled workers earn the same income as each other and the log difference between their income and that of the average nonspecialist is a constant over time.

8 Dynamic adjustment.

In this section we consider dynamic adjustment in an economy in which the labor shares at date t may not be the steady-state values ($n_{jt} \neq n_j^0$) and in which the preference parameters $\bar{q}_{ijt}$ and γ_{ijt} need not be the values associated with the steady-state path (denoted q_{ijt}^0 and γ_{ijt}^0). We specify $\bar{q}_{ijt} = \chi_{jt} q_{ijt}^0$ and $\gamma_{ijt} = \xi_{jt}\gamma_{ijt}^0$ where along the steady-state growth path the demand-shift parameters equal one ($\chi_{jt} = \xi_{jt} = 1$), and without loss of generality[3] $\chi_{1t} = \xi_{1t} = 1$. And while productivity grows at the constant rate g along the steady-state path ($\log X_{j,t+1}^0 = g + \log X_{jt}^0$), here we allow $X_{jt} = \zeta_{jt} X_{jt}^0$ with $\zeta_{jt} = 1$ along the steady-state path.

Proposition 3 established that the steady-state value n_1^0 is uniquely determined by the fixed demand share α_1 of good 1 and the costs of creating new goods. Thus the steady-state total share of skilled workers $1 - n_1^0$ is uniquely determined. For some questions it is also useful to think about the steady-state share n_j^0 for a particular good j as long as it is produced. This parameter governs the growth in demand for good j along the steady-state path by nonspecialists,

$$q_{ijt}^0 = \frac{[\alpha_1 + \tau(1-\alpha_1)]}{n_1^0} n_j^0 X_{jt}^0 \quad \text{for } i \in \mathcal{M}_{nt}, \tag{56}$$

and in the analogous expression for specialists. If the economy is always on the steady-state growth path, the meaning of n_j^0 is unambiguous, since $n_{jt} = n_j^0$ for all t and so n_j^0 was determined by the value of n_{jt} at the date when the good was first created. We now specify that the effort to create a new good succeeds in instilling consumers with a long-run value n_j^0 for a newly produced good given by the value of n_{jt} at the date when the good was introduced: $n_j^0 = n_{jt}$ if $j \in \mathcal{J}_{2t}^\sharp$. This implies a steady-state demand share given by

$$\alpha_j^0 = n_j^0(1-\alpha_1)/(1-n_1^0), \tag{57}$$

which need not equal the value of α_{jt} for the date t at which the good was first created. In the examples below, after a good is created, the value of n_{jt} may subsequently deviate from the initial n_j^0, but if the good survives long enough, eventually the labor share and demand share will return to n_j^0 and α_j^0.

Proposition 5. *At any point off the steady-state growth path:*
(a)

$$\bar{Q}_{jt}/2 = \chi_{jt} H_t n_j^0 X_{jt}^0 N_t \quad \textit{for } j \in \mathcal{J}_t \tag{58}$$

$$H_t = \frac{n_{1t}[\alpha_1 + \tau(1-\alpha_1)}{n_1^0} + \frac{(1-n_{1t})(1-\alpha_1)(1-\tau)}{1-n_1^0} \tag{59}$$

$$\frac{\partial H_t}{\partial n_{1t}} = \frac{\alpha_1 + \tau(1-\alpha_1) - n_1^0}{n_1^0(1-n_1^0)} \equiv \lambda_H; \tag{60}$$

[3] A lower demand for good 1 could equivalently be expressed as $\xi_{jt} > 1 \, \forall j \in \mathcal{J}_{2t}$.

(b) at $n_{1t} = n_1^0$*,* $H_t = 1$*. Its derivative* λ_H *is negative whenever*

$$\alpha_1 + \tau(1 - \alpha_1) < n_1^0, \tag{61}$$

which typical parameter values guarantee.

(c) the quantity of good j *that is produced at date* t *is*

$$Q_{jt} = \begin{cases} n_{1t} N_t \hat{X}_{1t} & \text{for } j = 1 \\ \min\{\bar{Q}_{jt}/2, n_{jt} N_t \zeta_{jt} X_{jt}^0\} & \text{for } j \in \mathcal{J}_{2t} \end{cases};$$

(d) the relative price of good j *at date* t *is*

$$\frac{P_{jt}}{P_{1t}} = \left(\frac{P_j^0}{P_1^0}\right)^2 \left(\frac{\alpha_1}{\alpha_j^0}\right) \left(\frac{\alpha_{jt}}{\alpha_j^0}\right) \xi_{jt} \left[\frac{\bar{Q}_{jt} - Q_{jt}}{\bar{Q}_{1t} - Q_{1t}}\right] \quad \text{for } j \in \mathcal{J}_{2t}; \tag{62}$$

(e) the share parameter for good j *is characterized by*

$$\alpha_{jt} = \frac{n_{jt}(1 - n_1^0)}{n_j^0(1 - n_{1t})} \alpha_j^0 \quad \text{for } j \in \mathcal{J}_{2t}; \tag{63}$$

(f) if good j *continues into* $t+1$*, the number of individuals specializing in* j *at* $t+1$ *is given by*

$$N_{j,t+1} = \max\left\{\frac{\bar{Q}_{j,t+1}}{2X_{j,t+1}}, N_{jt}\right\} \quad \text{for } j \in \mathcal{J}_{2,t+1}^\natural \tag{64}$$

$$n_{t+1}^\natural = \sum\nolimits_{j \in \mathcal{J}_{2,t+1}^\natural} N_{j,t+1}/N_{t+1}$$

and if $N_{j,t+1} = \bar{Q}_{j,t+1}/(2X_{j,t+1})$ *then*

$$\frac{N_{j,t+1}}{N_{j,t+1}^0} = \frac{\chi_{j,t+1} H_{t+1}}{\zeta_{j,t+1}}; \tag{65}$$

(g) after-tax income per individual specialized in good j *(*$Y_{jt} = (1-\tau)P_{jt}Q_{jt}/N_{jt}$*) is characterized by*

$$\frac{Y_{jt}}{P_{1t}} = \frac{Y_t^0 \xi_{jt}(1 - n_1^0) Q_{jt}(\bar{Q}_{jt} - Q_{jt}) Q_{1t}^0}{(1 - n_{1t})(Q_{jt}^0)^2(\bar{Q}_{1t} - Q_{1t})} \quad \text{for } j \in \mathcal{J}_{2t} \tag{66}$$

where the value of Y_{jt}/P_{1t} *along the steady-state growth path is the same for all skilled workers and is given by*

$$Y_t^0 = \frac{(1-\tau)(1-\alpha_1) n_1^0}{\alpha_1(1 - n_1^0)} \hat{X}_{1t}^0; \tag{67}$$

(h) compensation per unemployed individual is

$$C_t = \frac{\tau \sum_{j \in \mathcal{J}_{2t}} n_{jt} Y_{jt}}{n_{1t}(1 - h_{1t})}; \tag{68}$$

(i) the lifetime advantage of being skilled in good j relative to being unskilled is

$$\tilde{V}_{jt} = \log Y_{jt} - \log(P_{1t}\tilde{X}_{1t}) + \beta(1-k_X)\tilde{V}_{j,t+1} \quad \textit{for } j \in \mathcal{J}_{2t} \tag{69}$$

with good j endogenously discontinued after period t ($j \in \mathcal{J}_t^{\flat}$) if $\tilde{V}_{j,t+1} < 0$;

(j) if some individuals spend period t trying to create a new good, then

$$\log(P_{1t}X_{1t}^*) - \log C_t = -k_U + k_\pi \beta \tilde{V}_{j,t+1} \quad \textit{for } j \in \mathcal{J}_{2,t+1}^{\sharp}; \tag{70}$$

(k) if a positive fraction h_{jt} of unemployed workers seek to specialize in continuing good j, the fraction π_{jt} who are successful is characterized by

$$\pi_{jt} = \frac{N_{j,t+1} - N_{jt}}{(1-h_{1t})h_{jt}n_{1t}N_t}$$

$$\log(P_{1t}X_{1t}^*) - \log C_t = \pi_{jt}\beta\tilde{V}_{j,t+1} \tag{71}$$

$$h_{0t} = 1 - \sum\nolimits_{j \in \mathcal{J}_{2,t+1}^{\natural}} h_{jt}$$

for $j \in \mathcal{J}_{2,t+1}^{\natural}$ and h_{0t} the fraction seeking to create new goods;

(l) the fraction of the population in $t+1$ producing newly created goods is

$$n_{t+1}^{\sharp} = e^{-n}(1-h_{1t})h_{0t}n_{1t}k_\pi \tag{72}$$

and the fraction of the population that is unskilled is given by

$$n_{1,t+1} = 1 - n_{t+1}^{\sharp} - n_{t+1}^{\natural}. \tag{73}$$

(m) Define real GDP to be the ratio of current production evaluated at steady-state prices to steady-state production evaluated at steady-state prices:

$$Q_t = \frac{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}}{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0}. \tag{74}$$

This can equivalently be written as

$$Q_t = \sum\nolimits_{j\in\mathcal{J}_t} \alpha_j^0 (Q_{jt}/Q_{jt}^0) = \left(\frac{1-\alpha_1}{1-n_1^0}\right)\sum\nolimits_{j\in\mathcal{J}_{2t}}\left(\frac{Q_{jt}}{N_t X_{jt}^0}\right) + \left(\frac{\alpha_1}{n_1^0}\right)\left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right) n_{1t}. \tag{75}$$

In the special case when all goods are demand-constrained and $\chi_{jt} = \zeta_{jt} = 1 \; \forall j$, this becomes

$$Q_t = \frac{H_t(1-\alpha_1)}{1-n_1^0}\sum\nolimits_{j\in\mathcal{J}_{2t}} n_j^0 + \left(\frac{\alpha_1}{n_1^0}\right)\left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right) n_{1t}. \tag{76}$$

Discussion of Proposition 5. Result (a) establishes that an increase in n_{1t} results in lower total demand provided that $\alpha_1+\tau(1-\alpha_1) < n_1^0$. In interpreting this inequality, note that α_1 is the steady-state fraction of income that goes to unskilled individuals as a result of production of good 1 and $\tau(1-\alpha_1)$ is the fraction collected as unemployment compensation. If the sum of these is less than than n_1^0, the fraction of the population that is unskilled, then the average after-tax income of an unskilled individual along the steady-state path is less than that of someone who is skilled. This is all that is needed to conclude that $\lambda_H < 0$. This condition is almost guaranteed by Proposition 3, which established that $\tilde{V}^0 > 0$, meaning that the log income of skilled workers exceeds the expected log income of unskilled along the steady-state growth path. However, because of Jensen's Inequality, this is not quite enough to conclude that skilled income exceeds the expected income of the unskilled, which is the condition required by $\alpha_1 + \tau(1-\alpha_1) < n_1^0$. For most parameter values, Jensen's Inequality is not big enough to reverse the typical outcome. Appendix C provides sufficient conditions under which λ_H is necessarily negative. When this is the case, $\bar{Q}_{jt}$ is lower when the fraction of unskilled individuals is higher.

Note that steady-state real income Y_t^0 in (67) grows at the rate of productivity growth g common to $\hat{X}_{1t}^0$ and X_{jt}^0. However, the level of income for producers of good j does not depend on the particular productivity X_{jt}^0 (which determines the number of units that each worker produces). Income instead depends on α_j, the share of income consumers spend on j relative to n_j^0, the number of people producing good j. Along the steady-state growth path, $\alpha_j/n_j^0 = (1-\alpha_1)/(1-n_1^0) > 1$ is the same value for all goods j.

Note also that we defined real GDP in (74) as the ratio of current to steady-state output. Thus $Q_t > 1$ means a value of real GDP higher than steady state and $Q_t < 1$ means a value lower than steady state. As an example of using this result, consider the special case when there are no demand or productivity shocks ($\chi_{jt} = \zeta_{jt} = 1$), all goods are demand constrained ($Q_{jt} = \bar{Q}_{jt}/2$), and the population share of each good is the steady-state value ($n_{jt} = n_j^0$). In this case (76) becomes

$$Q_t = H_t \left(\frac{1-\alpha_1}{1-n_1^0}\right)(1-n_{1t}) + \left(\frac{\alpha_1}{n_1^0}\right)\left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right) n_{1t} \tag{77}$$

Note that $(1-\alpha_1)/(1-n_1^0)$, is greater than 1 and (α_1/n_1^0) is less than 1. Thus when $H_t = (\hat{X}_{1t}/\hat{X}_{1t}^0) = 1$,

$$\frac{\partial Q_t}{\partial n_{1t}} = \frac{\alpha_1}{n_1^0} - \frac{1-\alpha_1}{1-n_1^0} < 0.$$

Thus even if H_t and $(\hat{X}_{1t}/\hat{X}_{1t_0})$ were unity, a higher fraction of unskilled workers would mean lower GDP because fewer of the goods that consumers value are being produced. When $n_{1t} > n_1^0$, both $(\hat{X}_{1t}/\hat{X}_{1t}^0) < 1$ because when more individuals are unskilled, a higher fraction of them look for jobs,[4] and also $H_t < 1$ due to lower demand. Both these are additional factors pushing real GDP below 1 when $n_{1t} > n_1^0$.

Deviations from steady state. In exploring adjustment dynamics we will use linearization to approximate the values of variables off the steady-state growth path. Let $w_t^\dagger$ denote the deviation of the variable w_t or its log from the value on the steady-state growth path; specifically, $w_t^\dagger = \log w_t - \log w_t^0$ for $w_t = Q_{jt}, X_{1t}^*, \chi_{jt}, \xi_{jt}, \zeta_{jt}$, $w_t^\dagger = w_t - w^0$ for $w_t = n_{jt}$, $Y_{jt}^\dagger = \log(Y_{jt}/P_{1t}) - \log Y_t^0$, and $P_{jt}^\dagger = \log(P_{jt}/P_{1t}) - \log(P_{jt}^0/P_{1t}^0)$. Appendix B shows that for any specialized good j,

$$Q_{jt}^\dagger = \begin{cases} \chi_{jt}^\dagger + \lambda_H n_{1t}^\dagger & \text{if } \bar{Q}_{jt}/2 \le n_{jt} N_t X_{jt} \\ \zeta_{jt}^\dagger + \frac{n_{jt}^\dagger}{n_j^0} & \text{if } \bar{Q}_{jt}/2 > n_{jt} N_t X_{jt} \end{cases} \tag{78}$$

$$P_{jt}^\dagger = \xi_{jt}^\dagger + 2\chi_{jt}^\dagger + \frac{n_{jt}^\dagger}{n_j^0} + \frac{1}{1-n_1^0} n_{1t}^\dagger - Q_{jt}^\dagger + \frac{n_{1t}^\dagger}{n_1^0} + \lambda_5 X_{1t}^{*\dagger} \tag{79}$$

$$Y_{jt}^\dagger = \xi_{jt}^\dagger + 2\chi_{jt}^\dagger + \frac{1}{n_1^0(1-n_1^0)} n_{1t}^\dagger + \lambda_5 X_{1t}^{*\dagger}. \tag{80}$$

A demand shock arising from $\chi_{jt}^\dagger > 0$ shifts both the demand and marginal revenue curves out (see the bottom panel of Figure 2). If the good is demand constrained, the result is an increase in both output and price. If the good is supply constrained, only the price increases. By contrast, a slope demand shock $\xi_{jt}^\dagger > 0$ tilts the demand curve without changing $\bar{Q}_{jt}$, and only results in an increase in price regardless of the regime. A productivity shock $\zeta_{jt}^\dagger$ has no effect on income because it either has no effect on output and price or has offsetting effects on output and price; see Section 10 for more discussion.

Result (80) means that if two different goods j and k both experience the same proportionate demand shocks ($\chi_{jt}^\dagger = \chi_{kt}^\dagger$ and $\xi_{jt}^\dagger = \xi_{kt}^\dagger$), producers of the two goods will receive the identical income ($Y_{jt}^\dagger = Y_{kt}^\dagger$). This greatly simplifies analysis of the economy off the steady-state growth path in the examples below.

Adjustment dynamics in the absence of demand and productivity shocks. The following sections investigate economies in which demand or productivity shocks may influence the values of variables over periods $t_0, t_0+1, ..., t_0+M-1$, but all exogenous variables revert to their steady-state values after t_0+M. In every example, the initial conditions and subsequent sequence of shocks are all known with certainty from period $t = t_0$ on and

$$\chi_{jt} = \xi_{jt} = \zeta_{jt} = 1 \quad j \in \mathcal{J}_{2t} \text{ and } t \ge t_0 + M.$$

In this subsection we characterize dynamics for these examples beginning in period $t = t_0 + M$.

[4] If $n_{1t} > n_1^0$, then $X_{1t}^* > X_{1t}^{*0}$ and $\hat{X}_{1t} < \hat{X}_{1t}^0$.

Note from (80) that when $\xi_{jt} = \chi_{jt} = 1$, real income differs from the steady-state value Y_t^0 by the same multiple for every skilled worker. Let Y_t denote this common level of income in period t:

$$Y_{jt}/P_{1t} = Y_t \quad j \in \mathcal{J}_{2t} \text{ and } t \geq t_0 + M.$$

Looking forward from $t \geq t_0 + M$, all skilled workers face the same stream of future income prospects and thus there is a single lifetime benefit of specialization $\tilde{V}_t$ that is common to all skilled workers at t. The two key dynamic equations for $t \geq t_0 + M$ are then (73) and the common value for (69),

$$n_{1,t+1} = 1 - n_{t+1}^{\sharp} - n_{t+1}^{\natural} \tag{81}$$

$$\tilde{V}_t = \log(Y_t/P_{1t}) - \log \tilde{X}_{1t} + \beta(1-k_X)\tilde{V}_{t+1} \tag{82}$$

where $n_{t+1}^{\sharp}$ is the fraction of the population at $t+1$ producing newly created goods and $n_{t+1}^{\natural}$ the fraction producing continuing specialized goods.

