Certain and Uncertain Utility: The Allais Paradox and Five Decision Theory Phenomena*

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Abstract

In the study of decision making under risk, preferences are assumed to be continuous. We present a model of discontinuous preferences over certain and uncertain outcomes. Using existing parameter estimates for certain and uncertain utility, five important decision theory phenomena are discussed: the certainty effect, experimentally observed probability weighting, the uncertainty effect, extreme experimental risk aversion and quasi-hyperbolic discounting. All five phenomena can be resolved.

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1 Introduction

The Allais common consequence and common ratio paradoxes are known in decision theory as the primary departures from expected utility. Their appeal is that even without experimentation they ring true, and with experimentation they are found to be robust. The two paradoxes were proposed by Allais (1953) using the following hypothetical situations:[1]

1. *Common Consequence:*
   Situation A: Certainty of receiving 100 million.
   Situation B: 10% chance of 500 million; 89% chance of 100 million; 1% chance of nothing.

   Situation A': 11% chance of 100 million; 89% chance of nothing.
   Situation B': 10% chance of 500 million; 90% chance of nothing.

2. *Common Ratio:*
   Situation C: Certainty of receiving 100 milion.
   Situation D: 98% chance of 500 million; 2% chance of nothing.

   Situation C': 1% chance of 100 million; 99% chance of nothing.
   Situation D': 0.98% chance of 500 million; 99.02% chance of nothing.

Situations A and B share a common consequence of winning 100 million with probability 0.89. In situations A' and B' this common consequence is removed. Under expected utility, if A is preferred to B, then A' should be preferred to B', as the manipulation is only one of subtracting a common consequence. Situations C and D have a ratio of probabilities of 0.98. Situations C' and D' have a common ratio. Under expected utility if C is preferred to D, then C' should be preferred to D', as the manipulation is only one of dividing by 100.

Despite the predictions of expected utility, the majority of subjects choose A over B, B' over A' , C over D and D' over C' in similar problems (Kahneman and Tversky, 1979).[2] Allais' initial

[1]French francs were originally used as the currency of the paradoxes. No adjustment made for inflation.

[2]Perhaps the most revealing subject was Leonard Savage who chose A over B and B' over A' and concluded that his preferences were 'subtly' in error (Savage, 1954, p. 103). This may have left Paul Samuelson in an uncomfortable situation as he had stated just before, 'I sometimes feel that Savage and I are the only ones in the world who will give a consistent Bernoulli answer to questionnaires of the type that Professor Allais has been circulating' (Samuelson, 1952, p. 678).

motivation for the paradoxes was an intuition that expected utility's independence axiom was 'incompatible with the preference for security in the neighbourhood of certainty' (Allais, 2008, p. 4). Decision theorists have responded to this critique by relaxing the independence axiom and its implication of linearity in probabilities. The most important associated development is Cumulative Prospect Theory with its S-shaped probability weighting scheme (Tversky and Kahneman, 1992; Tversky and Fox, 1995). Why then, would Allais claim to the present that the paradoxes' true thrust is 'generally misunderstood' (Allais, 2008, p. 5).

One potential source of misunderstanding is that a preference for security in the 'neighborhood of certainty' represents only one half of Allais' intuition. Allais also claimed that 'far from certainty', individuals act as expected utility maximizers, valuing a gamble by the mathematical expectation of its utility outcomes (Allais, 1953).[3] Though the argument is vague as to the definitions of 'neighborhood of certainty' and 'far from certainty', such statements are revealing. In this light, the common ratio and common consequence effects read less like a general violation of linearity in probabilities and more like a local violation that appears as any particular outcome becomes close to perfectly certain. Indeed, if the violation is isolated very close to certainty, it may prove useful from a modeling perspective to represent it as a violation of continuity. Individuals could be modeled to exhibit discontinuous preferences over certain and uncertain outcomes. This is similar in spirit to the quasi-hyperbolic representation of discounting where preferences are discontinuous over immediate and future outcomes (Strotz, 1956; Phelps and Pollak, 1968; Laibson, 1997; O'Donoghue and Rabin, 1999). Just as a discontinuity at the present provides a simple, tractable method for representing diminishing impatience through time, a discontinuity at certainty provides a simple, tractable way to represent a disproportionate preference for security near to certainty. To motivate our discussion, we consider the following example:

[3] Allais' wording of 'far from certainty' is 'loin de certitude' (Allais, 1953, p.530, authors' translation).

Decision 1:
Situation P: Certainty of receiving 10 million.
Situation Q: 99% chance of 50 million; 1% chance of nothing.

Decision 2:
Situation P': 99% chance of 10 million; 1% chance of nothing.
Situation Q': 98% chance of 50 million; 2% chance of nothing.

Decision 3:
Situation P'': 98% chance of 10 million; 2% chance of nothing.
Situation Q'': 97% chance of 50 million; 3% chance of nothing.

All three situations share a common ratio of probabilities (with only slight rounding) of 0.99. Under expected utility, if P is preferred to Q, then P' should be preferred to Q', and P'' should be preferred to Q''.[4] Introspection suggests that a substantial proportion of individuals will violate expected utility by preferring P to Q and Q' to P'. However, one would not expect individuals who preferred P' to Q' to prefer Q'' to P''. That is, individuals may violate expected utility between decisions 1 and 2, but not between 2 and 3.[5]

In a survey of 134 University of California, San Diego undergraduate subjects, 52 made decision 1, 40 made decision 2, and 42 made decision 3.[6] While 42.3 percent of subjects preferred P to Q, only 22.5 percent preferred P' to Q', indicating a significant violation of expected utility ($z = 1.999,\ p = 0.046$)[7]. However, the 22.5 percent preferring P' to Q' is not

[4]Rounding error reinforces this prediction as $0.99 > 0.98/0.99 > 0.97/0.98$. If $U(10) > 0.99 \cdot U(50)$, then $U(10) > 0.99 \cdot U(50) > 0.98/0.99 \cdot U(50) > 0.97/0.98 \cdot U(50)$ by monotonicity. That is, if an individual prefers 10 million with certainty in decision 1, then the 10 million should grow more attractive in decision 2 and even more in decision 3.

[5]Importantly, this occurs in a region where probability weighting is believed to be sharply decreasing (Tversky and Kahneman, 1992; Tversky and Fox, 1995; Prelec, 1998), such that small changes in probabilities are associated with large changes in probability weights. Consider a probability weighting function, $\pi(p)$, monotonically increasing and S-shaped. The elegant probability weighting explanation of the common ratio effect is that the ratio of decision weights away from certainty is larger than the ratio of decision weights closer to certainty. This allows for $U(10) > \pi(0.99) \cdot U(50)$ and $\pi(0.99) \cdot U(10) < \pi(0.98) \cdot U(50)$; $U(10) < \pi(0.98)/\pi(0.99) \cdot U(50)$ as $\pi(0.98)/\pi(0.99) > \pi(0.99)$ when $\pi(\cdot)$ is S-shaped. The probability weighting logic is the same in 1) vs 2) and 2) vs 3) as $\pi(0.97)/\pi(0.98) > \pi(0.98)/\pi(0.99) > \pi(0.99)$ for an S-shaped, monotonic probability weighting function.

[6]The numbers are unbalanced because subjects with ID numbers ending in 0, 1, 2, and 3 were asked to make decision 1. Those with ID numbers ending in 4, 5 and 6 were asked to make decision 2 and those with ID numbers ending in 7, 8, and 9 were asked to make decision 3.

[7]The test statistic z corresponds to the null hypothesis, H_0, of equal proportions preferring P/P' across conditions.

significantly different from the 31.0 percent who preferred P'' to Q'', indicating that expected utility violations are less prevalent away from certainty ($z = -0.864$, $p = 0.388$).[8] Conlisk (1989) provides a complementary example, dramatically reducing common consequence effects by moving slightly away from certainty.