Another state variable that helps simplify calculations is $\bar{n}_t$, which is defined as the sum of the steady-state employment shares n_j^0 of all specialized goods that are produced at t: $\bar{n}_t = \sum_{j \in \mathcal{J}_{2t}} n_j^0$. After $t_0 + M$, a fraction $(1 - k_X)$ of goods in t survive to $t+1$, and the value of n_j^0 for these goods at $t+1$ is by definition is the same as in t. In addition, in our model the steady-state share of newly produced goods is the population share when they were first introduced: $n_j^0 = n_{jt}$ for $j \in \mathcal{J}_{2t}^{\sharp}$. Thus the equation of motion for $\bar{n}_t$ is

$$\bar{n}_{t+1} = (1 - k_X)\bar{n}_t + n_{t+1}^{\sharp} \quad t \geq t_0 + M. \tag{83}$$

When $\chi_{jt} = 1$, equation (58) implies that $\bar{Q}_{jt}/2 = H_t Q_{jt}^0$. The examples below all have the property that all goods have capacity to produce the profit-maximizing level $Q_{jt} = \bar{Q}_{jt}/2$ after $t \geq t_0 + M$. Recalling that $Q_{1t} = n_{1t} N_t \hat{X}_{1t}$, the level of real income common to all skilled workers is found from (66) to be

$$Y_t/P_{1t} = \frac{Y_t^0(1 - n_1^0)H_t^2 Q_{1t}^0}{(1 - n_{1t})(2H_t Q_{1t}^0 - n_{1t} N_t \hat{X}_{1t})} \quad t \geq t_0 + M. \tag{84}$$

Real unemployment compensation from (68) is

$$C_t/P_{1t} = \frac{\tau(1 - n_{1t})Y_t/P_{1t}}{n_{1t}(1 - h_{1t})} \quad t \geq t_0 + M. \tag{85}$$

The productivity threshold X_{1t}^* at which unskilled workers choose to produce good 1 is determined by (70):

$$\log X_{1t}^* - \log(C_t/P_{1t}) = -k_U + \beta k_\pi \tilde{V}_{t+1} \quad t \geq t_0 + M. \tag{86}$$

Since each specialization offers the same income, in equilibrium there is a common probability

π_t of succeeding in specializing in a continuing good, which from (71) is characterized by

$$\log X_{1t}^* - \log(C_t/P_{1t}) = \beta\pi_t\tilde{V}_{t+1}. \tag{87}$$

The number of openings with continuing goods is given by

$$O_t = n_{t+1}^{\natural}e^n N_t - (1-k_X)(1-n_{1t})N_t. \tag{88}$$

With $(1-h_{1t})(1-h_{0t})n_{1t}N_t$ individuals applying for these positions, the probability of success is

$$\pi_t = \frac{n_{t+1}^{\natural}e^n - (1-k_X)(1-n_{1t})}{(1-h_{1t})(1-h_{0t})n_{1t}} \quad t \geq t_0 + M. \tag{89}$$

We can solve this expression for h_{0t},

$$h_{0t} = 1 - \left[\frac{n_{t+1}^{\natural}e^n - (1-k_X)(1-n_{1t})}{(1-h_{1t})n_{1t}\pi_t}\right], \tag{90}$$

and substitute the result into (72) to find the fraction of the population at $t+1$ who are newly skilled:

$$n_{t+1}^{\sharp} = e^{-n}(1-h_{1t})n_{1t}k_\pi - [n_{t+1}^{\natural} - e^{-n}(1-k_X)(1-n_{1t})]k_\pi/\pi_t \quad t \geq t_0 + M. \tag{91}$$

In the examples below, any good that survives from t to $t+1$ will have the profit-maximizing level of employment at $t+1$: $n_{j,t+1} = H_{t+1}n_j^0$ for $j \in \mathcal{J}_{2,t+1}^{\natural}$. Summing over j gives the fraction of the population at $t+1$ producing continuing goods:

$$n_{t+1}^{\natural} = H_{t+1}(1-k_X)\bar{n}_t \quad t \geq t_0 + M. \tag{92}$$

Linearized adjustment dynamics in the absence of demand and supply shocks. Recall from Proposition 2 and Table 1 that $\log(1-h_{1t})$, $\log\tilde{X}_{1t}$, and $\log\hat{X}_{1t}$ are simple functions of $\log X_{1t}^*$ with derivatives λ_2, λ_3, and λ_5, respectively, and from (59) that H_t is a linear function of n_{1t} with derivative λ_H. Thus equations (81)-(87), (91), and (92) comprise a system of 9 dynamic equations in the 9 variables $(n_{1t}, \tilde{V}_t, \bar{n}_t, Y_t/P_{1t}, C_t/P_{1t}, X_{1t}^*, \pi_t, n_{t+1}^{\sharp}, n_{t+1}^{\natural})$. Let $z_{it}^{\dagger}$ denote the deviation at t of the ith of these variables from its value along the steady-state growth path.[5] The behavior of the system can be approximated by the following 9 linear equations (see Appendix B for details):

$$n_{1,t+1}^{\dagger} = -n_{t+1}^{\sharp\dagger} - n_{t+1}^{\natural\dagger} \tag{93}$$

$$\tilde{V}_t^{\dagger} = Y_t^{\dagger} - \lambda_3 X_{1t}^{*\dagger} + \beta(1-k_X)\tilde{V}_{t+1}^{\dagger} \tag{94}$$

[5] Specifically, $z_{it}^{\dagger} = z_{it} - z_i^0$ for $z_{it} = n_{1t}, \tilde{V}_t, \bar{n}_t, \pi_t, h_{0t}, n_{t+1}^{\sharp}, n_{t+1}^{\natural}$, $Y_t^{\dagger} = \log(Y_t/P_{1t}) - \log(Y_t^0)$, $C_t^{\dagger} = \log(C_t/P_{1t}) - \log(C_t^0)$, and $X_{1t}^{*\dagger} = \log(X_{1t}^*) - \log(X_{1t}^{*0})$.

$$\bar{n}^{\dagger}_{t+1} = (1-k_X)\bar{n}^{\dagger}_t + n^{\sharp\dagger}_{t+1} \tag{95}$$

$$Y^{\dagger}_t = \frac{1}{n^0_1(1-n^0_1)} n^{\dagger}_{1t} + \lambda_5 X^{*\dagger}_{1t} \tag{96}$$

$$C^{\dagger}_t = Y^{\dagger}_t - \frac{1}{n^0_1(1-n^0_1)} n^{\dagger}_{1t} - \lambda_2 X^{*\dagger}_{1t} \tag{97}$$

$$X^{*\dagger}_{1t} - C^{\dagger}_t = \beta k_\pi \tilde{V}^{\dagger}_{t+1} \tag{98}$$

$$X^{*\dagger}_{1t} - C^{\dagger}_t = \beta \pi^0 \tilde{V}^{\dagger}_{t+1} + \beta \tilde{V}^0 \pi^{\dagger}_t \tag{99}$$

$$\begin{aligned} n^{\sharp\dagger}_{t+1} = {} & e^{-n} k_\pi \left[1 - h^0_1 - \frac{1-k_X}{\pi^0}\right] n^{\dagger}_{1t} + e^{-n}(1-h^0_1) n^0_1 k_\pi \lambda_2 X^{*\dagger}_{1t} \\ & + \frac{[n^{\natural 0} - e^{-n}(1-k_X)(1-n^0_1)]k_\pi}{(\pi^0)^2} \pi^{\dagger}_t - \frac{k_\pi}{\pi^0} n^{\natural\dagger}_{t+1} \end{aligned} \tag{100}$$

$$n^{\natural\dagger}_{t+1} = (1-k_X)(1-n^0_1)\lambda_H n^{\dagger}_{1,t+1} + (1-k_X)\bar{n}^{\dagger}_t. \tag{101}$$

Solution algorithm. Define $z^{\dagger}_t = (z^{\dagger\prime}_{1t}, z^{\dagger\prime}_{2t})'$ for $z^{\dagger\prime}_{1t} = (n^{\dagger}_{1t}, \tilde{V}^{\dagger}_t, \bar{n}^{\dagger}_t)$ and $z^{\dagger\prime}_{2t} = (Y^{\dagger}_t, C^{\dagger}_t, X^{*\dagger}_{1t}, \pi^{\dagger}_t, n^{\sharp\dagger}_{t+1}, n^{\natural\dagger}_{t+1})$. Equations (93)-(101) comprise a system of the form

$$A z^{\dagger}_{1,t+1} = B z^{\dagger}_t. \tag{102}$$

This is a system in which the dependence between $z^{\dagger}_t$ and $z^{\dagger}_{t+1}$ is captured entirely by the three state variables $z^{\dagger}_{1,t+1}$. With linear operations we can eliminate $z^{\dagger}_{2t}$ from the right side of (102) to arrive at a system of the form

$$z^{\dagger}_{1,t+1} = \Phi z^{\dagger}_{1t}; \tag{103}$$

again see Appendix B for details. One of the eigenvalues of the (3×3) matrix Φ is greater than 1 and the other two are less than 1 in absolute value. The values of $n^{\dagger}_{1t}$ and $\bar{n}_t$ are predetermined, and the forward-looking variable $\tilde{V}_t$ is the value that causes $z^{\dagger}_{1t}$ to be a linear combination of the two eigenvectors of Φ associated with the stable eigenvalues.

In the examples below, the initial periods $t_0, t_0+1, ..., t_0+M-1$ may be governed by different dynamics from those in (93)-(101) as a result of demand or productivity shocks. However, in all the examples it is assumed that exogenous variables return to their steady-state values after a finite number of periods, and it can be shown that in each of these examples, all specialized goods will have exactly the capacity needed to meet demand after M periods. In this case (78) becomes

$$Q^{\dagger}_{jt} = \lambda_H n^{\dagger}_{1t} \quad j \in \mathcal{J}_{2,t},\ t \geq t_0 + M \tag{104}$$

and (102) turns out to govern dynamics after $t_0 + M$.

The dynamics over the first M periods are characterized by

$$\dot{A}_t \dot{z}^{\dagger}_{1,t+1} = \dot{B}_t \dot{z}^{\dagger}_t + \dot{c}_t \tag{105}$$

$$\dot{z}^{\dagger}_{1,t+1} = \dot{\Phi}_t \dot{z}^{\dagger}_{1t} + \dot{e}_t. \tag{106}$$

The initial state vector $\dot{z}^{\dagger}_{1t}$ includes $n^{\dagger}_{1t}, \tilde{V}_t$, and possibly other variables whose values at date t_0 are determined by initial conditions. These initial conditions along with the specified initial value $n^{\dagger}_{1t_0}$ and a conjectured value for $\tilde{V}_{t_0}$ imply a particular value $\dot{z}^{\dagger}_{1t_0}$ and thus from (106) a particular value for $\dot{z}^{\dagger}_{1,t_0+M}$ from which $z^{\dagger}_{1,t_0+M}$ can also be calculated. The rational-expectations solution for $\tilde{V}^{\dagger}_{t_0}$ is the value that causes $z^{\dagger}_{1,t_0+M}$ to be a linear combination of the stable eigenvectors of Φ. Since $z^{\dagger}_{1,t_0+M}$ is an affine function of $\dot{z}^{\dagger}_{1t_0}$, this pins down $\tilde{V}^{\dagger}_{t_0}$, the unknown element of $\dot{z}^{\dagger}_{1t_0}$. Now knowing $\dot{z}^{\dagger}_{1t_0}$, we can use (106) for calculate $\dot{z}^{\dagger}_{1,t_0+1}, ..., \dot{z}^{\dagger}_{1,t_0+M-1}$ and use (103) to calculate $z^{\dagger}_{1,t_0+M}, z^{\dagger}_{1,t_0+M+1}, ...$ From these we can find the remaining elements of $\dot{z}^{\dagger}_{t_0+1}, ..., \dot{z}^{\dagger}_{t_0+M-1}$ from (105) and $z^{\dagger}_{t_0+M}, z^{\dagger}_{t_0+M+1}$ from (102). With these we can calculate any other variable of interest. For example, the relative price of any specialized good can be found by substituting (104) into (79):

$$P^{\dagger}_{jt} = \left[\frac{1}{n^0_1(1-n^0_1)}\right] n^{\dagger}_{1t} + \lambda_5 X^{*\dagger}_{1t} \quad j \in \mathcal{J}_{2,t}, \ t \geq t_0 + M. \tag{107}$$

From (76), real GDP is given by

$$Q_t = \frac{H_t(1-\alpha_1)}{1-n^0_1}\bar{n}_t + \left(\frac{\alpha_1}{n^0_1}\right)\left(\frac{\hat{X}_{1t}}{\hat{X}^0_{1t}}\right) n_{1t} \tag{108}$$

$$Q^{\dagger}_t = \left[\frac{\alpha_1}{n^0_1} + (1-\alpha_1)\lambda_H\right] n^{\dagger}_{1t} + \alpha_1\lambda_5 X^{*\dagger}_{1t} + \frac{1-\alpha_1}{1-n^0_1}\bar{n}^{\dagger}_t. \tag{109}$$

9 Demand shocks.

In this section we consider an economy in which all exogenous and predetermined variables at t_0 are equal to their steady-state values, so $n_{jt} = n^0_j \ \forall j \in \mathcal{J}_{2t_0}$ with the single exception that the demand parameter $\chi_{jt} = \chi \neq 1$ for a fraction κ of the specialized goods produced at t_0. From (80) we see that the income of anyone producing a nonimpacted specialized good at t is still given by Y_t in (96) whereas anyone producing an impacted good has income Y^{χ}_t characterized by

$$Y^{\chi\dagger}_t = 2(\chi - 1) + Y^{\dagger}_t. \tag{110}$$

Let n^{χ}_t denote the fraction of the population at t specializing in goods for which $\chi_{jt} = \chi$ and n^c_t the fraction specializing in goods for which $\chi_{jt} = 1$. Then unemployment compensation is given by

$$C_t = \frac{\tau(n_t^c Y_t + n_t^\chi Y_t^\chi)}{n_{1t}(1 - h_{1t})}. \tag{111}$$

Baseline parameter values. Most of our numerical examples use the parameter values in Table 2. We assume that a period corresponds to one quarter, with n implying an annual population growth rate of 1% and β an annual discount rate of 2%. Note that taxes in this model are used solely to finance unemployment compensation, which is why we used a relatively low value ($\tau = 0.02$) for the marginal tax rate. At this rate, steady-state unemployment compensation is equal to about one-quarter of the average wage of unskilled workers.[6] Productivity for all workers grows at some fixed rate g (which does not affect any of the numbers reported here), and the proportional gap between the most productive and least productive unskilled individual ($S_t - R_t$) is constant at 1 for all t. There are huge gross flows out of and into employment in a typical month in the U.S. Davis, Faberman, and Haltiwanger (2006, Table 1) found that 10% of workers lose or quite their jobs each quarter, and the estimates in Ahn and Hamilton (2021) imply that 12% of employed individuals will be unemployed or out of the labor force 3 months later. Our value of $k_X = 0.02$ assumes that involuntary separations account for less than 1/5 of these observed gross flows. When the probability of successfully creating a new good is $k_\pi = 0.25$, the baseline parameters imply a steady-state unemployment rate of $u^0 = 5.1\%$. The discounted lifetime log income advantage of being skilled is $\tilde{V}^0 = 4.80$, which translates into a per-period flow advantage of $[1 - \beta(1 - k_X)]\tilde{V}^0 = 0.12$, or 12% higher after-tax incomes for skilled workers.

9.1 A transient drop in demand.

In our first example, the number of people producing each good starts out in period t_0 at the steady-state values ($n_{jt_0}^\dagger = 0$ for all $j \in \mathcal{J}_{t_0}$). In period t_0, one-quarter of the specialized goods ($\kappa = 0.25$) experience a 10% drop in demand in t_0 ($\chi = 0.9$) that lasts for only a single period with χ_{jt} for all goods j returning to unity for $t \geq t_0 + 1$. From (78), the demand-impacted goods lower their output by 10% ($Q_{t_0}^{\chi\dagger} = \chi - 1$). From (79), if there were no change in the fraction of unskilled workers who are unemployed, impacted goods would also lower their price by 10% in period t_0 ($P_{t_0}^{\chi\dagger} = \chi - 1 + \lambda_5 X_{1t_0}^{*\dagger}$), with $n_{1t_0}^\dagger = n_{jt_0}^\dagger = 0$ for this example. The 10% drop in both price and quantity account for the 20% drop in income described in (110). Since the shock lasts for only one period, the lifetime advantage at $t = t_0$ of one of the impacted specialists $\tilde{V}_{t_0}^\chi$ differs from that of nonimpacted specialists $\tilde{V}_{t_0}$ by $\tilde{V}_{t_0}^{\chi\dagger} = \tilde{V}_{t_0}^\dagger + 2(\chi - 1)$. The decision of impacted units to continue is determined solely by $\tilde{V}_{t_0+1}^\chi = \tilde{V}_{t_0+1}$, so no impacted unit has an incentive to discontinue. The decision of how many workers to add in $t_0 + 1$ is based on (64), which again from (58) is the same for all j. If (96) and (94) are understood

[6] The ratio of unemployment compensation (39) to average unskilled income $(P_{1t}^0 \hat{X}_{1t}^0)/(n_1^0 h_1^0)$ is $[h_1^0 \tau(1 - \alpha_1)]/[\alpha_1(1 - h_1^0)] = 0.23$.

to characterize the values for producers of goods that are not impacted by the demand shock, equations (93)-(101) thus all hold for $t > t_0$ and almost all hold for $t = t_0$ as well with the single exception that (97) for $t = t_0$ is replaced by a linearization of (112):

$$C^{\dagger}_{t_0} = 2(\chi - 1)\kappa + Y^{\dagger}_{t_0} - \lambda_2 X^{*\dagger}_{1t_0} - \frac{1}{n^0_1(1 - n^0_1)} n^{\dagger}_{1t_0}. \tag{112}$$

Expression (112) reflects the fact that the lower income of demand-impacted specialists ($2(\chi - 1) = -0.2$) reduces unemployment compensation by -0.2κ in equilibrium. Appendix B shows that real GDP in the initial period is

$$Q^{\dagger}_{t_0} = (1 - \alpha_1)\kappa(\chi - 1) + \alpha_1 \lambda_5 X^{*\dagger}_{1t_0}. \tag{113}$$

Output of the impacted goods falls by χ, and nonimpacted goods have no ability and no incentive to increase production. Equations (93)-(101), (107) and (109) all hold as written for all $t > t_0$, so for this example $M = 1$ and expression (112) is the only way in which the dynamic equations for the first date differ from the usual adjustment dynamics. In terms of the notation in expression (105), for this example $\dot{A}_{t_0} = A$, $\dot{B}_{t_0} = B$, and $\dot{c}_{t_0} = (0, 0, 0, 0, -2(\chi - 1)\kappa, 0, 0, 0, 0)'$.

The solid green curves in Figure 5 plot the time paths of key variables from period t_0 on. Real GDP (Panel E) falls by 1.4% in t_0 but almost completely recovers by $t_0 + 1$. The reason that real GDP does not quite completely return to the steady-state growth path in $t_0 + 1$ is that lower taxes collected from specialized workers in t_0 mean lower unemployment compensation, which induces slightly more unskilled workers to produce good 1 in t_0 (Panel C). Because fewer unskilled workers spent t_0 trying to develop a skill, the number of unskilled workers in $t_0 + 1$ is very slightly above steady state (Panel A). However, these persistent effects are quite small in size, and to a first approximation the effects of the demand shock are limited to the single initial period in which $\chi_{jt_0} \neq 1$, in which GDP essentially falls by the size of the drop in demand.

9.2 A transient increase in demand.

Consider next the case in which a fraction $\kappa = 0.25$ of specialized goods experience a 10% increase in demand ($\chi = 1.1$) in period t_0 with demand returning to normal in $t_0 + 1$. Because these goods would be producing at capacity in t_0 if $\chi = 1$, they do not increase production but instead increase price by about 20%. Equations (78) and (79) in this case imply $Q^{\chi\dagger}_{t_0} = 0$ and $P^{\chi\dagger}_{t_0} = 2(\chi - 1) + \lambda_5 X^{*\dagger}_{1t_0}$. If $X^{*}_{1t_0}$ did not change, there would be a 20% increase in the incomes of workers specialized in the now-favored good. There is no direct boost to GDP from the demand shock, and (113) for this case becomes $Q^{\dagger}_{t_0} = \alpha_1 \lambda_5 X^{*\dagger}_{1t_0}$. Unemployment compensation is still given by (112). In this example, unemployment compensation rises in t_0 because of the higher tax revenues from some of the skilled workers. The dynamic effects of the shock are described by the dashed blue lines in Figure 5. The higher unemployment compensation induces an increased fraction of the unskilled to try to develop a specialty (Panel C). This means less production of good 1, and with no added production of any specialized good, GDP actually falls in t_0. Some of these individuals succeed in producing new goods, leading to a values of n_{1,t_0+1} and $\tilde{V}_{t_0+1}$ very slightly below the steady state. The latter means less incentive to specialize, and the economy eventually returns to steady state. As in Example 9.1, the changes resulting from unemployment compensation are quite small. To a first approximation, a transitory increase in demand has no real effects.