The above demonstration helps to pin down the Allais intuition. Away from certainty in decisions 2 and 3, individuals behave roughly consistently. However, when forced to compare situations with certainty and uncertainty together in decision 1, individuals exhibit a disproportionate preference for certain outcomes. Such intuition also carries through to experimental studies. Violations of expected utility are found to be substantially less prevalent when all outcomes are uncertain (Camerer, 1992; Harless and Camerer, 1994; Starmer, 2000). Additionally, a growing body of evidence, at odds with both expected utility and probability weighting, suggests that utility at or near certainty may differ from utility away from certainty (Gneezy, List and Wu, 2006; Simonsohn, 2009; Andreoni and Sprenger, 2009b). In this paper, we demonstrate that allowing for discontinuous preferences at certainty can provide a tractable representation of Allais' intuition and explain a variety of decision theory phenomena including the certainty effect, experimentally observed probability weighting, the uncertainty effect, extreme risk aversion over small stakes, and quasi-hyperbolic time preferences.

The paper proceeds as follows: Section 2 discusses continuity of preferences over certainty and uncertainty and presents a simple discontinuous model of preferences. Section 3 summarizes several prior studies suggestive of differences between certain and uncertain utility. In Section 4, these results are applied to the five decision theory phenomena mentioned above. Section 5 provides a brief discussion and conclusion.

[8] A potential counterpoint is that at probability 0.99, all of the 'action' is effectively taken out of the probability weighting function. This interpretation is at odds with evidence on the shape of the probability weighting function, but is similar in spirit to the violation in continuity we discuss below. In Section 4.1, we discuss probability weighting results in more detail.

2 Continuity over Certainty and Uncertainty

Most textbook treatments of expected utility include an axiom for continuity of preferences.[9] With fixed outcomes, preferences are represented as an ordering over probability distributions, or lotteries. Let $\mathcal{L}$ represent the space of lotteries and let $\succeq$ be a complete, reflexive and transitive preference ordering.

Definition (Continuity): Preferences are continuous if the sets $\{p \in [0,1] : p \circ x \oplus (1-p) \circ y \succeq z\}$ and $\{p \in [0,1] : p \circ x \oplus (1-p) \circ y \preceq z\}$ are closed for all $x, y, z \in \mathcal{L}$

Assume there exists a best lottery, b, and a worst lottery, w, such that for any lottery $z \in \mathcal{L}$, $b \preceq z \preceq w$. For a given lottery z, there exists a probability mixture of w and b that is in the better than set and another probability mixture of w and b that is in the worse than set. This implies that there exists a third mixture, p_z, such that $p_z \circ b \oplus (1-p_z) \circ w \sim z$.

Importantly, continuity implies that for every certain outcome, a degenerate lottery, there exists a probability mixture of the best and worst elements of $\mathcal{L}$, a non-degenerate lottery, that is indifferent. This further implies that certain and uncertain utility must be functionally identical. The proof is by contrapositive. That is, if certain and uncertain utility are not functionally identical then one can construct a case which violates continuity.

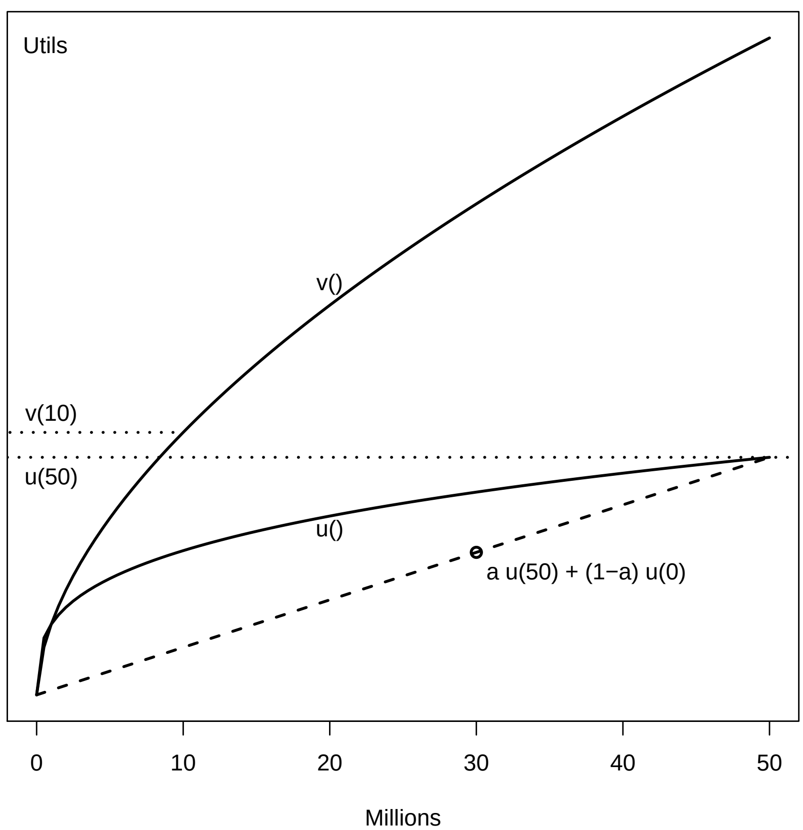
Let there be a fixed set of outcomes. Let $v(x)$ represent utility under certainty and $u(x)$ represent utility under uncertainty and assume that both are increasing in x, but allow $u(x)$ to differ from $v(x)$. Additionally, let there be standard expected utility away from certainty such that for any gamble yielding x_i with probability p_i for $i = 1, 2, ..., S$ with $p_i < 1 \; \forall \; i$, total utility is represented by $\sum_i^S p_i u(x_i)$.

For example, take the set of outcomes $\{0, 10, 50\}$ and the set of lotteries (p_1, p_2, p_3) such that $p_1 + p_2 + p_3 = 1$ and $p_i \leq 1, \; i = 1, 2, 3$. Consider the degenerate lottery of 10 with

[9] The definition and treatment here is that of Varian (1992) and is provided primarily for reference.

certainty, so $(p_1, p_2, p_3) = (0, 1, 0)$. The best lottery is 50 with certainty, $(0, 0, 1)$, and the worst lottery is 0 with certainty, $(1, 0, 0)$. To satisfy continuity, one must find a probability mixture $(a, 1-a)$, $0 \leq a \leq 1$ such that $a \cdot u(50) + (1-a) \cdot u(0) = 1 \cdot v(10)$. Note that the mixture, because it is a non-degenerate lottery, is necessarily evaluated using the uncertain utility function, $u(\cdot)$, and the certain outcome of 10 is evaluated with $v(\cdot)$. The value $a \cdot u(50) + (1-a) \cdot u(0) < u(50)$ as long as $a < 1$ (a true mixture). If $u(50) < v(10)$, continuity is violated. Figure 1 demonstrates the logic.

Figure 1: Violating Continuity with $u(\cdot) \neq v(\cdot)$



Note: By allowing for a difference between certain utility, $v(\cdot)$, and uncertain utility, $u(\cdot)$, continuity is violated. No probability mixture, $(a, 1-a), a < 1$, exists satisfying $v(10) = a \times u(50) + (1-a) \times u(0)$.

The example discussed above yields exactly Allais' violation of the common ratio property of expected utility. A disproportionate preference for security at certainty is represented by $v(10) > u(50)$. An individual would choose 10 with certainty over 50 with probability 0.99 if $v(10) > u(50)$. However, an individual would choose 50 with probability 0.98 over 10 with probability 0.99 if $0.98 \cdot u(50) > 0.99 \cdot u(10)$. The common consequence violation noted by Allais can be generated with a very similar argument.

A simple discontinuous utility function can parsimoniously represent the preferences discussed above. Let there be a $1 \times S$ vector of outcomes: $X = (x_1, x_2, ..., x_S)$. The set of lotteries over these outcomes, $\mathcal{L}$, can be partitioned into $\mathcal{L}_D$, the set of degenerate lotteries, and $\mathcal{L}_N$ the set of non-degenerate lotteries. Note that $\mathcal{L}_D$ has exactly S elements, one for each possible degenerate lottery over the S outcomes. Let x_j represent the degenerate lottery outcome associated with a given element of $\mathcal{L}_D$, and let $(p_{N1}, p_{N2}, ..., p_{NS})$ represent a given element of $\mathcal{L}_N$. For a given lottery L, we define the following discontinuous utility function:

$$W(X, L) = \left\{ \begin{array}{lll} v(x_j) & if & L \in \mathcal{L}_D \\ \sum_{i=1}^{S} p_{Ni} \times u(x_i) & if & L \in \mathcal{L}_N \end{array} \right\}$$

Note that if $u(\cdot) = v(\cdot)$, $W(\cdot)$ reduces to standard expected utility.[10] Continuity is violated if $u(x_i) \neq v(x_i)$ for a given $x_i \in X$. If $v(x) > u(x)$ for $x > 0$, then individuals exhibit a disproportionate preference for certainty.