9.3 A large, persistent but isolated drop in demand.

Next consider the case of a 50% drop in demand that affects only 5% of specialized goods ($\chi = 0.5$, $\kappa = 0.05$). Note that the total size of the shock to demand is the same as in Example 9.1 ($\kappa(\chi - 1) = 0.025$ in both examples), but in Example 9.3 the drop in demand is concentrated on a relatively small number of goods. If the low demand lasted only for a single period, the results would be identical to those in Example 9.1. Here however we consider a shock that lasts for $D = 8$ periods. We take the shock to be isolated in the sense that new goods created beginning in $t_0 + 1$ all enjoy the steady-state demand level $\chi_{jt} = 1$. From (110) the period t log income for workers specializing in the impacted good would differ from that of other specialized workers by $2(\chi - 1)$ for each $t = t_0, ..., t_0 + D - 1$. From (94) this means that the lifetime advantage as of date t from being specialized in an impacted good differs from the steady-state advantage $\tilde{V}^0$ by

$$\tilde{V}_t^{\chi\dagger} = \tilde{V}_t^{\dagger} + \beta^X_{t_0+D-t} 2(\chi - 1) \tag{114}$$

$$\beta^X_{t_0+D-t} = \begin{cases} \sum_{s=0}^{t_0+D-t-1} [\beta(1-k_X)]^s & t = t_0, ..., t_0 + D - 1 \\ 0 & t \geq t_0 + D \end{cases} . \tag{115}$$

The value of $\tilde{V}_t^{\chi+}$ for this numerical example is plotted in black in Panel D of Figure 5, with the horizontal black dashed line drawn at $\tilde{V}^0 + \tilde{V}_t^{\chi\dagger} = 0$, the level at which producers of the impacted good would be indifferent between waiting out the period of low demand and abandoning their specialty. For this numerical example, $\tilde{V}^0 + \tilde{V}_{t_0+1}^{\chi\dagger} < 0$, meaning that workers who had specialized in the impacted good in period t_0 would be better off joining the pool of unskilled beginning in period $t_0 + 1$ rather than wait for demand for their good to pick up. These workers will produce their specialized good in t_0 and then join the pool of unskilled beginning in $t_0 + 1$. This means that only a fraction $(1 - k_X)(1 - \kappa)$ of the specialized goods that were produced in t_0 survive to $t_0 + 1$. Expressions (83) and (92) for $t = t_0$ in this example become

$$\bar{n}_{t_0+1} = (1-k_X)(1-\kappa)\bar{n}_{t_0} + n^{\sharp}_{t_0+1}$$

$$n^{\natural}_{t_0+1} = H_{t_0+1}(1-k_X)(1-\kappa)\bar{n}_{t_0}.$$

The number of openings (88) in t_0 is

$$O_{t_0} = n^{\natural}_{t_0+1}e^{n}N_{t_0} - (1-k_X)(1-\kappa)(1-n_{1t_0})N_{t_0}$$

leading (91) to be replaced by

$$n^{\sharp}_{t_0+1} = e^{-n}(1-h_{1t_0})n_{1t_0}k_{\pi} - [n^{\natural}_{t_0+1} - e^{-n}(1-k_X)(1-\kappa)(1-n_{1t_0})]k_{\pi}/\pi_{t_0}.$$

After linearization this results in adding the following terms to the right sides of equations (95), (100), and (101), respectively:

$$-\kappa(1-k_X)(1-n^0_1)$$

$$-\frac{\kappa e^{-n}(1-k_X)(1-n^0_1)k_{\pi}}{\pi^0} + \frac{\kappa e^{-n}(1-k_X)k_{\pi}}{\pi^0}n^{\dagger}_{1t_0} + \frac{\kappa e^{-n}(1-k_X)(1-n^0_1)k_{\pi}}{(\pi^0)^2}\pi^{\dagger}_{t_0}$$

$$-\kappa(1-k_X)(1-n^0_1) - \kappa(1-k_X)(1-n^0_1)\lambda_H n^{\dagger}_{1,t_0+1} - \kappa(1-k_X)\bar{n}^{\dagger}_{t_0}.$$

Also as in equation (112) we add $2(\chi-1)\kappa$ to the right side of (97).

Beginning in $t = t_0+1$, all the impacted goods are gone and newly created goods face steady-state demand parameters, so that the dynamic equations for the economy for $t = t_0+1, t_0+2, ...$ are exactly the same as in (93)-(101). Hence this is another example in which dynamics revert to the time-invariant system (93)-(101) after $M = 1$ period.[7] However, the effects of the shock at t_0 have not gone away, because the extra inflow of workers into the pool of unskilled causes n_{1,t_0+1} to be higher than the steady-state value n^0_1, shown in Panel A of figure 4. This causes real GDP in (109) (Panel E) to be lower in t_0+1 both because fewer people are producing high-value goods and because lower-skilled workers have lower overall demand for goods. Panel F shows the relative price that would maximize profits for impacted units if they were to continue in operation beyond t_0. However, because $\tilde{V}^{\chi}_{t_0+1} < 0$, they have all dropped out and these goods are no longer produced after t_0. Thus the same-sized shock to demand (Examples 9.1 and 9.3) can have a bigger effect if it results in significant numbers of skilled workers losing their jobs, as in this example.

This example offers one possible explanation for why goods are always being discontinued

[7] For this example, the time dependence is captured by the $t = t_0$ values of $\dot{A}_{t_0}, \dot{B}_{t_0}, \dot{c}_{t_0}$. Although expression (114) exhibits time-dependence up until $t = t_0 + D_1 - 1$, this equation is only relevant for the decision impacted workers make in period t_0. For the parameter values in this example, workers abandon their specialty and these goods are no longer produced after period t_0, so the only economically relevant equations after $t > t_0$ all take the form of (93)-(101).

along the steady-state growth path. Suppose that along the steady-state growth path, each period t a fraction k_X of specialized producers learn that demand for their particular product is going to experience a decrease of the magnitude considered here beginning in $t+1$. Producers of those goods would have an incentive to abandon their specialty after producing in t, and choose to return to the pool of unskilled workers. Thus a shock of the kind considered in Example 9.3 could be viewed as occurring all the time in this model. Everyone takes into account the likelihood that it will eventually happen to them through the parameter k_X that enters every decision. Example 9.3 could be viewed as exploring what happens when these regular demand shocks affect a larger fraction of goods than usual and catch the producers of these goods by surprise in t_0.

9.4 The role of technological frictions.

How long the adjustment process requires depends on how hard it is to create new goods. Consider an economy in which the probability of successfully creating a new specialized good is 0.60 rather than 0.25 in our baseline parameterization. With lower technological frictions to developing new goods, there is a lower equilibrium unemployment rate ($u^0 = 2.4\%$ versus 5.1% for the baseline parameters) and a lower equilibrium advantage to being skilled ($\tilde{V}^0 = 0.7$ versus 4.8 for the baseline case). In an economy with more modest technological frictions, workers would be much quicker to abandon their skill under adverse conditions (see the horizontal dashed red line in Figure 5D). Although the surge in unskilled workers in t_0+1 is the same as in Example 9.3, the economy recovers more quickly; the largest stable eigenvalue of Φ is 0.89 for the baseline parameters but only 0.79 when $k_\pi = 0.6$. Thus technological frictions are the key determinant of the Keynesian multiplier effect.

9.5 A persistent drop in demand for new and existing goods.

The assumption in Examples 9.3 and 9.4 was that newly created goods were immune from the lower demand that hit existing goods, with the result that a surge in new good creation was a key factor mitigating the economic downturn. In reality, starting a new business may be harder than usual when the economy is weak. To study this possibility, we now consider a demand shock that affects both new and existing goods. As in Example 9.1, we suppose that 25% of existing goods at t_0 experience a 10% drop in demand ($\kappa = 0.25$, $\chi = 0.9$). Unlike that example, here we assume that the drop lasts for $D = 5$ periods and also affects any goods that are newly created in $t_0, ..., t_0 + D - 1$. In this case, the returns to creating a new good will be determined not by $\tilde{V}_{t+1}$ but by $\tilde{V}^{\chi}_{t+1}$. For the first $D-1$ periods the condition for new good creation (98) becomes

$$X^{*\dagger}_{1t} - C^{\dagger}_t = \beta k_\pi \tilde{V}^{\dagger}_{t+1} + \beta^X_{t_0+D-t-1} 2(\chi - 1) \quad t = t_0, ..., t_0 + D - 2$$

for β_j^X given by (115). The attractiveness of specializing in a nonimpacted good is still $\tilde{V}_{t+1}$, so that the equilibrium probability π_t of obtaining one of those positions is still determined by (87). As in the baseline model, we assume that successfully creating a new good provides the new good with a steady-state demand based on the number of people who initially created it, represented by $n_j^0 = n_{jt}$. No demand-impacted good, whether continuing or newly created, will have an incentive to hire new workers during the first $D-1$ periods. Assuming that all goods continue to face the steady-state probability k_X of being forced to discontinue, the fraction of the population that produces demand-impacted goods thus evolves according to

$$n_{t+1}^{\chi} = e^{-n}(1-k_X)n_t^{\chi} + n_{t+1}^{\sharp} \quad t = t_0, ..., t_0 + D - 2$$

starting from $n_{t_0}^{\chi} = \kappa(1 - n_1^0)$. Nonimpacted goods hire a fraction of the population $n_{t+1}^c = H_{t+1}(1-k_X)\bar{n}_t^c$ where $\bar{n}_t^c = (1-k_X)^{t-t_0}(1-\kappa)(1-n_1^0)$, leading to a total number of skilled workers given by $1 - n_{1,t+1} = n_{t+1}^{\chi} + n_{t+1}^c$. After period $t_0 + D$, the dynamics revert to the system (81)-(92). For details see Appendix B.

The cyan curves in Figure 5 show adjustment dynamics for this example. The drop in production of impacted specialized goods in t_0 by itself would lower GDP by 1.5%, just as in Examples 9.1 and 9.3-9.4. However, in this case unskilled workers recognize at t_0 the lower probability of successfully specializing in an existing good and limited returns from producing a new impacted good. More of them respond by choosing to produce good 1 rather than try to develop a skill, with only $1 - h_{1t_0} = 6.1\%$ of them unemployed in t_0 compared to the steady-state value of 11.5%. This results in an increased production of good 1 in period t_0 that turns out to completely offset the lost production of specialized goods, so that real GDP at t_0 is about the same as the steady-state value. The lower-than-normal rate of skill accumulation results in a buildup in the fraction of unskilled over time, and this eventually brings GDP to 2.3% below steady-state by $t_0 + 4$. The buildup in n_{1t} also increases the incentive to try to specialize, and by $t_0 + 4$ a higher fraction than normal of the unskilled are searching for work. The demand shock is gone for all goods beginning in $t_0 + 5$, resulting in a sharp rebound in real GDP. However, GDP is still 0.9% below normal in $t_0 + 5$ and only gradually returns to the steady-state growth path as the surplus of unemployed eventually develop skills.

9.6 Shocks to ξ_{jt}.

Up to this point we have been discussing shocks to the preference parameter χ_{jt}, which results in a vertical shift of the demand curve for good j and changes the profit-maximizing level of output $\bar{Q}_{jt}/2$ (see Figure 2). Consider now a shock to the parameter ξ_{jt}, which changes the vertical intercept of the demand curve but leaves the horizontal intercept and the profit-maximizing level of output unchanged. From equation (78), this has no direct effect on output regardless of whether the good is demand- or supply-constrained. Instead, from (79) and (80), a 10% increase in ξ_{jt} leads to a 10% increase in price and income for good j. A 10% decrease in ξ_{jt} leads to 10% decreases in P_{jt} and Y_{jt}, again regardless of the regime. The changes in income will result in a change in tax receipts and unemployment compensation which would have general-equilibrium effects on h_{1t} and n_{1t}, but these would be secondary contributions of the size noted in Example 9.1.

It is possible that if a drop in ξ_{jt} is large enough and lasts long enough, the drop in income for producers of the good would be sufficiently large to persuade them to discontinue production. Note that the coefficient on χ_{jt} in the income equation (80) is twice as big as the coefficient on ξ_{jt}, so it would take roughly twice as big a shock to ξ_{jt} to cause the good to be discontinued. For shocks of moderate size, a change in ξ_{jt} would affect relative prices but have negligible effects on any real quantities.

10 Productivity shocks.

In this section we consider an economy that begins period t_0 with all exogenous and predetermined variables equal to their steady-state values with the exception that the productivity parameter $\zeta_{jt} = \zeta \neq 1$ for a fraction κ of the specialized goods in production in date t_0.

10.1 A transient drop in productivity.

Consider first a 10% drop in productivity that affects 25% of the goods in t_0 ($\zeta = 0.9$, $\kappa = 0.25$) with productivity returning to normal in $t_0 + 1$. Impacted goods are supply-constrained in t_0 and from (78) each lower their output by 10% ($Q_{t_0}^{\zeta\dagger} = \zeta - 1$). From (79) this means they raise their price by 10% ($P_{t_0}^{\zeta\dagger} = -(\zeta - 1) + \lambda_5 X_{1t_0}^{*\dagger}$) which from (80) means no change in income ($Y_{t_0}^{\zeta+} = Y_{t_0}^{\dagger}$). The last equation also means that there is no change in unemployment compensation for t_0, and the elements of $z_t^{\dagger}$ are determined by equations (93)-(101) for all t. With reduced production of the impacted goods and no increase in the production of any other goods, real GDP in t_0 falls by $Q_{t_0}^{\dagger} = (1 - \alpha_1)\kappa(\zeta - 1) + \alpha_1\lambda_5 X_{1t_0}^{*\dagger}$ with $X_{1t_0}^{*\dagger} = 0$, and then returns exactly to the steady-state growth path in $t_0 + 1$.

The time path of key variables are shown in the solid black curves in Figure 6. Apart from lacking the modest effects on n_{1,t_0+1} arising from the lower unemployment compensation in t_0 in Example 9.1, the effects of a productivity shock in Figure 6 are essentially the same as those of a demand shock in Figure 5. Note in particular we could not use measured productivity as a way to distinguish between demand and supply shocks. In Example 9.1, productivity of impacted goods falls by $Q_{jt_0}^{\dagger} - N_{jt_0}^{\dagger} = (\chi - 1)$, whereas in Example 10.1, productivity of impacted goods falls by $Q_{jt_0}^{\dagger} - N_{jt_0}^{\dagger} = (\zeta - 1)$. One could not tell by looking at the behavior of output, employment, or productivity whether output fell because it is harder to produce the good or because fewer people want to buy it. The one variable that could be used to distinguish these two shocks is the relative price in Panel F of Figures 5 and 6. A demand shock results in lower output and lower price, whereas a supply shock results in lower output and higher price.

10.2 A transient increase in productivity.

Consider next the case in which a fraction $\kappa = 0.25$ of the goods in production at t_0 experience a 10% increase in productivity ($\zeta = 1.1$), with conditions again returning to normal beginning in $t_0 + 1$. Although more goods could be produced in t_0, no one has an incentive to do so, since $\bar{Q}_{jt_0}/2$ is still the profit-maximizing level of production. There is no incentive to change prices, and no incentive to make any changes for the future since conditions at $t_0 + 1$ will be back to steady state. Thus in this example, the increase in productivity has no effect on any real or nominal variable at any date.

10.3 A persistent decrease in productivity.

Even if a drop in productivity is persistent, according to equation (80) it would not lead to a change in the relative income received by producers of the impacted goods. This is because (80) is derived from a linearization around the steady-state growth path. Along the steady-state growth path, the elasticity of demand is unity meaning that the decrease in output equals the increase in price. As the productivity shock becomes larger, the elasticity of demand is pushed above unity, and the linearization becomes less accurate. If we wanted to examine the effects of a very large shock to productivity, we should use the exact expression for income (66) instead of the linearization (80). Apart from general-equilibrium feedback arising from changes in n_{1t} or Q_{1t}, the direct effects of either demand shocks χ_{jt} or supply shocks ζ_{jt} on income Y_{jt} in (66) come through their implications for Q_{jt} or $\bar{Q}_{jt}$:

$$Y_{jt} \propto Q_{jt}(\bar{Q}_{jt} - Q_{jt}).$$

In the case of a demand shock hitting a demand-constrained good, $Q_{jt} = \bar{Q}_{jt}/2 = \chi_{jt} H_t n_j^0 X_{1t}^0 N_t$. When $n_{1t} = n_1^0$, this means $Y_{jt} \propto \chi_{jt}^2$ and $\Delta \log Y_{jt} = 2\Delta \log \chi_{jt}$. Hence the coefficient on $\chi_{jt}^\dagger$ in the linearization (80) is in fact the same as the coefficient in an exact representation, since $\log Y_{jt}$ is an exact linear function of $\log \chi_{jt}$ in the demand-constrained case.

By contrast, in the case of a productivity shock hitting a supply-constrained good, $Q_{jt} = n_{jt} N_t \zeta_{jt} X_{jt}^0$ and $\bar{Q}_{jt} = 2H_t n_j^0 X_{jt}^0 N_t$. Thus when $n_{jt} = n_j^0$ and $n_{1t} = n_1^0$, we have $Y_{jt} \propto \zeta_{jt}(2-\zeta_{jt})$ and

$$\Delta \log Y_{jt} = \log[\zeta_{jt}(2 - \zeta_{jt})]. \tag{116}$$

For the 10% productivity drop in Example 10.1, $\Delta \log Y_{jt} = \log[(0.9)(1.1)] = -0.01$, a little below the value of zero assumed in the linearization.

To examine the effects of larger productivity drops, we can use (116) instead of the linear approximation $\Delta \log Y_{jt} \simeq 0$. In this case we would describe unemployment compensation in t_0 by

$$C^{\dagger}_{t_0} = \kappa \log[\zeta(2-\zeta)] + Y^{\dagger}_{t_0} - \lambda_2 X^{*\dagger}_{1t_0} - \frac{1}{n^0_1(1-n^0_1)} n^{\dagger}_{1t_0}.$$

If the productivity shock persists for D periods, the relative income and lifetime advantage of impacted relative to nonimpacted goods

$$Y^{\zeta\dagger}_t = \log[\zeta(2-\zeta)] + Y^{\dagger}_t$$

$$\tilde{V}^{\zeta\dagger}_t = \tilde{V}^{\dagger}_t + \beta^X_{t_0+D-t} \log[\zeta(2-\zeta)]$$

for β^X_t given by (115) and $t = t_0, ..., t_0 + D - 1$. For $\zeta = 0.5$, $\log[\zeta(2-\zeta)] = -0.29$ and a productivity shock of this size that persisted for $D = 8$ periods would not be sufficiently big to persuade workers to surrender their specialty. However, an 80% drop in productivity ($\zeta = 0.2$) that lasted for 8 periods would induce production of impacted goods to be discontinued. This is indicated by the solid black curves in Figure 6. Note that in order to scale this example so that the period t_0 shock is the same as in the other examples, we assume that the fraction of goods experiencing the 80% drop in productivity is $\kappa = 0.03$. Although the overall patterns are similar to those in the persistent demand Example 9.3, the effects for $t > t_0$ are smaller because a smaller fraction of specialized workers are displaced.