Critical efforts have been made to provide an axiomatic basis for such a discontinuous representation of preferences (see Neilson, 1992; Schmidt, 1998; Diecidue, Schmidt and Wakker, 2004). These results demonstrate that with additional continuity or independence assumptions, $W(X, L)$ can represent standard expected utility preferences with a disproportionate preference for certainty. However, the represented preferences will violate stochastic dominance (Schmidt, 1998; Diecidue et al., 2004). Though one could circumvent dominance with an 'editing' argument similar to that of Kahneman and Tversky (1979), this leaves the above

[10]One need not posit standard expected utility as the baseline when $u(\cdot) = v(\cdot)$. Prospect Theory probability weighting is also continuous in probabilities. We choose this baseline for two reasons: 1) to have a model with a small deviation from standard expected utility and 2) to capture the stylized fact that away from certainty individuals act mainly in line with expected utility.

model of preferences with the somewhat undesirable property of violating dominance without such additional elements. As such, in the words of Diecidue et al. (2004), 'the interest of the model is descriptive and lies in its psychological plausibility.'

It is important to note that allowing for discontinuous preferences over certainty and uncertainty is not standard in the study of decision making under risk. However, in models of time discounting such preferences frequently arise. Quasi-hyperbolic discounting (Strotz, 1956; Phelps and Pollak, 1968; Laibson, 1997; O'Donoghue and Rabin, 1999) is discontinuous at the present. In such models, individuals discount between the present and one future period with a low discount factor, $\beta\delta$, and between subsequent periods with a higher discount factor, δ alone.

The model of quasi-hyperbolic time preferences is considered an elegant and powerful simplification for several reasons. First, only a single parameter is added to standard exponential discounting; second, many behavioral anomalies can be reconciled. Third, the null hypothesis of dynamically consistent preferences can be tested. Our goal is similar. Consider, for example, standard CRRA utility of $v(x) = x^{\alpha}$. The discontinuity we envision is of the form $u(x) = x^{\alpha-\beta}$, $\alpha > \beta > 0$. A single parameter is added to a standard model, many behavioral anomalies can be reconciled, and the null hypothesis of $\beta = 0$ can be tested. In the next section we show several results in support of modeling such a discontinuity in risk preferences.

3 Experimental Results Suggesting Discontinuity

A growing body of literature is suggestive of different preferences over certainty and uncertainty. Expected utility is generally found to perform well when all outcomes are uncertain (Conlisk, 1989; Camerer, 1992; Harless and Camerer, 1994; Starmer, 2000; Andreoni and Harbaugh, 2009).[11] However, violations of expected utility abound when individuals are asked to consider certain and uncertain outcomes together. Generally, the direction of these violations indicate a disproportionate preference for certainty. Of course, these findings alone do not establish

[11]There do exist some identified violations even when all things are uncertain. For examples and discussion, see the noted citations and Wu and Gonzalez (1996).

differential treatment of certain and uncertain utility. Importantly, there exists a collection of studies which test more directly whether certain and uncertain utility should be treated as identical.

A puzzling and surprising result is the 'uncertainty effect' documented by Gneezy et al. (2006) and reproduced by Simonsohn (2009). In Gneezy et al. (2006), 60 undergraduate subjects at the University of Chicago are randomly assigned to one of three conditions in equal numbers. Subjects were asked to provide their willingness to pay (WTP) for a \$100 gift certificate to a local bookstore, or for a \$50 gift certificate to the bookstore, or for a lottery with 50% chance of winning a \$100 gift certificate and 50% chance of winning a \$50 gift certificate to the bookstore.[12] The average WTP for the \$50 gift certificate in condition 2 was significantly higher than the WTP for the lottery in condition 3.

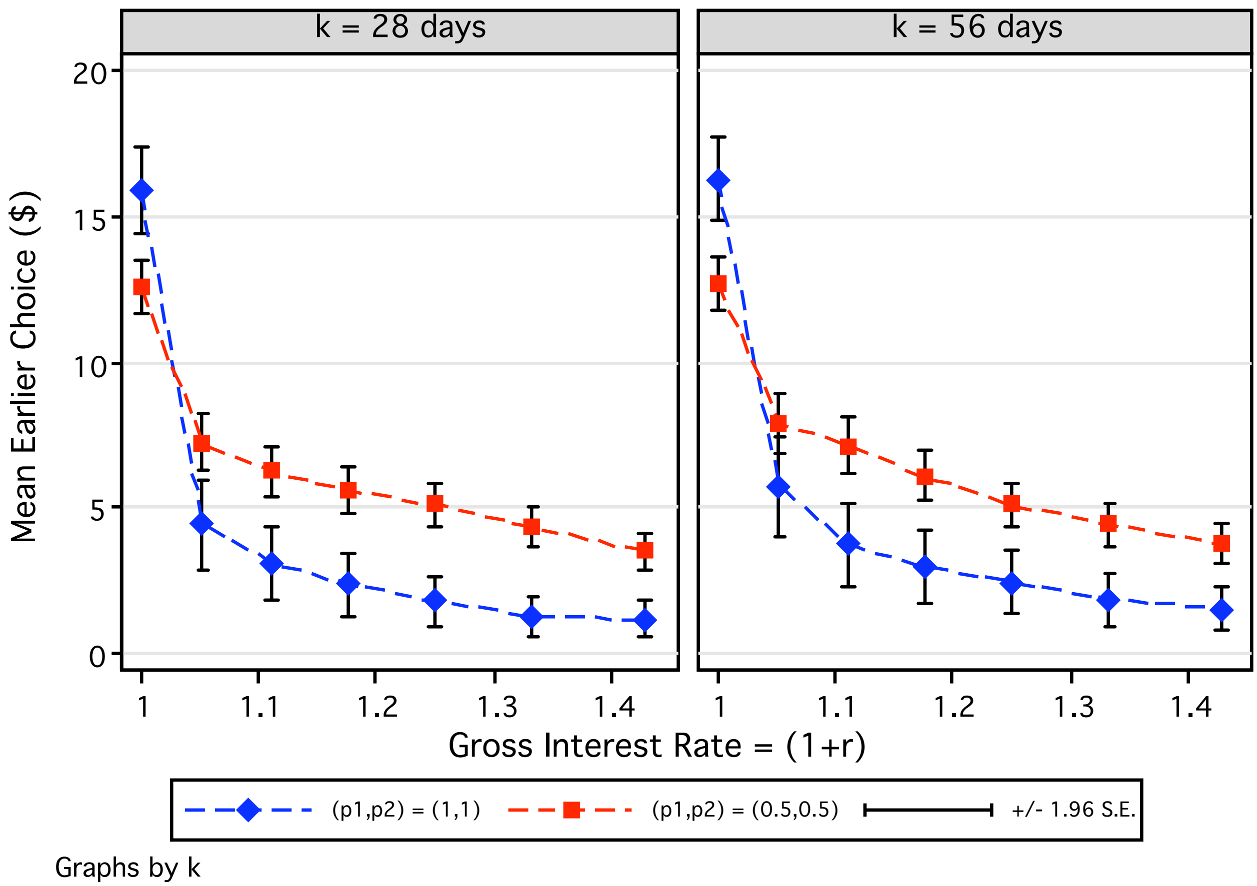
The between-subject behavior of valuing a lottery lower than its worst possible outcome violates expected utility and Prospect Theory probability weighting.[13] The authors suggest that this 'uncertainty effect' should be interpreted as a violation of an 'internality axiom' that 'the value of a risky prospect must lie between the value of that prospect's highest and lowest outcome' (Gneezy et al., 2006, p. 1284). This internality axiom, which has also been called 'betweenness' (Camerer and Ho, 1994) is closely related to continuity. If preferences are discontinuous with a disproportionate preference for certainty, there will exist a degenerate lottery, say the \$50 gift certificate, that is preferred to some probability mixture from the strictly and weakly better than sets. As such, violations of the internality axiom could be generated by violations of continuity. The results of Gneezy et al. (2006) are potentially suggestive of certain outcomes being assessed differently than uncertain outcomes. Indeed Gneezy et al. (2006) argue that subjects may 'code' uncertain lotteries differently than certain outcomes and apply a direct premium for certainty akin to the disproportionate preference $v(x) > u(x)$ for