11 Discussion.

In order to focus as clearly as possible on the role of specialization in determining the level of economic activity, this paper abstracted from many details that play a key role in economic fluctuations. Here labor was the only input, with specialization taking the form of training and assembling a dedicated team of workers. Specialized capital is an even more important commitment for most businesses (Ramey and Shapiro, 1998 and 2001). Production moreover typically depends on inputs purchased from other firms that themselves specialize to be able to provide those goods or services, amplifying the forces studied here through network connections (Baqaee, 2018 and Baqaee and Farhi, 2019). This paper completely ignored financial frictions, even though they appear to be a key factor in many economic downturns. And although nominal frictions played no role in this model, they could well be an additional factor in amplifying economic downturns.

By focusing on just a single source of specialization and a single technological friction, the hope was to shed light on the interaction between specialization and demand as a fundamental short-run determinant of the level of GDP.

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Table 1. Ranges and derivatives of key variables.

	Variable	Value at	Value at	Sign of	Steady-state
	Z_t	$\log X^*_{1t} = R_t$	$\log X^*_{1t} = S_t$	$\partial Z_t / \partial \log X^*_{1t}$	derivative
(1)	$\log X^*_{1t}$	R_t	S_t	> 0	
(2)	$\log(1 - h_{1t})$	$-\infty$	0	> 0	$\lambda_2 = \frac{1}{\log X^{*0}_{1t} - R_t}$
(3)	$\log \tilde{X}_{1t}$	$\frac{S_t + R_t}{2}$	S_t	> 0	$\lambda_3 = \frac{\log X^{*0}_{1t} - R_t}{S_t - R_t}$
(4)	$\hat{X}_{1t}$	$\frac{\exp(S_t) - \exp(R_t)}{S_t - R_t}$	0	< 0	
(5)	$\log \hat{X}_{1t}$	$> \frac{S_t + R_t}{2}$	$-\infty$	< 0	$\lambda_5 = \frac{-X^{*0}_{1t}}{\hat{X}^0_{1t}(S_t - R_t)}$
(6)	h_{Yt}	$-\infty$	∞	> 0	
(7)	$\tilde{V}_t$	finite	$-\infty$	< 0	

Table 2. Parameter values used in baseline calculations.

Exogenous parameters

parameter	**meaning**
$\alpha_1 = 0.4$	steady-state expenditure share of good 1
$\tau = 0.02$	marginal tax rate
$\beta = 0.995$	discount rate
$k_U = 0.2$	utility cost of trying to create new good
$k_\pi = 0.25$	probability of successfully creating new good
$k_X = 0.02$	fraction of goods discontinued each period in steady state
$n = 0.0025$	population growth rate
$R_{t_0} = 1$	initial lowest log productivity of unskilled workers
$S_{t_0} = 2$	initial highest log productivity of unskilled workers

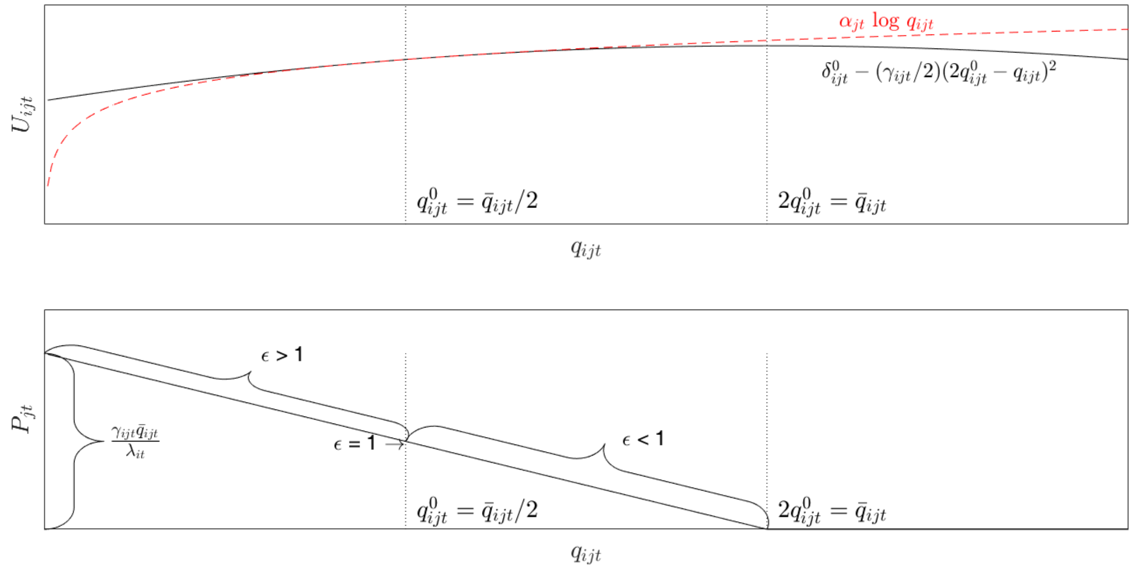
Derived parameters

parameter	**meaning**
$\lambda_2 = 8.668$	elasticity of unskilled unemployment $(1 - h_{1t})$ with respect to threshold X^*_{1t}
$\lambda_3 = 0.1154$	elasticity of flow-value of unskilled $\tilde{X}_{1t}$ with respect to threshold X^*_{1t}
$\lambda_5 = -0.7032$	elasticity of productivity of unemployed $\hat{X}_{1t}$ with respect to threshold X^*_{1t}
$\lambda_H = -0.1281$	semi-elasticity of demand parameter $\bar{Q}_{jt}$ with respect to fraction of unskilled n_{1t}

Steady-state values of endogenous variables

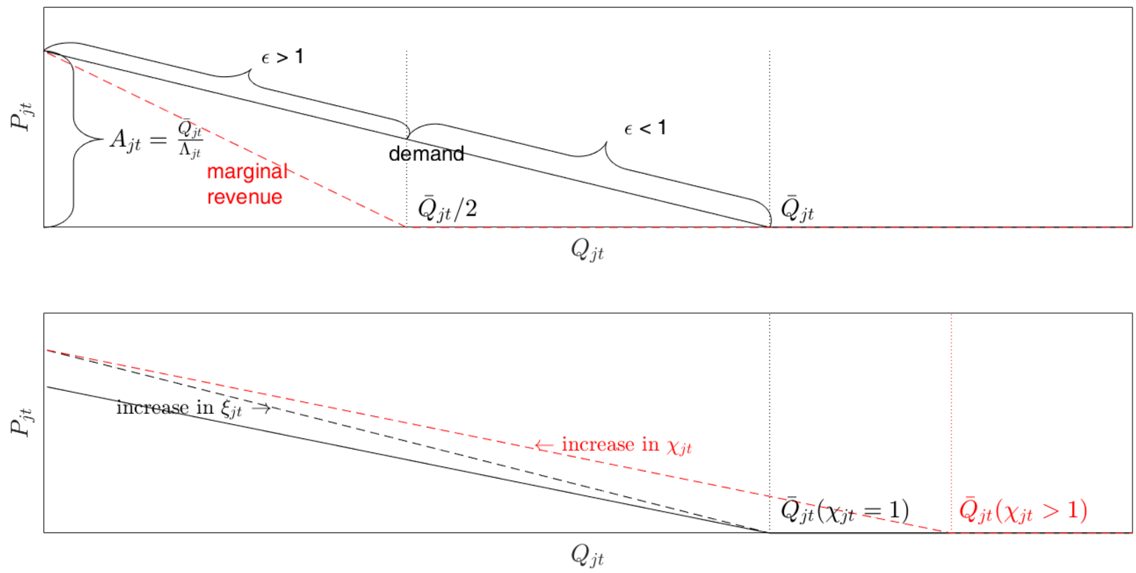
variable	**meaning**
$n^0_1 = 0.4436$	fraction of population without skills
$\log X^{*0}_{1t_0} = 1.1154$	initial productivity threshold for unskilled workers to produce good 1
$1 - h^0_1 = 0.1154$	fraction of unskilled workers who are unemployed
$u^0 = 0.0512$	fraction of population who are unemployed
$\pi^0 = 0.2082$	probability of successfully becoming specialized in an existing good
$\tilde{V}^0 = 4.8032$	discounted lifetime log income differential between skilled and unskilled

Figure 1. Individual utility and demand curves.



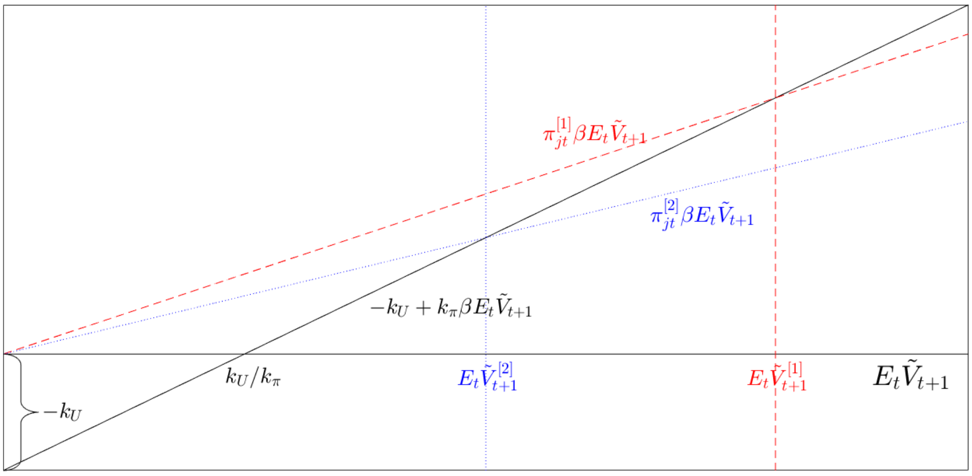
Notes to Figure 1. Top panel: logarithmic preferences and quadratic approximation. Bottom panel: demand curve associated with quadratic preferences.

Figure 2. Market demand curves.



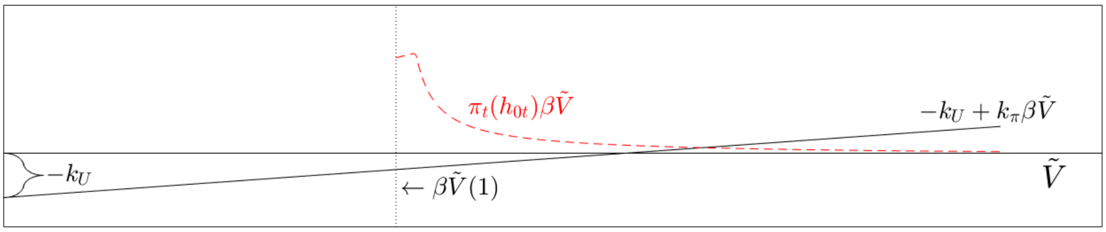
Notes to Figure 2. Top panel: market demand and marginal revenue. Bottom panel: effects of shifts in ξ_{jt} and χ_{jt}.

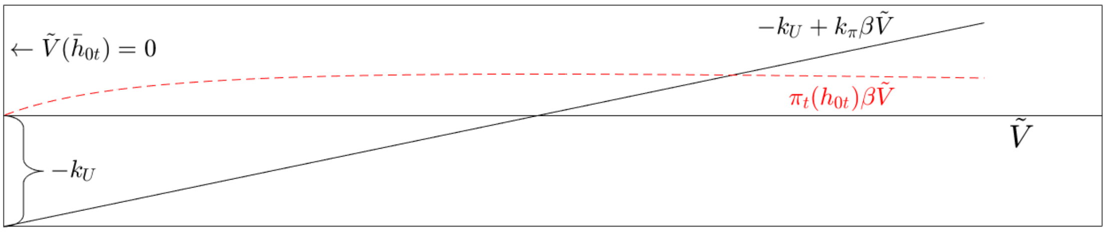
Figure 3. Benefit of creating new good versus specializing in existing good.



Notes to Figure 3. Horizontal axis: expected advantage of specialization ($E_t\tilde{V}_{t+1}$). Vertical axis: benefit to trying to create new good (solid black) and of specializing in existing good for two different values of π_{jt} (dashed red and dotted blue).

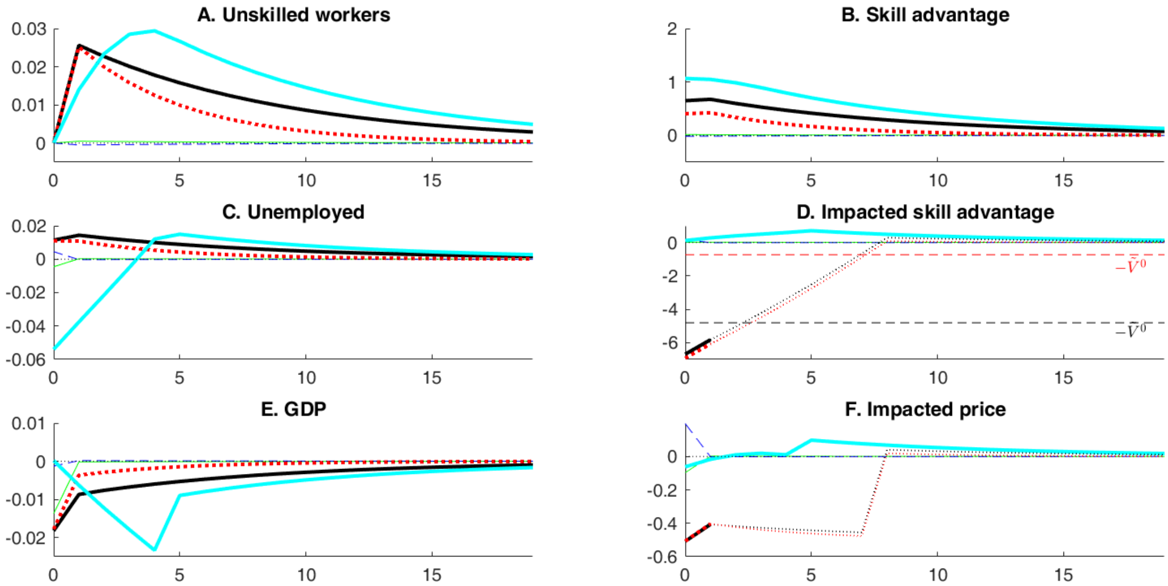
Figure 4. Value of specialization solved in terms of h_{0t} and $X_{1t}^*(h_{0t})$.





Notes to Figure 4. Each value of h_{0t} implies a steady-state fraction of nonspecialized labor and thus particular values of $X_{1t}^*(h_{0t})$ and $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$. Each point on the horizontal axis corresponds to a particular value of h_{0t} and its implied $X_{1t}^*(h_{0t})$ and $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$ with that value of $\tilde{V}$ plotted on the horizontal axis. Note that h_{0t} decreasing and X_{1t}^* increasing as we move to the right along the horizontal axis. The vertical axis plots the value of trying to create a new good (in black) or seeking to specialize in an existing good (in dashed red) as a function of that $\tilde{V}(h_{0t}, X_{1t}^*(h_{0t}))$. The two panels correspond to different parameter configurations depending on whether $\tilde{V}(1, X_{1t}^*(1))$ is positive (top panel) or nonpositive (bottom panel).

Figure 5. Effects of demand shocks.



Notes to Figure 5. Horizontal axis: number of periods since t_0. Vertical axis: difference of variable from value on the steady-state growth path. Panel A: fraction of population without a high-skilled, high-paying job ($n_{1t}^{\dagger}$). Panel B: lifetime advantage of specializing in goods that do not experience demand shock ($\tilde{V}_t^{\dagger}$). Panel C: fraction of unskilled workers who are unemployed ($-h_{1t}^{\dagger}$). Panel D: lifetime advantage of specializing in goods that experience demand shock ($\tilde{V}_t^{\chi\dagger}$). Horizontal lines in Panel D indicate levels at which goods cease production after t_0 if $\tilde{V}_{t_0+1}^{\chi\dagger}$ is below that value. Panel E: log of real GDP ($Q_t^{\dagger}$). Panel F: log of relative price that would maximize profits for goods that experience demand shocks ($P_{jt}^{\chi\dagger}$). Solid green (Example 9.1): 25% of specialized goods experience 10% drop in demand that only lasts for period t_0. Dashed blue (Example 9.2): 25% of specialized goods experience 10% increase in demand that only lasts for period t_0. Solid black (Example 9.3): 5% of specialized goods experience 50% drop in demand that would last for 8 periods if goods remained in production. Dotted red (Example 9.4): same as Example 9.3 except with lower level of technological frictions ($k_\pi = 0.6$). Solid cyan (Example 9.5): 25% of specialized goods and all newly created goods experience a 10% drop in demand that lasts for 5 periods.

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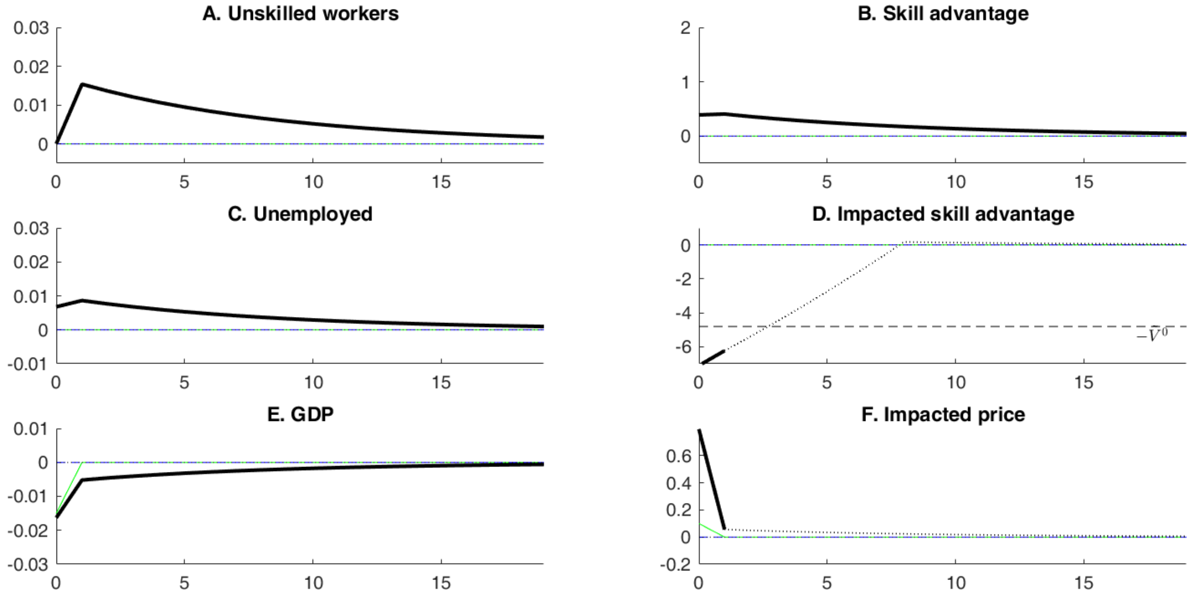
Figure 6. Effects of productivity shocks.



Notes to Figure 6. Horizontal axis: number of periods since t_0. Vertical axis: difference of variable from value on the steady-state growth path. Panel A: fraction of population without a high-skilled, high-paying job ($n_{1t}^{\dagger}$). Panel B: lifetime advantage of specializing in goods that do not experience productivity shock ($\tilde{V}_t^{\dagger}$). Panel C: fraction of unskilled workers who are unemployed ($-h_{1t}^{\dagger}$). Panel D: lifetime advantage of specializing in goods that experience productivity shock ($\tilde{V}_t^{\chi\dagger}$). Horizontal lines in Panel D indicate levels at which goods cease production after t_0 if $\tilde{V}_{t_0+1}^{\chi\dagger}$ is below that value. Panel E: log of real GDP ($Q_t^{\dagger}$). Panel F: log of relative price that would maximize profits for goods that experience productivity shocks ($P_{jt}^{\chi\dagger}$). Solid green (Example 10.1): 25% of specialized goods experience 10% drop in productivity that only lasts for period t_0. Dashed blue (Example 10.2): 25% of specialized goods experience 10% increase in productivity that only lasts for period t_0. Solid black (Example 10.3): 3% of specialized goods experience 80% drop in productivity that would last for 8 periods if goods remained in production.

Appendix A. Proofs of propositions

Proof of Proposition 1.

Substituting expressions (4)-(5) into (6),

$$\frac{\alpha_{jt}}{(q^0_{kjt})^2}(2q^0_{kjt} - q_{ijt}) = \lambda_{it} P_{jt} \quad i \in \mathcal{M}_{kt},\ j \in \mathcal{J}_t.$$

Averaging across all individuals in $\mathcal{M}_{kt}$ and using (9), $\alpha_{jt}/q^0_{kjt} = P_{jt} \int_{i\in\mathcal{M}_{kt}} \lambda_{it}/R_{kt}$ and

$$\alpha_{jt} = \left[\int_{i\in\mathcal{M}_{kt}} \lambda_{it}/R_{kt}\right]\left[P_{jt}\int_{i\in\mathcal{M}_{kt}} q_{ijt}/R_{kt}\right]. \tag{A1}$$

Summing across goods $j \in \mathcal{J}_t$ and using the fact that $\sum_{j\in\mathcal{J}_t} \alpha_{jt} = 1$,

$$1 = \left[\int_{i\in\mathcal{M}_{kt}} \lambda_{it}/R_{kt}\right]\left[\int_{i\in\mathcal{M}_{kt}} \sum\nolimits_{j\in\mathcal{J}_t} P_{jt}q_{ijt}/R_{kt}\right].$$

But since $\sum_{j\in\mathcal{J}_t} P_{jt}q_{ijt} = y_{it}$ for each i, $1 = \left[\int_{i\in\mathcal{M}_{kt}} \lambda_{it}/R_{kt}\right]\left[\int_{i\in\mathcal{M}_{kt}} y_{it}/R_{kt}\right]$, allowing (A1) to be written

$$\alpha_{jt} = \frac{P_{jt}\int_{i\in\mathcal{M}_{kt}} q_{ijt}/R_{kt}}{\int_{i\in\mathcal{M}_{kt}} y_{it}/R_{kt}}.$$

Proof of Proposition 2.

Let $z \sim U(R,S)$: $f(z) = (S-R)^{-1}$ for $z \in [R,S]$. Then:

(a)

$$P(z \geq z^*) = \int_{z^*}^{S} \frac{1}{S-R}dz = \left.\frac{z}{S-R}\right|_{z^*}^{S} = \frac{S-z^*}{S-R};$$

(b)

$$\begin{aligned}
\tilde{z} &= E(z|z \geq z^*)P(z \geq z^*) + z^*P(z < z^*) \\
&= \int_{z^*}^{S} \frac{z}{S-R}dz + z^*\int_{R}^{z^*} \frac{1}{S-R}dz = \frac{1}{S-R}\left.\frac{z^2}{2}\right|_{z^*}^{S} + \frac{z^*}{S-R}\,z|_R^{z^*} \\
&= \frac{S^2 - z^{*2}}{2(S-R)} + \frac{z^*(z^*-R)}{S-R} = \frac{S^2 - 2Rz^* + z^{*2}}{2(S-R)}
\end{aligned}$$

$$\frac{d\tilde{z}}{dz^*} = \frac{2z^* - 2R}{2(S-R)} > 0 \quad \forall z^* > R;$$

(c)

$$\int_{z^*}^{S} \frac{\exp(z)}{S-R}dz = \left.\frac{\exp(z)}{S-R}\right|_{z^*}^{S} = \frac{\exp(S) - \exp(z^*)}{S-R};$$

(d)

$$h_{1,t+1} = \frac{S_{t+1} - \log X^*_{1,t+1}}{S_{t+1} - R_{t+1}} = \frac{S_t + g - (\log X^*_{1t} + g)}{(S_t + g) - (R_t + g)} = \frac{S_t - \log X^*_{1t}}{S_t - R_t} = h_{1t};$$

(e)

$$\begin{aligned}\log \tilde{X}_{1,t+1} &= \frac{(S_t + g)^2 - 2(R_t + g)(\log X^*_{1t} + g) + (\log X^*_{1t} + g)^2}{2[(S_t + g) - (R_t + g)]} \\ &= \frac{S_t^2 - 2R_t \log X^*_{1t} + (\log X^*_{1t})^2}{2(S_t - R_t)} + \frac{2S_t g + g^2 - 2g \log X^*_{1t} - 2gR_t - 2g^2 + 2g \log X^*_{1t} + g^2}{2(S_t - R_t)} \\ &= \log \tilde{X}_{1t} + \frac{g(2S_t - 2R_t)}{2(S_t - R_t)} = \log \tilde{X}_{1t} + g;\end{aligned}$$

(f)

$$\hat{X}_{1,t+1} = \frac{\exp(S_t + g) - X^*_{1t} \exp(g)}{(S_t + g) - (R_t + g)} = \exp(g) \left[\frac{\exp(S_t) - X^*_{1t}}{S_t - R_t}\right] = \exp(g)\hat{X}_{1t}.$$

Proof of Proposition 3. Write expression (51) as

$$\frac{1 - n_{1t}}{n_{1t}} = \frac{k_\pi}{k_X e^n}(1 - h_{1t})h_{0t} \tag{A2}$$

and substitute this result into (42):

$$\begin{aligned}\tilde{V}(h_{0t}, X^*_{1t}) = \left[\frac{1}{1 - \beta(1 - k_X)}\right] &\left\{\log \left[\frac{(1 - \tau)(1 - \alpha_1)}{\alpha_1}\right]\right. \\ &\left. - \log \left[\frac{k_\pi}{k_X e^n}\right] - \log(1 - h_{1t}) - \log h_{0t} + \log \hat{X}_{1t} - \log \tilde{X}_{1t}\right\}.\end{aligned} \tag{A3}$$

Condition (45) can be written

$$h_{Yt}(X^*_{1t}) = -k_U + k_\pi \beta \tilde{V}(h_{0t}, X^*_{1t}) \tag{A4}$$

where $h_{Yt}(X^*_{1t})$ denotes the function of X^*_{1t} given in (44).

From rows (2) and (5) of Table 1, as $\log X^*_{1t}$ increases from R_t to S_t, the left side of (A4) monotonically increases from $-\infty$ to ∞. For fixed $h_{0t} > 0$, the right side monotonically decreases from ∞ to $-\infty$. Thus given any $h_{0t} \in (0, 1)$, there exists a unique $\log X^*_{1t} \in (R_t, S_t)$ at which condition (A4) holds, that is, for which conditions (51) and (45) simultaneously hold. Denote this solution $X^*_{1t}(h_{0t})$.

From (A3), a larger value of h_{0t} lowers the right side of (A4) and thus is associated with a lower value of X^*_{1t}: $\partial X^*_{1t}(h_{0t})/\partial h_{0t} < 0$. As $h_{0t} \to 0$, $-\log h_{0t} \to \infty$ and $\log(X^*_{1t}(h_{0t}))$ is driven to S_t. Since $h_{Yt}(X^*_{1t})$ in (44) is monotonically increasing in X^*_{1t} and since $X^*_{1t}(h_{0t})$ is monotonically decreasing in h_{0t}, it follows that $h_{Yt}(X^*_{1t}(h_{0t}))$ is a monotonically decreasing function of h_{0t}. By the definition of $X^*_{1t}(h_{0t})$, we know that

$$h_{Yt}(X^*_{1t}(h_{0t})) = -k_U + k_\pi \beta \tilde{V}(h_{0t}, X^*_{1t}(h_{0t})) \tag{A5}$$

holds for all h_{0t}. Monotonicity of the left side of (A5) as a function of h_{0t} implies that the right side is also a monotonically decreasing function of h_{0t}.

Next consider the incentives for applying for a position with continuing enterprises. Substituting (A2) into (48),

$$\pi_t(h_{0t}) = \frac{(1-k_X)(1-e^{-n})k_\pi}{k_X} \frac{h_{0t}}{(1-h_{0t})}, \tag{A6}$$

allowing us to write (49) as

$$-k_U + \beta k_\pi \tilde{V}(h_{0t}, X^*_{1t}(h_{0t})) = \beta \pi_t(h_{0t}) \tilde{V}(h_{0t}, X^*_{1t}(h_{0t})). \tag{A7}$$

Recalling that $\tilde{V}(h_{0t}, X^*_{1t}(h_{0t}))$ is a monotonically decreasing function of h_{0t}, consider two cases. Suppose first that $\tilde{V}(h_{0t}, X^*_{1t}(h_{0t}))$ is positive at its lowest point ($h_{0t} = 1$). Note from (A6) that $\pi_t = 1$ at this point. With $\tilde{V}$ positive and $\pi_t = 1 > k_\pi$, the right side of (A7) must be larger than the left side at the lowest possible value for $\tilde{V}$, namely $\tilde{V}(1, X^*_{1t}(1))$. As h_{0t} decreases below 1, $\tilde{V}$ monotonically increases and π_t monotonically decreases, the latter eventually reaching 0 as $h_{0t} \to 0$. Thus there exists a unique value $h^0_{0t} \in (0,1)$ at which (A7) holds; see the top panel of Figure 4. This h^0_{0t} implies a unique $X^*_{1t}(\bar{h}_{0t})$, a unique $h_{1t}(X^*_{1t}(\bar{h}_{0t})$ and thus a unique $n_{1t}(\bar{h}_{0t})$ from (A2). By construction $(X^*_{1t}(h^0_{0t}, n_{1t}(h^0_{0t}), h^0_{0t})$ satisfy (50), (49) and (45).

Alternatively, suppose that $\tilde{V}(1, X^*_{1t}(1))$ is negative (see the bottom panel of Figure 4). Since $\tilde{V}$ is monotonically decreasing in h_{0t} and goes to ∞ as $h_{0t} \to 0$, there exists a unique $\bar{h}_{0t} \in (0,1)$ at which $\tilde{V}(\bar{h}_{0t}, X^*_{1t}(\bar{h}_{0t})) = 0$. At this point the right side of (A7) is zero and the left side is negative. As h_{0t} decreases below $\bar{h}_{0t}$, the value of $\tilde{V}$ increases without bound while the magnitude $\pi_t(h_{0t})$ eventually goes to 0. Thus there again exists a unique h^0_{0t} for which condition (A7) holds and for which $(X^*_{1t}(h^0_{0t}, n_{1t}(h^0_{0t}), h^0_{0t})$ simultaneously satisfies (50), (49) and (45).

Proof of Proposition 4.

(a) Let $(X^*_{1t_0}, n^0_1, h^0_0)$ be the unique solution to (45), (49) and (50) for date t_0. Then $(e^g X^*_{1t_0}, n^0_1, h^0_0)$ solve these three equations for date t_0+1, as can be verified as follows. From proposition 2, $X^*_{1,t_0+1} = e^g X^*_{1t_0}$ would imply $h_{1,t_0+1} = h^0_1 = (S_{t_0} - \log X^*_{1t_0})/(S_{t_0} - R_{t_0})$, $\log \hat{X}_{1,t_0+1} = g + \log \hat{X}_{1t_0}$ and $\log \tilde{X}_{1,t_0+1} = g + \log \tilde{X}_{1t_0}$ establishing from (44) that $h_{Y,t_0+1} = h_{Yt_0}$ and from (42) that $\tilde{V}_{t_0+1} = \tilde{V}_{t_0}$. Hence (45), (49) and (50) are all satisfied at date t_0+1, confirming that $(e^g X^*_{1t_0}, n^0_1, h^0_0)$ is the solution. By induction, $(e^{g(t-t_0)} X^*_{1t_0}, n^0_1, h^0_0)$ is the solution for all t.

(b) Of the $J_{2t_0} = k_J/k_X$ goods at initial date t_0, $k_X J_{2t_0} = k_J$ will no longer be produced beginning in $t_0 + 1$. And since $h_{0t_0} > 0$, k_J new goods (one of each type) will begin being produced in $t_0 + 1$. Thus $J_{2,t_0+1} = J_{2t_0}$ and by induction J_{2t} is constant for all t.

(c) Along the steady-state growth path, a fraction $(1 - \alpha_1)(1 - \tau)$ of total income Y_t is earned by skilled workers and the remaining $[\alpha_1 + \tau(1 - \alpha_1)]Y_t$ is received by unskilled. Each of these groups on average spends a fraction α_{jt} of their income on good j. Since $n_{jt}N_tX_{jt}$ units of good j get produced, $(1 - \alpha_1)(1 - \tau)n_{jt}N_tX_{jt}$ units of good j are consumed by specialists and the remaining $[\alpha_1 + \tau(1 - \alpha_1)]n_{jt}N_tX_{jt}$ by nonspecialists. Dividing the first expression by the total number of skilled workers $(1 - n_{1t})N_t$ gives result (c). Result (i) below verifies that this is in fact the same number for all skilled workers.

(d) Dividing nonspecialist total spending $[\alpha_1 + \tau(1 - \alpha_1)]n_{jt}N_tX_{jt}$ by the total number of these individuals $n_{1t}N_t$ gives result (d). Since productivities x_{it} are drawn independently over time, this is the average spending by nonspecialists and is the level of consumption for nonspecialists q^0_{njt} along the steady-state growth path.

(e) Since total demand for good j in (8) is the sum across a growing population N_t, it grows at rate n from population growth plus g from productivity growth. Thus (46) requires $n_{jt}N_t$ to grow at rate n implying that n_{jt} is constant at n_j. From expression (35) this implies that α_{jt} is constant as well.

(f) The ratio of nominal spending on good j to that for good 1 is $(P_{jt}Q_{jt})/(P_{1t}Q_{1t}) = \alpha_{jt}/\alpha_1$. Since $Q_{jt} = n_{jt}N_tX_{jt}$, $P_{jt}/P_{1t} = (\alpha_{jt}n_{1t}\hat{X}_{1t})/(\alpha_1 n_{jt}X_{jt})$. Since $\hat{X}_{1t}$ and X_{jt} both grow at rate g, the ratio $\hat{X}_{1t}/X_{jt}$ is constant over time, as is α_{jt}/n_{jt} from results (c) and (e).

(g) This follows immediately from applying results (c) and (d) to expression (8).

(h) From results (c) and (d) and the constant population growth we see that

$$\frac{\bar{Q}_{j,t+1}}{2} = [n^0_1 q^0_{njt}e^g + (1 - n^0_1)q^0_{sjt}e^g]N_te^n = \frac{\bar{Q}_{jt}}{2}e^{g+n}.$$

Applying (46) and taking logs establishes result (h).

(i) Total spending on good j is $P_{jt}Q_{jt} = \alpha_{jt}Y_t$, so the after-tax income per person producing good j is

$$\frac{(1 - \tau)P_{jt}Q_{jt}}{n_{jt}N_t} = \frac{\alpha_{jt}Y_t(1 - \tau)}{n_{jt}N_t}. \tag{A8}$$

Equation (52) establishes that at date t_0 this magnitude is

$$\frac{(1 - \tau)P_{jt_0}Q_{jt_0}}{n_{jt_0}N_{t_0}} = \frac{Y_{t_0}(1 - \tau)(1 - \alpha_1)}{(1 - n^0_1)N_{t_0}}$$

which is the same for all $j \in \mathcal{J}_{2t_0}$. Thus the stated initial conditions imply that all skilled workers earn the same income at date t_0. The income for a worker producing good j at date

t is $P_{jt}X_{jt}$, which from result (f) is $e^{g(t-t_0)}$ times the income that individual received at date t_0, the same constant factor for each j.

For goods that are produced for the first time in period t, substituting condition (34) into (A8) gives

$$\frac{(1-\tau)P_{jt}Q_{jt}}{n_{jt}N_t} = \frac{(1-\alpha_1)Y_t(1-\tau)}{(1-n_1^0)N_t} \quad j \in \mathcal{J}_{2t}^{\sharp},$$

which is the same for each $j \in \mathcal{J}_{2t}^{\sharp}$ and the same as the income received by those producing continuing specialized goods.

Proof of Proposition 5.

(a) Note that

$$\bar{q}_{ijt} = \begin{cases} \chi_{jt}q_{sjt}^0 & \text{for } i \in \mathcal{M}_{st} \\ \chi_{jt}q_{njt}^0 & \text{for } i \in \mathcal{M}_{nt} \end{cases}$$

where $\mathcal{M}_{st}$ and $\mathcal{M}_{nt}$ denote the sets of skilled and unskilled workers, respectively. From (4), $\bar{q}_{ijt} = 2\chi_{jt}q_{ijt}^0$ and (8) and Proposition 4c-e then give

$$\begin{aligned} \bar{Q}_{jt} &= 2\chi_{jt}[n_{1t}q_{njt}^0 + (1-n_{1t})q_{sjt}^0]N_t \\ &= 2\chi_{jt}\left[\frac{n_{1t}[\alpha_1+\tau(1-\alpha_1)}{n_1^0} + \frac{(1-n_{1t})(1-\alpha_1)(1-\tau)}{1-n_1^0}\right] n_j^0 X_{jt}^0 N_t \end{aligned} \tag{A9}$$

establishing (58).

(b)

$$\begin{aligned} \frac{\partial H_t}{\partial n_{1t}} &= \frac{\alpha_1+\tau(1-\alpha_1)}{n_1^0} - \frac{1-[\alpha_1+\tau(1-\alpha_1)]}{1-n_1^0} \\ &= \frac{(1-n_1^0)[\alpha_1+\tau(1-\alpha_1)] - n_1^0 + n_1^0[\alpha_1+\tau(1-\alpha_1)]}{n_1^0(1-n_1^0)} = \left[\frac{\alpha_1+\tau(1-\alpha_1)-n_1^0}{n_1^0(1-n_1^0)}\right] \end{aligned} \tag{A10}$$

whose sign is the same as that of $\alpha_1+\tau(1-\alpha_1)-n_1^0$, as claimed.

The inequality $[\alpha_1+\tau(1-\alpha_1)] < n_1^0$ holds whenever the average income of the unskilled is less than the average income of the skilled. We know from Proposition 3 that the average log income of the unskilled is less than the average log income of the skilled. Online Appendix C establishes sufficient conditions to bound Jensen's Inequality to ensure that this also means that the average income of the unskilled is less the average income of the skilled.

(c) This simply restates (26) and (11).