[12] WTP was elicited using the Becker, DeGroot, Marschak mechanism (Becker, Degroot and Marschak, 1964), 5% of subjects had their choices actualized and were given \$100 to purchase the gift certificate or lottery.

[13] Both of these models respect 'betweenness' (Camerer and Ho, 1994), such that lotteries must be valued at some weighted average of the valuations of their outcomes.

$x > 0$.[14]

Figure 2: Aggregate Behavior Under Certainty and Uncertainty



Note: The figure presents aggregate behavior for $N = 80$ subjects under two conditions: $(p_1, p_2) = (1, 1)$, i.e. no risk, in blue; and $(p_1, p_2) = (0.5, 0.5)$, i.e. 50% chance sooner payment would be sent *and* 50% chance later payment would be sent, in red. $t = 7$ *days* in all cases, $k \in \{28, 56\}$ *days*. Error bars represent 95% confidence intervals, taken as $+/- 1.96$ standard errors of the mean. Test of H_0 : Equality across conditions: $F_{14,2212} = 15.66$, $p < .001$.

In Andreoni and Sprenger (2009b), we present a discounted expected utility violation that is also suggestive of differences between certain and uncertain utility. Using Andreoni and Sprenger (2009a) Convex Time Budgets (CTB) and a within-subject design, 80 subjects are

[14] The authors express this premium as follows: 'An individual posed with a lottery that involves equal chance at a \$50 and \$100 gift certicate might code this lottery as a \$75 gift certicate plus some risk. She might then assign a value to a \$75 gift certicate (say \$35), and then reduce this amount (to say \$15) to account for the uncertainty' (Gneezy et al., 2006, p. 1291).

asked to make intertemporal allocation decisions in two principal decision environments. In the first decision environment, sooner and later payments are made 100% of the time.[15] In the second decision environment, sooner payments are made 50% of the time and later payments are made 50% of the time (determined by rolls of two ten-sided die). The prediction from standard discounted expected utility is that allocations should be identical across the two situations. This is due to the common ratio of probabilities across the two conditions. Figure 2 reproduces Figure 2 of Andreoni and Sprenger (2009b), presenting the sooner payment allocation decisions. Allocations in the two decision environments differ dramatically, violating discounted expected utility. Additionally, the pattern of results cannot be explained by either standard probability weighting or temporally dependent probability weighting.[16] In estimates of utility parameters, utility function curvature is found to be markedly more pronounced when all payments are risky as opposed to when all payments are certain, suggesting a disproportionate preference for certainty, $v(x) > u(x)$ for $x > 0$.[17] For experimental payment values of around \$20, certain utility is estimated as the Stone-Geary utility function $v(x) = (x - \omega)^{\alpha}$, $\hat{\alpha} = 0.988$ ($s.e.$ 0.002); $\hat{\omega} = 2.414$ (0.418), uncertain utility is estimated as $u(x) = (x - \omega)^{\alpha-\beta}$, $\hat{\beta} = 0.105$ (0.017), and the null hypothesis of $\beta = 0$ is rejected.

These studies indicate that modeling utility as different for certain and uncertain outcomes may help to explain decision theory phenomena that remain anomalous in expected utility and probability weighting models. Additionally, many experimental methodologies use certainty equivalence techniques, asking individuals to compare certain and uncertain outcomes. If certain and uncertain utility are different, this may help to explain some of the conjectured violations of expected utility. In the following section, we discuss five decision theory phenomena existent in the literature that can be explained by allowing certain and uncertain utility to differ. To demonstrate the effects we use the parameter estimates obtained in Andreoni and Sprenger (2009b) in hopes of convincing readers that the difference between certain and uncer-

[15] See Andreoni and Sprenger (2009b) for efforts made to equate transaction costs.