(d) Consumer i's first-order condition (6) is

$$\gamma_{ijt}(\bar{q}_{ijt} - q_{ijt}) = \lambda_{it}P_{jt}. \tag{A11}$$

For $\mathcal{M}_{nt}$ the set of unskilled individuals and any $i \in \mathcal{M}_{nt}$, we have $\gamma_{ijt} = \xi_{jt}\alpha_{jt}/(q_{njt}^0)^2$ and $\bar{q}_{ijt} = 2\chi_{jt}q_{njt}^0$, both constant across $i \in \mathcal{M}_{nt}$. Taking the expected value of (A11) across

$i \in \mathcal{M}_{nt}$ gives

$$\frac{\xi_{jt}\alpha_{jt}}{(q^0_{njt})^2}(2\chi_{jt}q^0_{njt} - q_{njt}) = \lambda_{nt}P_{jt} \tag{A12}$$

where $q_{njt} = E(q_{ijt}|i \in \mathcal{M}_{nt})$ and $\lambda_{nt} = E(\lambda_{it}|i \in \mathcal{M}_{nt})$. Dividing (A12) by the value for $j = 1$,

$$\frac{\xi_{jt}\alpha_{jt}(2\chi_{jt}q^0_{njt} - q_{njt})}{\alpha_1(2q^0_{n1t} - q_{n1t})} = \frac{P_{jt}(q^0_{njt})^2}{P_{1t}(q^0_{n1t})^2}. \tag{A13}$$

From Proposition 4d and 4f,

$$\frac{q^0_{njt}}{q^0_{n1t}} = \frac{n^0_j X^0_{jt}}{n^0_1 \hat{X}^0_{1t}} = \frac{\alpha^0_j P^0_1}{\alpha_1 P^0_j}.$$

Substituting this into (A13) gives

$$\left(\frac{\alpha_{jt}}{\alpha^0_j}\right)\xi_{jt}(2\chi_{jt}q^0_{njt} - q_{njt}) = \frac{\alpha^0_j}{\alpha_1}\left(\frac{P^0_1}{P^0_j}\right)^2(2q^0_{n1t} - q_{n1t})\left(\frac{P_{jt}}{P_{1t}}\right). \tag{A14}$$

Analogous calculations for skilled workers give

$$\left(\frac{\alpha_{jt}}{\alpha^0_j}\right)\xi_{jt}(2\chi_{jt}q^0_{sjt} - q_{sjt}) = \frac{\alpha^0_j}{\alpha_1}\left(\frac{P^0_1}{P^0_j}\right)^2(2q^0_{s1t} - q_{s1t})\left(\frac{P_{jt}}{P_{1t}}\right). \tag{A15}$$

Note that total consumption of good j is given by

$$Q_{jt} = [n_{1t}q_{njt} + (1 - n_{1t})q_{sjt}]\,N_t. \tag{A16}$$

Multiplying (A14) by $n_{1t}N_t$, adding the result to $(1 - n_{1t})N_t$ times (A15) and using (A9) and (A16), we conclude

$$\left(\frac{\alpha_{jt}}{\alpha^0_j}\right)\xi_{jt}(\bar{Q}_{jt} - Q_{jt}) = \frac{\alpha^0_j}{\alpha_1}\left(\frac{P^0_1}{P^0_j}\right)^2(\bar{Q}_{1t} - Q_{1t})\left(\frac{P_{jt}}{P_{1t}}\right).$$

Rearranging gives (62).

To check this expression, as a special case, at a point on the steady-state path, $\xi_{jt} = 1$ and $(\bar{Q}_{jt} - Q_{jt}) = n^0_j N_t X^0_{jt}$, for which (62) would become $P_{jt}/P_{1t} = (P^0_j/P^0_1)^2(\alpha_1 n^0_j N_t X^0_{jt})/(\alpha^0_j n^0_1 N_t \hat{X}^0_{1t}) = P^0_j/P^0_1$ with the last equality following from (55).

(e) This is obtained by taking the ratio of (34) or (35) to the steady-state value.

(f) Expression (65) follows from (58):

$$\frac{\bar{Q}_{j,t+1}/2}{X_{j,t+1}} = \frac{\chi_{j,t+1}H_{t+1}n^0_j X^0_{j,t+1}N_{t+1}}{X_{j,t+1}} = \chi_{j,t+1}H_{t+1}\left(\frac{X^0_{j,t+1}}{X_{j,t+1}}\right)N^0_{j,t+1}.$$

(g) From (62) and (63),

$$Y_{jt} = (1-\tau)P_{1t}\left(\frac{P_j^0}{P_1^0}\right)^2\left(\frac{\alpha_1}{\alpha_j^0}\right)\left(\frac{n_{jt}(1-n_1^0)}{n_j^0(1-n_{1t})}\right)\xi_{jt}\left[\frac{\bar{Q}_{jt}-Q_{jt}}{\bar{Q}_{1t}-Q_{1t}}\right]\frac{Q_{jt}}{n_{jt}N_t}. \tag{A17}$$

From (55) we know

$$\frac{P_j^0}{P_1^0} = \frac{\alpha_j^0 n_1^0 \hat{X}_{1t}^0}{\alpha_1 n_j^0 X_{jt}^0} = \frac{\alpha_j^0 Q_{1t}^0}{\alpha_1 Q_{jt}^0} \tag{A18}$$

allowing (A17) to be written

$$\frac{Y_{jt}}{P_{1t}} = (1-\tau)\frac{Q_{1t}^0}{Q_{jt}^0}\frac{P_j^0}{P_1^0}\frac{1-n_1^0}{1-n_{1t}}\xi_{jt}\left[\frac{\bar{Q}_{jt}-Q_{jt}}{\bar{Q}_{1t}-Q_{1t}}\right]\frac{Q_{jt}}{n_j^0 N_t}. \tag{A19}$$

Using (57) we can also conclude from (A18) that $P_j^0/P_1^0 = [(1-\alpha_1)n_1^0\hat{X}_{1t}^0]/[\alpha_1(1-n_1^0)X_{jt}^0]$. Substituting this into (A19) and rearranging,

$$\frac{Y_{jt}}{P_{1t}} = (1-\tau)\frac{(1-\alpha_1)n_1^0}{\alpha_1(1-n_1^0)}\frac{1-n_1^0}{1-n_{1t}}\xi_{jt}\frac{(\bar{Q}_{jt}-Q_{jt})/Q_{jt}^0}{(\bar{Q}_{1t}-Q_{1t})/Q_{1t}^0}\frac{Q_{jt}}{n_j^0 N_t X_{jt}^0}\hat{X}_{1t}^0. \tag{A20}$$

Along the steady-state growth path, $n_{1t} = n_1^0$, $\xi_{jt} = 1$, $\bar{Q}_{jt} - Q_{jt} = Q_{jt}^0$, $\bar{Q}_{1t} - Q_{1t} = Q_{1t}^0$, and $Q_{jt}^0 = n_j^0 N_t X_{jt}^0$. Thus from (A20) the steady-state real income of skilled workers is given by (67). Substituting (67) into (A20) gives (66).

Results (h)-(l) restate expressions from elsewhere in the paper.

(m) Notice from $\alpha_j^0 = P_{jt}^0 Q_{jt}^0 / \sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0$ that

$$Q_t = \frac{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0 (Q_{jt}/Q_{jt}^0)}{\sum_{j\in\mathcal{J}_t} P_{jt}^0 Q_{jt}^0} = \sum\nolimits_{j\in\mathcal{J}_t} \alpha_j^0 (Q_{jt}/Q_{jt}^0).$$

Recall also that for $j \in \mathcal{J}_{2t}$, $Q_{jt}^0 = n_j^0 N_t X_{jt}^0$ and from (57) that $\alpha_j^0/n_j^0 = (1-\alpha_1)/(1-n_1^0)$. Using these results along with (26) we conclude that

$$\begin{aligned}\sum\nolimits_{j\in\mathcal{J}_t} \alpha_j^0 (Q_{jt}/Q_{jt}^0) &= \sum\nolimits_{j\in\mathcal{J}_{2t}}\left(\frac{\alpha_j^0}{n_j^0}\right)\left(\frac{Q_{jt}}{N_t X_{jt}^0}\right) + \alpha_1\left(\frac{Q_{1t}}{Q_{1t}^0}\right)\\ &= \left(\frac{1-\alpha_1}{1-n_1^0}\right)\sum\nolimits_{j\in\mathcal{J}_{2t}}\left(\frac{Q_{jt}}{N_t X_{jt}^0}\right) + \left(\frac{\alpha_1}{n_1^0}\right)\left(\frac{\hat{X}_{1t}}{\hat{X}_{1t}^0}\right)n_{1t}.\end{aligned}$$

Note that if $\chi_{jt} = \zeta_{jt} = 1$ and $Q_{jt} = \bar{Q}_{jt}/2$, then $Q_{jt} = H_t n_j^0 N_t X_{jt}^0$ so $Q_{jt}/(N_t X_{jt}^0) = H_t n_j^0$ and (75) becomes (76).

Appendix B. Details on adjustment dynamics (online)

Define $w_t^\dagger = \log w_t - \log w_t^0$ for $w_t = \bar{Q}_{jt}, Q_{jt}, X_{1t}^*, Y_{jt}, C_t, X_{jt}, Q_t, \chi_{jt}, \xi_{jt}, \zeta_{jt}$ (recalling that $\log Q_t^0 = \log \chi_{jt}^0 = \log \xi_{jt}^0 = 0$); $w_t^\dagger = w_t - w^0$ for $w_t = \alpha_{jt}, n_{jt}, h_{0t}, \tilde{V}_{jt}$; $P_{jt}^\dagger = \log(P_{jt}/P_{1t}) - \log(P_{jt}^0/P_{1t}^0)$; $\tilde{Y}_{jt}^\dagger = \log Y_{jt} - \log Y_{jt}^0 - [\log(P_{1t}X_{1t}^*) - \log(P_{1t}^0\tilde{X}_{1t}^0)]$; $\lambda_2, \lambda_3, \lambda_5$ the derivatives in Table 1, λ_H the derivative in Proposition 5A.

Linearized version of Proposition 5.

Evaluating derivatives of Proposition 5 along the steady-state growth path and taking deviations from steady state results in

$$\bar{Q}_{jt}^\dagger = \chi_{jt}^\dagger + \lambda_H n_{1t}^\dagger \quad j \in \mathcal{J}_t \tag{B1}$$

$$Q_{1t}^\dagger = \frac{1}{n_1^0} n_{1t}^\dagger + \lambda_5 X_{1t}^{*\dagger} \tag{B2}$$

$$Q_{jt}^\dagger = \begin{cases} \chi_{jt}^\dagger + \lambda_H n_{1t}^\dagger & \text{if } j \in \mathcal{J}_{2t} \text{ and } \bar{Q}_{jt}/2 \leq n_{jt}N_tX_{jt} \\ \frac{n_{jt}^\dagger}{n_j^0} + \zeta_{jt}^\dagger & \text{if } j \in \mathcal{J}_{2t} \text{ and } \bar{Q}_{jt}/2 > n_{jt}N_tX_{jt} \end{cases} \tag{B3}$$

$$P_{jt}^\dagger = \frac{\alpha_{jt}^\dagger}{\alpha_j^0} + \xi_{jt}^\dagger + 2\chi_{jt}^\dagger - Q_{jt}^\dagger + \frac{n_{1t}^\dagger}{n_1^0} + \lambda_5 X_{1t}^{*\dagger} \quad j \in \mathcal{J}_{2t} \tag{B4}$$

$$\frac{\alpha_{jt}^\dagger}{\alpha_j^0} = \frac{n_{jt}^\dagger}{n_j^0} + \frac{1}{1-n_1^0} n_{1t}^\dagger \quad j \in \mathcal{J}_{2t} \tag{B5}$$

$$n_{j,t+1}^\dagger = \begin{cases} n_j^0 \chi_{j,t+1}^\dagger + n_j^0 \lambda_H n_{1,t+1}^\dagger - n_j^0 X_{j,t+1}^\dagger & \text{if } j \in \mathcal{J}_{2t} \text{ and } \bar{Q}_{j,t+1}/2 \geq X_{j,t+1}N_{jt} \\ n_{jt}^\dagger - n & \text{if } j \in \mathcal{J}_{2t} \text{ and } \bar{Q}_{j,t+1}/2 < X_{j,t+1}N_{jt} \end{cases} \tag{B6}$$

$$Y_{jt}^\dagger = \xi_{jt}^\dagger + \frac{1}{n_1^0(1-n_1^0)} n_{1t}^\dagger + 2\chi_{jt}^\dagger + \lambda_5 X_{1t}^{*\dagger} \quad j \in \mathcal{J}_{2t} \tag{B7}$$

$$\tilde{V}_{jt}^\dagger = Y_{jt}^\dagger - \lambda_3 X_{1t}^{*\dagger} + \beta(1-k_X)\tilde{V}_{j,t+1}^\dagger \quad j \in \mathcal{J}_{2t}. \tag{B8}$$

To derive result (B4) we used the fact that for all $j \in \mathcal{J}_t$, $\bar{Q}_{jt}^0 = 2Q_{jt}^0$ establishing

$$\begin{aligned} \log(\bar{Q}_{jt} - Q_{jt}) &\simeq \log Q_{jt}^0 + \frac{1}{Q_{jt}^0}[(\bar{Q}_{jt} - \bar{Q}_{jt}^0) - (Q_{jt} - Q_{jt}^0)] \\ &= \log Q_{jt}^0 + \frac{2}{\bar{Q}_{jt}^0}(\bar{Q}_{jt} - \bar{Q}_{jt}^0) - \frac{Q_{jt} - Q_{jt}^0}{Q_{jt}^0} \\ &= \log Q_{jt}^0 + 2\bar{Q}_{jt}^\dagger - Q_{jt}^\dagger \end{aligned} \tag{B9}$$

$$P_{jt}^\dagger = \frac{\alpha_{jt}}{\alpha_j^0} + \xi_{jt}^\dagger + 2\bar{Q}_{jt}^\dagger - Q_{jt}^\dagger - 2\bar{Q}_{1t}^\dagger + Q_{1t}^\dagger.$$

Result (B4) then follows from (B1) and (B9). Similarly to derive (B7) we used (B9) along with

$$Y^\dagger_{jt} = \xi^\dagger_{jt} + \frac{1}{1-n^0_1} n^\dagger_{1t} + Q^\dagger_{jt} + 2\bar{Q}^\dagger_{jt} - Q^\dagger_{jt} - 2\bar{Q}^\dagger_{1t} + Q^\dagger_{1t}.$$

For $\tilde{Y}_{jt} = \log Y_{jt} - \log(P_{1t}\tilde{X}_{1t})$ and $\tilde{Y}^\dagger_{jt} = Y^\dagger_{jt} - \tilde{X}^\dagger_{1t} = Y^\dagger_{jt} - \lambda_3 X^{*\dagger}_{1t}$ it follows from (B7) that

$$\tilde{Y}^\dagger_{jt} = \xi^\dagger_{jt} + 2\chi^\dagger_{jt} + \frac{1}{n^0_1(1-n^0_1)} n^\dagger_{1t} + (\lambda_5 - \lambda_3) X^{*\dagger}_{1t} \quad j \in \mathcal{J}_{2t}. \tag{B10}$$

Details for Examples 9.1-9.4.

Summary of linearized system. The equations entering (102) are :

$$-n^{\sharp\dagger}_{t+1} - n^{\natural\dagger}_{t+1} = n^\dagger_{1,t+1} \tag{B11}$$

$$\tilde{V}^\dagger_t - Y^\dagger_t + \lambda_3 X^{*\dagger}_{1t} = \beta(1-k_X)\tilde{V}^\dagger_{t+1} \tag{B12}$$

$$(1-k_X)\bar{n}^\dagger_t + n^{\sharp\dagger}_{t+1} = \bar{n}^\dagger_{t+1} \tag{B13}$$

$$\frac{1}{n^0_1(1-n^0_1)} n^\dagger_{1t} - Y^\dagger_t + \lambda_5 X^{*\dagger}_{1t} = 0 \tag{B14}$$

$$\frac{1}{n^0_1(1-n^0_1)} n^\dagger_{1t} - Y^\dagger_t + C^\dagger_t + \lambda_2 X^{*\dagger}_{1t} = 0 \tag{B15}$$

$$-C^\dagger_t + X^{*\dagger}_{1t} = \beta k_\pi \tilde{V}^\dagger_{t+1} \tag{B16}$$

$$-C^\dagger_t + X^{*\dagger}_{1t} - \beta\tilde{V}^0 \pi^\dagger_t = \beta\pi^0 \tilde{V}^\dagger_{t+1} \tag{B17}$$

$$\begin{aligned} & e^{-n} k_\pi \left[1 - h^0_1 - \frac{1-k_X}{\pi^0}\right] n^\dagger_{1t} + e^{-n}(1-h^0_1) n^0_1 k_\pi \lambda_2 X^{*\dagger}_{1t} \\ & + \frac{[n^{\natural 0} - e^{-n}(1-k_X)(1-n^0_1)]k_\pi}{(\pi^0)^2} \pi^\dagger_t - n^{\sharp\dagger}_{t+1} - \frac{k_\pi}{\pi^0} n^{\natural\dagger}_{t+1} = 0 \end{aligned} \tag{B18}$$

$$-(1-k_X)\bar{n}^\dagger_t + n^{\natural\dagger}_{t+1} = (1-k_X)(1-n^0_1)\lambda_H n^\dagger_{1,t+1} \tag{B19}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 1-k_X & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{n^0_1(1-n^0_1)} & 0 & 0 & -1 & 0 & \lambda_5 & 0 & 0 & 0 \\ \frac{1}{n^0_1(1-n^0_1)} & 0 & 0 & -1 & 1 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -\beta\tilde{V}^0 & 0 & 0 \\ b_{81} & 0 & 0 & 0 & 0 & b_{86} & b_{87} & -1 & -k_\pi/\pi^0 \\ 0 & 0 & -(1-k_X) & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{B20}$$

$$b_{81} = e^{-n} k_\pi \left[1 - h_1^0 - \frac{1 - k_X}{\pi^0} \right] \quad \text{(B21)}$$

$$b_{86} = e^{-n}(1 - h_1^0) n_1^0 k_\pi \lambda_2 \quad \text{(B22)}$$

$$b_{87} = \frac{[n^{\natural 0} - e^{-n}(1 - k_X)(1 - n_1^0)] k_\pi}{(\pi^0)^2} \quad \text{(B23)}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \beta(1 - k_X) & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta k_\pi & 0 \\ 0 & \beta \pi^0 & 0 \\ 0 & 0 & 0 \\ (1 - k_X)(1 - n_1^0)\lambda_H & 0 & 0 \end{bmatrix}. \quad \text{(B24)}$$

For Examples 9.1-9.2, $M = 1$, $\dot{z}_{1t}^\dagger = z_{1t}^\dagger$, $\dot{z}_{2t}^\dagger = z_{2t}^\dagger$, $\dot{B}_{t_0} = B$, $\dot{A}_{t_0} = A$, $\dot{c}_{t_0} = (0, 0, 0, 0, -2(\chi - 1)\kappa, 0, 0, 0, 0)'$. Examples 9.3 and 9.4 are the same except

$$\dot{c}_{3t_0} = -\kappa(1 - k_X)(1 - n_1^0)$$

$$\dot{c}_{5t_0} = -2(\chi - 1)\kappa$$

$$\dot{c}_{8t_0} = \frac{-\kappa(1 - k_X)(1 - n_1^0) k_\pi}{\pi^0}$$

$$\dot{c}_{9t_0} = \kappa(1 - k_X)(1 - n_1^0)$$

$$\dot{b}'_{8t_0} = b'_{8t_0} + (\kappa e^{-n}(1 - k_X) k_\pi / \pi^0, 0, 0, 0, 0, 0, \kappa e^{-n}(1 - k_X)(1 - n_1^0) k_\pi / (\pi^0)^2, 0, 0)$$

$$\dot{b}'_{9t_0} = b'_{9t_0} + (0, 0, \kappa(1 - k_X), 0, 0, 0, 0, 0, 0)$$

$$\dot{a}'_{9t_0} = a_{9t_0} + (-\kappa(1 - k_X)(1 - n_1^0)\lambda_H, 0, 0)$$

where for example $\dot{b}'_{8t_0}$ and b'_{8t_0} denotes the 8th rows of $\dot{B}_{t_0}$ and B_{t_0}, respectively.

Details of solution algorithm. Expression (105) can be written in partitioned form as

$$\begin{bmatrix} \underset{(3\times3)}{\dot{A}_{11,t}} \\ \underset{(6\times3)}{\dot{A}_{21,t}} \end{bmatrix} \dot{z}_{1,t+1}^\dagger = \begin{bmatrix} \underset{(3\times3)}{\dot{B}_{11,t}} & \underset{(3\times6)}{\dot{B}_{12,t}} \\ \underset{(6\times3)}{\dot{B}_{21,t}} & \underset{(6\times6)}{\dot{B}_{22,t}} \end{bmatrix} \begin{bmatrix} \dot{z}_{1t}^\dagger \\ \dot{z}_{2t}^\dagger \end{bmatrix} + \begin{bmatrix} \underset{(3\times1)}{\dot{c}_{1t}} \\ \underset{(6\times1)}{\dot{c}_{2t}} \end{bmatrix}. \quad \text{(B25)}$$

Premultiplying by $\left[\begin{array}{cc} I_3 & -\dot{B}_{12,t}\dot{B}_{22,t}^{-1} \end{array} \right]$ results in

$$\underset{(3\times3)}{\hat{A}_{11,t}}\underset{(3\times1)}{\dot{z}_{1,t+1}^{\dagger}} = \underset{(3\times3)}{\hat{B}_{11,t}}\ \underset{(3\times1)}{\dot{z}_{1t}^{\dagger}} + \underset{(3\times1)}{\hat{c}_{1t}} \tag{B26}$$

where $\hat{B}_{11,t} = \dot{B}_{11,t} - \dot{B}_{12,t}\dot{B}_{22,t}^{-1}\dot{B}_{21,t}$, $\hat{A}_{11,t} = \dot{A}_{11,t} - \dot{B}_{12,t}\dot{B}_{22,t}^{-1}\dot{A}_{21,t}$, $\hat{c}_{1t} = \dot{c}_{1t} - \dot{B}_{12,t}\dot{B}_{22,t}^{-1}\dot{c}_{2t}$. Expression (B26) can in turn be written as (106) for $\dot{\Phi}_t = \hat{A}_{11,t}^{-1}\hat{B}_{11,t}$, $\dot{e}_t = \hat{A}_{11,t}^{-1}\hat{c}_{1t}$. Analogous operations allow us to rewrite (102) in the form (103).

Expression (106) implies that

$$\dot{z}_{1,t_0+M}^{\dagger} = \dot{\Lambda}_M \dot{z}_{1t_0}^{\dagger} + \dot{g}_M. \tag{B27}$$

Here $\dot{\Lambda}_m$ and $\dot{g}_M$ are found by iterating on $\dot{\Lambda}_\ell = \dot{\Phi}_{t_0+\ell-1}\dot{\Lambda}_{\ell-1}$ for $\ell = 1, ..., M$ starting from $\dot{\Lambda}_0 = I_2$ and $\dot{g}_\ell = \dot{e}_{t_0+\ell-1} + \dot{\Phi}_{t_0+\ell-1}\dot{g}_{\ell-1}$ starting from $\dot{g}_0 = 0$. Let v_1 and v_2 be the eigenvectors of Φ in (103) associated with the eigenvalues below 1. The rational-expectations solution requires $z_{1,t_0+M}^{\dagger} = \delta_1 v_1 + \delta_2 v_2$ for some scalars δ_1 and δ_2. Recall that for Examples 9.1-9.4, $\dot{z}_t^{\dagger} = z_t^{\dagger}$ and partition $\dot{\Lambda}_M = \left[\begin{array}{ccc} \dot{\lambda}_{M1} & \dot{\lambda}_{M2} & \dot{\lambda}_{M3} \end{array} \right]$. The rational-expectations solution thus requires

$$\dot{\lambda}_{M1} n_{1t_0}^{\dagger} + \dot{\lambda}_{M2} \tilde{V}_{t_0}^{\dagger} + \dot{\lambda}_{M3} \bar{n}_{t_0}^{\dagger} + \dot{g}_M = \delta_1 v_1 + \delta_2 v_2$$

$$\left[\begin{array}{ccc} v_1 & v_2 & -\dot{\lambda}_{M2} \end{array} \right] \left[\begin{array}{c} \delta_1 \\ \delta_2 \\ \tilde{V}_{t_0}^{\dagger} \end{array} \right] = \dot{\lambda}_{M1} n_{1t_0}^{\dagger} + \dot{\lambda}_{M3} \bar{n}_{t_0}^{\dagger} + \dot{g}_M$$

$$\left[\begin{array}{c} \delta_1 \\ \delta_2 \\ \tilde{V}_{t_0}^{\dagger} \end{array} \right] = \left[\begin{array}{ccc} v_1 & v_2 & -\dot{\lambda}_{M2} \end{array} \right]^{-1} (\dot{\lambda}_{M1} n_{1t_0}^{\dagger} + \dot{\lambda}_{M3} \bar{n}_{t_0}^{\dagger} + \dot{g}_M). \tag{B28}$$

Given the specified initial values for $n_{1t_0}^{\dagger}$ and $\bar{n}_{t_0}^{\dagger}$, the third element of this vector is the rational-expectations solution for $\tilde{V}_{t_0}^{\dagger}$. With this we now know $z_{1t_0}^{\dagger} = (n_{1t_0}^{\dagger}, \tilde{V}_{t_0}^{\dagger}, \bar{n}_{t_0}^{\dagger})'$, and can iterate on $z_{1,t+1}^{\dagger} = \dot{\Phi}_t z_{1t}^{\dagger} + \dot{e}_t$ for $t = t_0, ..., t_0 + M - 1$ and on $z_{1,t+1}^{\dagger} = \Phi z_{1t}^{\dagger}$ for $t \geq t_0 + M$, thereby finding the value of $z_{1t}^{\dagger}$ for every t.. From this solution we can then calculate

$$z_{2t}^{\dagger} = \dot{B}_{22,t}^{-1}(-\dot{B}_{21,t} z_{1t}^{\dagger} + \dot{c}_{2t} + \dot{A}_{21,t} z_{1,t+1}^{\dagger}) \qquad t = t_0, ..., t_0 + M - 1$$

$$z_{2t}^{\dagger} = B_{22}^{-1}(-B_{21} z_{1t}^{\dagger} + A_{21} z_{1,t+1}^{\dagger}) \quad t = t_0 + M, ...$$

One can verify numerically that the sequences that result from this algorithm generate the forward-looking solution characterized by

$$\tilde{V}_t^{\dagger} = \tilde{Y}_t^{\dagger} + \beta(1 - k_X)\tilde{V}_{t+1}^{\dagger} = \sum\nolimits_{s=0}^{S-1} [\beta(1 - k_X)]^s \tilde{Y}_{t+s} + [\beta(1 - k_X)]^S \tilde{V}_{t+s}.$$

Real GDP for Examples 9.1-9.5. Let $\mathcal{J}_{2t}^{\chi}$ denote the set of demand-impacted goods and $\mathcal{J}_{2t}^{c}$ non-impacted specialized goods. From (58),

$$\frac{Q_{jt}}{N_t X_{jt}^0} = \begin{cases} \chi H_t n_j^0 & \text{for } j \in \mathcal{J}_{2t}^{\chi} \\ H_t n_j^0 & \text{for } j \in \mathcal{J}_{2t}^{c} \end{cases}.$$

Thus using (75),

$$\sum\nolimits_{j \in \mathcal{J}_{2t}} \frac{Q_{jt}}{N_t X_{jt}^0} = H_t \left[\chi \sum\nolimits_{j \in \mathcal{J}_{2t}^{\chi}} n_j^0 + \sum\nolimits_{j \in \mathcal{J}_{2t}^{c}} n_j^0 \right] = H_t [\chi \bar{n}_t^{\chi} + \bar{n}_t^c]$$

$$Q_t = \frac{1-\alpha_1}{1-n_1^0} H_t [\chi \bar{n}_t^{\chi} + \bar{n}_t^c] + \frac{\alpha_1}{n_1^0} \frac{\hat{X}_{1t}}{\hat{X}_{1t}^0} n_{1t}. \tag{B29}$$

Define $Q_t(n_{1t}, \bar{n}_t^{\chi}, X_{1t}^*)$ to be the function in (B29), from which

$$Q_t(n_1^0, n_{t_0}^{\chi}, X_{1t}^{*0}) - 1 = \frac{1-\alpha_1}{1-n_1^0} [\chi n_{t_0}^{\chi} + \bar{n}_t^c - (1-n_1^0)]$$

$$\begin{aligned} Q_t^{\dagger} = {} & \frac{1-\alpha_1}{1-n_1^0} [\chi n_{t_0}^{\chi} + \bar{n}_t^c - (1-n_1^0)] + \left[\frac{1-\alpha_1}{1-n_1^0} (\chi n_{t_0}^{\chi} + \bar{n}_t^c) \lambda_H + \frac{\alpha_1}{n_1^0} \right] n_{1t}^{\dagger} \\ & + \frac{1-\alpha_1}{1-n_1^0} \chi \bar{n}_t^{\chi\dagger} + \alpha_1 \lambda_5 X_{1t}^{*\dagger}. \end{aligned} \tag{B30}$$

For $t = t_0$, $n_{1t_0} = n_1^0$, $n_{t_0}^{\chi} = \kappa(1-n_1^0)$, $\bar{n}_{t_0}^c = (1-\kappa)(1-n_1^0)$, in which case (B30) becomes (109) as a special case.

Details for Example 9.5.

Equations characterizing initial dynamics. In this example we need to keep track of the fraction of the population specializing in impacted and nonimpacted goods (n_t^{χ} and n_t^c, respectively) and what the fractions would be if each good employed its steady-state level n_j^0 ($\bar{n}_t^{\chi} = \sum_{j \in \mathcal{J}_{2t}^{\chi}} n_j^0$ and $\bar{n}_t^c = \sum_{j \in \mathcal{J}_{2t}^{c}} n_j^0$). The value of $\bar{n}_t^c$ evolves independently of all other variables, since goods in J_{2t}^c started out with $n_{jt_0} = n_j^0$ and a fraction k_X of these disappear each period,

$$\bar{n}_{t+1}^c = (1-k_X)\bar{n}_t^c \quad t = t_0, ..., t_0 + D - 2$$

starting from $\bar{n}_{t_0}^c = (1-\kappa)(1-n_1^0)$. The other three magnitudes $(n_t^{\chi}, n_t^c, \bar{n}_t^{\chi})$ influence and respond to other variables during the initial periods as described below.

Equation (81) is replaced for the initial periods by

$$n_{1,t+1} = 1 - n_{t+1}^{\chi} - n_{t+1}^c \quad t = t_0, ..., t_0 + D - 2. \tag{B31}$$

Equation (82) continues to describe the lifetime advantage of nonimpacted goods,

$$\tilde{V}_t = \log(Y_t/P_{1t}) - \log \tilde{X}_{1t} + \beta(1-k_X)\tilde{V}_{t+1} \quad t = t_0, t_0+1, ..., \tag{B32}$$

while equation (83) is replaced by two new state equations. All newly created goods in this example enter with initial $n_{jt} = n_j^0$ and impacted demand, and no impacted goods do any new hiring while demand remains depressed:

$$\bar{n}_{t+1}^{\chi} = (1-k_X)\bar{n}_t^{\chi} + n_{t+1}^{\sharp} \quad t = t_0, ..., t_0+D-2 \tag{B33}$$

$$n_{t+1}^{\chi} = e^{-n}(1-k_X)n_t^{\chi} + n_{t+1}^{\sharp} \quad t = t_0, ..., t_0+D-2. \tag{B34}$$

Expression (84) continues to describe the income of nonimpacted workers,

$$Y_t/P_{1t} = \frac{Y_t^0(1-n_1^0)H_t^2 Q_{1t}^0}{(1-n_{1t})(2H_t Q_{1t}^0 - n_{1t}N_t\hat{X}_{1t})} \quad t = t_0, t_0+1, ... \tag{B35}$$

with income of impacted workers given by

$$Y_t/P_{1t} = \chi^2 Y_t/P_{1t} \quad t = t_0, ..., t_0+D-1. \tag{B36}$$

Unemployment compensation is given by (111):

$$C_t/P_{1t} = \frac{\tau[n_t^c(Y_t/P_{1t}) + n_t^{\chi}(Y_t^{\chi}/P_{1t})]}{n_{1t}(1-h_{1t})} \quad t = t_0, ..., t_0+D-1. \tag{B37}$$

The income earned from newly created goods for the first D periods is Y_t^{χ} rather than Y_t, so (86) is replaced by

$$\log X_{1t}^* - \log(C_t/P_{1t}) = -k_U + \beta k_\pi \tilde{V}_{t+1} + \beta_{t_0+D-t-1}^X[\log\chi^2 - 1] \quad t = t_0, ..., t_0+D-2 \tag{B38}$$

for β_j^X given by (115). By contrast, the advantage of specializing in an existing good is still $\tilde{V}_t$ so (87) still holds:

$$\log X_{1t}^* - \log(C_t/P_{1t}) = \beta\pi_t\tilde{V}_{t+1} \quad t = t_0, t_0+1, ... \tag{B39}$$

Openings while demand is low come only from nonimpacted goods, so (88) becomes $O_t = n_{t+1}^c e^n N_t - (1-k_X)n_t^c N_t$ leading (91) to be replaced by

$$n_{t+1}^{\sharp} = e^{-n}(1-h_{1t})n_{1t}k_\pi - \frac{[n_{t+1}^c - e^{-n}(1-k_X)n_t^c]k_\pi}{\pi_t} \quad t = t_0, ..., t_0+D-2. \tag{B40}$$

Equation (92) is replaced by the equation for hiring by nonimpacted goods:

$$n^c_{t+1} = H_{t+1}(1-k_X)\bar{n}^c_t \quad t = t_0, ..., t_0 + D - 2. \tag{B41}$$

Linearization of equations for first $D-1$ periods. Equations (B31)-(B41) comprise a system of 11 equations in the 11 variables $(n_{1t}, \tilde{V}_t, \bar{n}^\chi_t, n^\chi_t, Y_t/P_{1t}, Y^\chi_t/P_{1t}, C_t/P_{1t}, X^*_{1t}, \pi_t, n^\sharp_{t+1}, n^c_{t+1})$. To approximate initial dynamics, we linearize these for $t = t_0, ..., t_0 + D - 2$ around $(n^0_1, \tilde{V}^0, \bar{n}^\chi_{t_0} = \kappa(1-n^0_1), n^\chi_{t_0} = \kappa(1-n^0_1), Y^0_t, Y^0_t, C^0_t, X^{*0}_{1t}, \pi^0, \tilde{n}^{\sharp 0}, n^c_{t_0} = (1-\kappa)(1-n^0_1))$ for $\tilde{n}^{\sharp 0}$ the value of (B40) at the point of linearization:

$$\tilde{n}^{\sharp 0} = e^{-n}(1-h^0_1)n^0_1 k_\pi - \frac{[1-e^{-n}(1-k_X)]n^c_{t_0}k_\pi}{\pi^0}.$$

Defining $\tilde{n}^{\sharp\dagger}_{t+1} = n^\sharp_{t+1} - \tilde{n}^{\sharp 0}$, $n^{c\dagger}_t = n^c_t - n^c_{t_0}$, $n^{\chi\dagger}_t = n^\chi_t - n^\chi_{t_0}$, $\bar{n}^{\chi\dagger}_t = \bar{n}^\chi_t - \bar{n}^\chi_{t_0}$, and other $w^\dagger_t$ variables as deviations from their usual steady-state values, the linearized system for $t = t_0, ..., t_0+D-2$ is

$$n^\dagger_{1,t+1} = -n^{\chi\dagger}_{t+1} - n^{c\dagger}_{t+1} \tag{B42}$$

$$\tilde{V}^\dagger_t = Y^\dagger_t - \lambda_3 X^{*\dagger}_{1t} + \beta(1-k_X)\tilde{V}^\dagger_{t+1} \tag{B43}$$

$$\bar{n}^{\chi\dagger}_{t+1} = -k_X \bar{n}^\chi_{t_0} + \tilde{n}^{\sharp 0} + (1-k_X)\bar{n}^{\chi\dagger}_t + \tilde{n}^{\sharp\dagger}_{t+1} \tag{B44}$$

$$n^{\chi\dagger}_{t+1} = [e^{-n}(1-k_X) - 1]n^\chi_{t_0} + \tilde{n}^{\sharp 0} + e^{-n}(1-k_X)n^{\chi\dagger}_t + \tilde{n}^{\sharp\dagger}_{t+1} \tag{B45}$$

$$Y^\dagger_t = \frac{1}{n^0_1(1-n^0_1)}n^\dagger_{1t} + \lambda_5 X^{*\dagger}_{1t} \tag{B46}$$

$$Y^{\chi\dagger}_t = Y^\dagger_t + 2(\chi - 1) \tag{B47}$$

$$C^\dagger_t = 2(\chi-1)\kappa - \frac{1}{n^0_1(1-n^0_1)}n^\dagger_{1t} + Y^\dagger_t - \lambda_2 X^{*\dagger}_{1t} \tag{B48}$$

$$X^{*\dagger}_{1t} - C^\dagger_t = \beta k_\pi \tilde{V}^\dagger_{t+1} + \beta^X_{t_0+D-t-1}2(\chi-1) \tag{B49}$$

$$X^{*\dagger}_{1t} - C^\dagger_t = \beta\pi^0\tilde{V}^\dagger_{t+1} + \beta\tilde{V}^0\pi^\dagger_t \tag{B50}$$

$$\begin{aligned}\tilde{n}^{\sharp\dagger}_{t+1} &= e^{-n}(1-h^0_1)k_\pi n^\dagger_{1t} + e^{-n}(1-h^0_1)n^0_1 k_\pi \lambda_2 X^{*\dagger}_{1t} + \frac{[1-e^{-n}(1-k_X)]n^c_{t_0}k_\pi}{(\pi^0)^2}\pi^\dagger_t \\ &\quad - \frac{k_\pi}{\pi^0}n^{c\dagger}_{t+1} + \frac{e^{-n}(1-k_X)k_\pi}{\pi^0}n^{c\dagger}_t \\ &= e^{-n}(1-h^0_1)k_\pi n^\dagger_{1t} + e^{-n}(1-h^0_1)n^0_1 k_\pi \lambda_2 X^{*\dagger}_{1t} + \frac{[1-e^{-n}(1-k_X)]n^c_{t_0}k_\pi}{(\pi^0)^2}\pi^\dagger_t \\ &\quad - \frac{k_\pi}{\pi^0}n^{c\dagger}_{t+1} - \frac{e^{-n}(1-k_X)k_\pi}{\pi^0}n^\dagger_{1t} - \frac{e^{-n}(1-k_X)k_\pi}{\pi^0}n^{\chi\dagger}_t\end{aligned} \tag{B51}$$

$$n^{c\dagger}_{t+1} = -n^c_{t_0} + (1-k_X)\bar{n}^c_t + \lambda_H(1-k_X)\bar{n}^c_t n^\dagger_{1,t+1}. \tag{B52}$$

To derive (B48) we used

$$\log(C_t/P_{1t}) = \log\tau + \log[n_t^c(Y_t/P_{1t}) + n_t^\chi(Y_t^\chi/P_{1t})] - \log n_{1t} - \log(1-h_{1t})$$

$$\left.\frac{\partial\log(C_t/P_{1t})}{\partial n_t^c}\right|_{n_t^c=n_{t_0}^c, n_t^\chi=n_{t_0}^\chi, Y_t/P_{1t}=Y_t^\chi/P_{1t}=Y_t^0} = \frac{Y_t^0}{(n_{t_0}^c+n_{t_0}^\chi)Y_t^0} = \frac{1}{1-n_1^0} = \left.\frac{\partial\log(C_t/P_{1t})}{\partial n_t^\chi}\right|_{\dots}$$

$$\left.\frac{\partial\log(C_t/P_{1t})}{\partial(Y_t^\chi/P_{1t})}\right|_{\dots} = \frac{n_{t_0}^\chi}{(n_{t_0}^c+n_{t_0}^\chi)Y_t^0} = \frac{\kappa}{Y_t^0}$$

$$\begin{aligned} C_t^\dagger &= \frac{1}{1-n_1^0}(n_t^{c\dagger}+n_t^{\chi\dagger}) + (1-\kappa)Y_t^\dagger + \kappa Y_t^{\chi\dagger} - \frac{1}{n_1^0}n_{1t}^\dagger - \lambda_2 X_{1t}^{*\dagger} \\ &= -\frac{1}{n_1^0(1-n_1^0)}n_{1t}^\dagger + Y_t^\dagger + 2(\chi-1)\kappa - \lambda_2 X_{1t}^{*\dagger} \end{aligned}$$

where the last line follows from (B42) and (B47). Note also that since the value of $\bar{n}_t^c$ is known exactly for $t = t_0, ..., t_0 + D - 1$ and since H_{t+1} exactly equals $1 + \lambda_H(n_{1,t+1} - n_1^0)$, no approximation was involved in arriving at (B52):

$$n_{t+1}^c = [1 + \lambda_H(n_{1,t+1} - n_1^0)](1-k_X)\bar{n}_t^c$$

$$n_{t+1}^c - n_{t_0}^c = -n_{t_0}^c + (1-k_X)\bar{n}_t^c + \lambda_H(1-k_X)\bar{n}_t^c(n_{1,t+1} - n_1^0).$$

Note that since we already substituted out $Y_t^{\chi\dagger}$ in writing equations (B48) and (B49), equation (B47) is unnecessary, and equations (B42)-(B46) and (B48)-(B52) comprise a system of 10 equations for $t = t_0, ..., t_0 + D - 2$ of the form of (105) for $\dot{z}_{1t}^{\dagger\prime} = (n_{1t}^\dagger, \tilde{V}_t^\dagger, \bar{n}_t^{\chi\dagger}, n_t^{\chi\dagger})$ and $\dot{z}_{2t}^{\dagger\prime} = (Y_t^\dagger, C_t^\dagger, X_{1t}^{*\dagger}, \pi_t^\dagger, \tilde{n}_{t+1}^{\sharp\dagger}, n_{t+1}^{c\dagger})$ with

$$\underset{(10\times10)}{\dot{B}_t} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 1-k_X & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-n}(1-k_X) & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{n_1^0(1-n_1^0)} & 0 & 0 & 0 & -1 & 0 & \lambda_5 & 0 & 0 & 0 \\ \frac{1}{n_1^0(1-n_1^0)} & 0 & 0 & 0 & -1 & 1 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & -\beta\tilde{V}^0 & 0 & 0 \\ \dot{b}_{91} & 0 & 0 & \dot{b}_{94} & 0 & 0 & \dot{b}_{97} & \dot{b}_{98} & -1 & -k_\pi/\pi^0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dot{b}_{91} = e^{-n}k_\pi\left[1 - h_1^0 - \frac{(1-k_X)}{\pi^0}\right]$$

$$\dot{b}_{94} = -\frac{e^{-n}(1-k_X)k_\pi}{\pi^0}$$

$$\dot{b}_{97} = e^{-n}(1-h_1^0)n_1^0 k_\pi \lambda_2$$

$$\dot{b}_{98} = \frac{[1-e^{-n}(1-k_X)]k_\pi n_{t_0}^c}{(\pi^0)^2}$$

$$\underset{(10\times 1)}{\dot{c}_t} = \begin{bmatrix} 0 \\ 0 \\ -k_X \bar{n}_{t_0}^\chi + \tilde{n}^{\sharp 0} \\ [e^{-n}(1-k_X)-1]n_{t_0}^\chi + \tilde{n}^{\sharp 0} \\ 0 \\ -2(\chi-1)\kappa \\ -\beta_{t_0+D-t-1}^X 2(\chi-1) \\ 0 \\ 0 \\ n_{t_0}^c - (1-k_X)\bar{n}_t^c \end{bmatrix}$$

$$\underset{(10\times 4)}{\dot{A}_t} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta(1-k_X) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta k_\pi & 0 & 0 \\ 0 & \beta\pi^0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_H(1-k_X)\bar{n}_t^c & 0 & 0 & 0 \end{bmatrix}.$$

Solution algorithm. Given the initial values $n_{1t_0}^\dagger = \bar{n}_{t_0}^{\chi\dagger} = n_{t_0}^{\chi\dagger} = 0$ and a conjectured value of $\tilde{V}_{t_0}^\dagger$, this system implies as in (B27) a value for $\dot{z}_{1,t_0+D-1}^\dagger = \dot{\Lambda}_{D-1}\dot{z}_{1,t_0}^\dagger + \dot{g}_{D-1}$ as an affine function of $\dot{z}_{1,t_0}^\dagger$. In period t_0+D-1, weak demand conditions persist so that $C_{t_0+D-1}^\dagger$ continues to take the form of (B48). But the forward-looking variables anticipate that demand shocks will be gone in t_0+D and a 9-variable system similar to (102) characterizes the relation between variables in dates t_0+D-1 and t_0+D. This system is described by the three state variables $z_{1,t_0+D-1}^{\dagger\prime} = (n_{1,t_0+D-1}^\dagger, \tilde{V}_{t_0+D-1}^\dagger, \bar{n}_{t_0+D-1}^\dagger)$. We know the first two of these from the first two elements of $\dot{z}_{1,t_0+D-1}^\dagger$ and the third can be found from the fact that

$$\bar{n}_{t_0+D-1} = \bar{n}_{t_0+D-1}^\chi + \bar{n}_{t_0+D-1}^c$$

$$\bar{n}_{t_0+D-1} - \bar{n}^0 = (\bar{n}_{t_0}^\chi - \bar{n}^0) + (\bar{n}_{t_0+D-1}^\chi - \bar{n}_{t_0}^\chi) + \bar{n}_{t_0+D-1}^c$$

$$
\begin{aligned}
\bar{n}^{\dagger}_{t_0+D-1} &= (\bar{n}^{\chi}_{t_0} - \bar{n}^0) + \bar{n}^{\chi\dagger}_{t_0+D-1} + \bar{n}^{c}_{t_0+D-1} \\
&= \kappa(1-n^0_1) - (1-n^0_1) + \bar{n}^{\chi\dagger}_{t_0+D-1} + (1-k_X)^{D-1}(1-\kappa)(1-n^0_1) \\
&= \bar{n}^{\cdot}_{D-1} + \bar{n}^{\chi\dagger}_{t_0+D-1}
\end{aligned} \tag{B53}
$$

for $\bar{n}^{\cdot}_{D-1} = -[1-(1-k_X)^{D-1}](1-\kappa)(1-n^0_1)$. Thus the third element of $z^{\dagger}_{1,t_0+D-1}$ is equal to the third element of $\dot{z}^{\dagger}_{1,t_0+D-1}$ plus $\bar{n}^{\cdot}_{D-1}$. For $z^{\dagger\prime}_{1t} = (n^{\dagger}_{1t}, \tilde{V}^{\dagger}_t, \bar{n}^{\dagger}_t)$, $z^{\dagger\prime}_{2t} = (Y^{\dagger}_t, C^{\dagger}_t, X^{*\prime}_{1t}, \pi^{\dagger}_t, n^{\sharp\dagger}_{t+1}, n^{\natural\dagger}_{t+1})$, and $z^{\dagger\prime}_t = (z^{\dagger\prime}_{1t}, z^{\dagger\prime}_{2t})$ we then have

$$
\underset{(9\times3)}{A}\ \underset{(3\times1)}{z^{\dagger}_{1,t_0+D}} = \underset{(9\times9)}{B}\ \underset{(9\times1)}{z^{\dagger}_{t_0+D-1}} + \underset{(9\times1)}{c_{t_0+D-1}} \tag{B54}
$$

for A and B given by (B24) and (B20) and

$$
c'_{t_0+D-1} = (0,0,0,0,-2(\chi-1)\kappa,0,0,0,0).
$$

We then know $z^{\dagger}_{1,t_0+D} = \Phi z^{\dagger}_{1,t_0+D-1} + e_{t_0+D}$ where e_{t_0+D} is calculated as described in (B26), meaning

$$
\underset{(3\times1)}{z^{\dagger}_{1,t_0+D}} = \underset{(3\times4)}{\dot{\Lambda}_D}\ \underset{(4\times1)}{\dot{z}^{\dagger}_{1,t_0}} + \underset{(3\times1)}{\dot{g}_D}
$$

$$
\underset{(3\times4)}{\dot{\Lambda}_D} = \begin{bmatrix} \underset{(3\times3)}{\Phi} & \underset{(3\times1)}{0} \end{bmatrix} \underset{(4\times4)}{\dot{\Lambda}_{D-1}}
$$

$$
\underset{(3\times1)}{\dot{g}_D} = \underset{(3\times4)}{\dot{\Lambda}_D}\ (\underset{(4\times1)}{\dot{g}_{D-1}} + \underset{(4\times1)}{\dot{n}_{D-1}}) + \underset{(3\times1)}{e_{t_0+D}}
$$

$$
\dot{n}_{D-1} = (0,0,\bar{n}^{\cdot}_{D-1},0)'.
$$

This expresses $z^{\dagger}_{1,t_0+D}$ as an affine function of $\dot{z}^{\dagger}_{1,t_0}$. We then choose $\tilde{V}^{\dagger}_{t_0}$ to be the value that causes $z^{\dagger}_{1,t_0+D}$ to be a linear combination of the two eigenvectors of Φ associated with the stable eigenvalues. Since for this example $\tilde{V}^{\dagger}_{t_0}$ is the only nonzero element in $\dot{z}^{\dagger}_{1,t_0}$, the solution is found from the third element of the vector

$$
\begin{bmatrix} \delta_1 \\ \delta_2 \\ \tilde{V}^{\dagger}_{t_0} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & -\dot{\lambda}_{D2} \end{bmatrix}^{-1} \dot{g}_D
$$

for $\dot{\lambda}_{D2}$ the second column of $\dot{\Lambda}_D$ and v_1 and v_2 the stable eigenvectors of Φ.

Now knowing $\tilde{V}^{\dagger}_{t_0}$, we can calculate $\dot{z}^{\dagger}_{1,t_0+1}, ..., \dot{z}^{\dagger}_{t_0+D-1}$ using (105), $z^{\dagger}_{1,t_0+D-1}$ using (B53), and $z^{\dagger}_{1,t_0+D}$ using (B54), and $z^{\dagger}_{1,t_0+D+1}, z^{\dagger}_{t_0+D+2}, ...$ using (103). Knowing now the full sequence of state variables, we can calculate $\dot{z}^{\dagger}_{t_0}, ..., \dot{z}^{\dagger}_{t_0+D-1}$ using (105) and $z^{\dagger}_{t_0+D-1}, z^{\dagger}_{t_0+D}, ...$ using (B54) and (102).

Details for Examples 10.1-10.3.

Real GDP for Examples 10.1-10.3. Notice

$$Q_{jt_0} = \begin{cases} \zeta n_{jt_0} N_{t_0} X^0_{jt_0} & \text{for impacted goods} \\ n_{jt_0} N_{t_0} X^0_{jt_0} & \text{for nonimpacted goods} \end{cases}.$$

Thus in this case (75) becomes

$$\begin{aligned} Q_{t_0} &= \frac{1-\alpha_1}{1-n^0_1}[(1-\kappa)+\kappa\zeta](1-n_{1t_0}) + \left(\frac{\alpha_1}{n^0_1}\right)\left(\frac{\hat{X}_{1t_0}}{\hat{X}^0_{1t_0}}\right) n_{1t_0} \\ &= (1-\alpha_1)[(1-\kappa)+\kappa\zeta] + \alpha_1 \left(\frac{\hat{X}_{1t_0}}{\hat{X}^0_{1t_0}}\right). \end{aligned}$$

Linearized,

$$Q^\dagger_{t_0} = (1-\alpha_1)\kappa(\zeta-1) + \alpha_1\lambda_5 X^{*\dagger}_{1t_0}.$$

Appendix C. Bounds on Jensen's Inequality (online)

Proposition C1. *Let*

$$\delta_t = \log\left[\frac{\exp(S_t) - \exp(R_t)}{S_t - R_t}\right] - \left[\frac{S_t + R_t}{2}\right] \tag{C1}$$

where $\delta_t = \delta$ is constant along the steady-state growth path. If

$$\frac{[1 - \beta(1 - k_X)]k_U}{\beta k_\pi} > \delta, \tag{C2}$$

then

$$\alpha_1 + \tau(1 - \alpha_1) < n_1^0. \tag{C3}$$

Proof of Proposition C1.

We first show that $\delta_t = \delta$ is constant along the steady-state growth path:

$$\begin{aligned}
\delta_{t+1} &= \log\left[\frac{[\exp(S_t + g) - \exp(R_t + g)]}{S_t + g - (R_t + g)}\right] - \left[\frac{S_t + g + R_t + g}{2}\right] \\
&= g + \log\left[\frac{\exp(S_t) - \exp(R_t)}{S_t - R_t}\right] - \left[\frac{S_t + R_t}{2}\right] - \frac{2g}{2} \\
&= \delta_t.
\end{aligned}$$

Let $I_t^0 = \sum_{j \in \mathcal{J}_t} P_{jt}^0 Q_{jt}^0 / P_{1t}^0$ denote steady-state real national income. The skilled receive a total share $(1 - \alpha_1)(1 - \tau)$ and the unskilled $\alpha_1 + \tau(1 - \alpha_1)$, and thus per capita receive

$$Y_t^0 = \frac{(1 - \alpha_1)(1 - \tau)}{1 - n_1^0} I_t^0$$

$$\bar{Y}_{1t}^0 = \frac{\alpha_1 + \tau(1 - \alpha_1)}{n_1^0} I_t^0.$$

As in (A10),

$$Y_t^0 - \bar{Y}_{1t}^0 = \frac{n_1^0 - [\alpha_1 + \tau(1 - \alpha_1)]}{n_1^0(1 - n_1^0)} I_t^0$$

so (C3) holds whenever $Y_t^0 > \bar{Y}_{1t}^0$. Note that $\bar{Y}_{1t}^0$ could alternatively be calculated as

$$\bar{Y}_{1t}^0 = \int_{\log X_{1t}^*}^{S_t} \frac{\exp(z)dz}{S_t - R_t} + \frac{C_t^0}{P_{1t}^0} \int_{R_t}^{\log X_{1t}^*} \frac{dz}{S_t - R_t}.$$

Expression (43) and Proposition 3 established that

$$h_Y^0 = \log X_{1t}^{*0} - \log(C_t^0 / P_{1t}^0) > 0$$

so

$$\bar{Y}_{1t}^{0} < \int_{\log X_{1t}^{*}}^{S_t} \frac{\exp(z)dz}{S_t - R_t} + X_{1t}^{*} \int_{R_t}^{\log X_{1t}^{*}} \frac{dz}{S_t - R_t} = \tilde{Y}_{1t}^{0}. \tag{C4}$$

Thus if $Y_t^0 > \tilde{Y}_{1t}^0$, then also $Y_t^0 > \bar{Y}_{1t}^0$. Thus the proof will be complete if we can show that (C2) implies that $Y_t^0 > \tilde{Y}_{1t}^0$.

From (45) and (41),

$$h_Y^0 = -k_U + k_\pi \left[\frac{\beta}{1 - \beta(1 - k_X)} \right] \log \tilde{Y}^0 \tag{C5}$$

where from (40), $\log \tilde{Y}^0 = \log Y_t^0 - \log \tilde{X}_{1t}^0$. Since $h_Y^0 > 0$, (C5) implies

$$\log \tilde{Y}^0 > \frac{[1 - \beta(1 - k_X)]k_U}{\beta k_\pi}.$$

Condition (C2) then establishes that $\log \tilde{Y}^0 > \delta$ meaning $\log Y_t^0 > \log \tilde{X}_{1t}^0 + \delta$. Thus we will have succeeded in showing that $\log Y_t^0 > \log \tilde{Y}_{1t}^0$ if we show that $\log \tilde{Y}_{1t}^0 < \log \tilde{X}_{1t}^0 + \delta$. From (C4) and (24), this means establishing

$$\log \left[\int_{\log X_{1t}^{*}}^{S_t} \frac{\exp(z)dz}{S_t - R_t} + X_{1t}^{*} \int_{R_t}^{\log X_{1t}^{*}} \frac{dz}{S_t - R_t} \right] < \int_{\log X_{1t}^{*}}^{S_t} \frac{zdz}{S_t - R_t} + \log X_{1t}^{*} \int_{R_t}^{\log X_{1t}^{*}} \frac{dz}{S_t - R_t} + \delta. \tag{C6}$$

For $z^* = \log X_{1t}^*$ define the functions

$$k(z^*) = \int_{z^*}^{S} \frac{\exp(z)dz}{S - R} + \exp(z^*) \int_{R}^{z^*} \frac{dz}{S - R}$$

$$Q(z^*) = \log[k(z^*)] - \int_{z^*}^{S} \frac{zdz}{S - R} - z^* \int_{R}^{z^*} \frac{dz}{S - R}$$

whose derivatives are

$$\frac{dk(z^*)}{dz^*} = \frac{-\exp(z^*)}{S - R} + \frac{\exp(z^*)}{S - R} + \exp(z^*) \int_{R}^{z^*} \frac{dz}{S - R} = \exp(z^*) \int_{R}^{z^*} \frac{dz}{S - R}$$

$$\begin{aligned} \frac{dQ(z^*)}{dz^*} &= \frac{\exp(z^*)}{k(z^*)} \int_{R}^{z^*} \frac{dz}{S - R} + \frac{z^*}{S - R} - \frac{z^*}{S - R} - \int_{R}^{z^*} \frac{dz}{S - R} \\ &= \left[\int_{R}^{z^*} \frac{dz}{S - R} \right] \left[\frac{\exp(z^*)}{k(z^*)} - 1 \right]. \end{aligned}$$

Since $k(z^*) \geq \exp(z^*)$, this derivative is negative, meaning this function reaches its maximum at the lowest possible value of z^*, namely $z^* = R$,

$$Q(z^*) \leq Q(R) = \log\left[\int_R^S \frac{\exp(z)dz}{S-R}\right] - \int_R^S \frac{zdz}{S-R} = \log\left[\frac{\exp(S)-\exp(R)}{S-R}\right] - \left[\frac{S+R}{2}\right] = \delta,$$

which is $\log E(x_{it}) - E[\log x_{it}]$ when $\log x_{it} \sim U(R,S)$. From the definition of $Q(\log X_{1t}^{*0})$, this means

$$\log\left[\int_{\log X_{1t}^{*0}}^S \frac{\exp(z)dz}{S-R} + \exp(\log X_{1t}^{*0}) \int_R^{\log X_{1t}^{*0}} \frac{dz}{S-R}\right] - \int_{\log X_{1t}^{*0}}^S \frac{zdz}{S-R} - \log X_{1t}^{*0} \int_R^{\log X_{1t}^{*0}} \frac{dz}{S-R} < \delta$$

establishing (C6).

Note that (C2) is a sufficient, but not a necessary, condition to guarantee (C3). Typically (C3) also holds even when (C2) does not.