[16] See Andreoni and Sprenger (2009b) for discussion.

[17] Discounting is found to be virtually identical across the two conditions and very similar to Andreoni and Sprenger (2009a).

tain utility need not be implausibly large to reconcile anomalous results. All demonstrations are conducted including the Stone-Geary minimum parameter, ω. All results are maintained when imposing the restriction $\omega = 0$. See Andreoni and Sprenger (2009b) for discussion and further examples.

4 Applications: Five Phenomena of Decision Theory

Allowing certain and uncertain utility to be different with a disproportionate preference for certainty can account for five important decision theory phenomena: the certainty effect, experimentally observed probability weighting, the uncertainty effect, experimentally observed extreme risk aversion, and quasi-hyperbolic discounting. Readers will notice that although the present section is titled 'Five Phenomena', there are only four subsections. This is because one of the five decision theory phenemona we discuss is trivially generated by a difference between certain and uncertain utility.

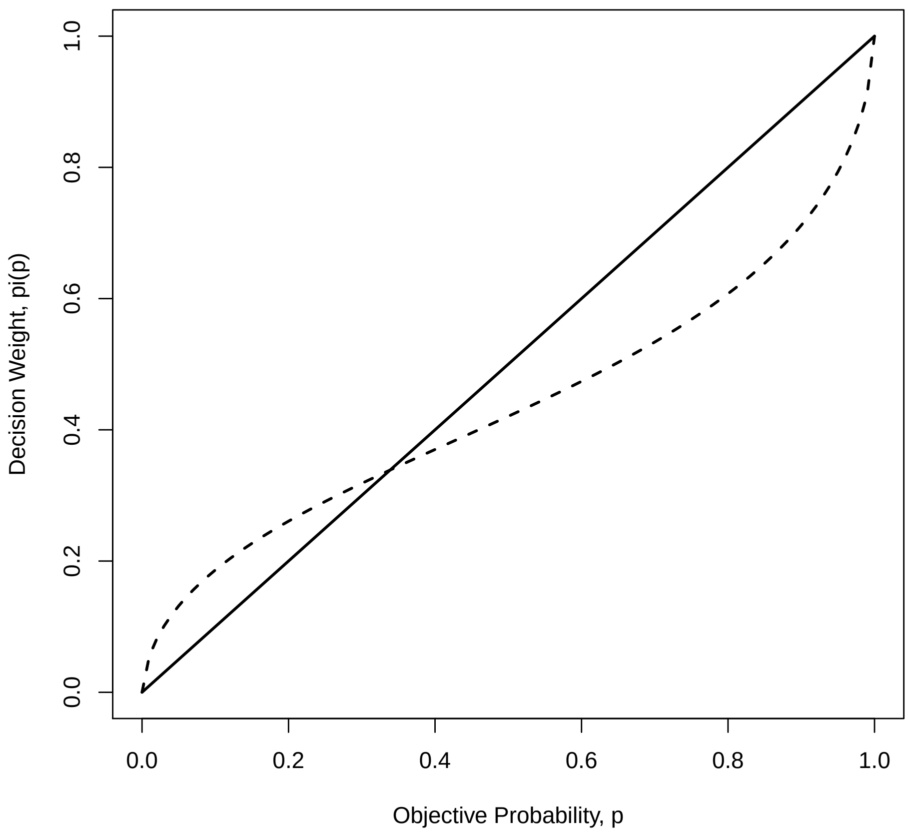
The 'certainty effect' is the robust finding, frequently derived from intuitions of the Allais Paradox, that when certain options are available, they are disproportionately preferred. Allowing certain and uncertain utility to differ with the assumption that $v(x) > u(x)$ for $x > 0$ provides the certainty effect trivially. Certain options are assumed to be disproportionately preferred via the functional difference between $u(\cdot)$ and $v(\cdot)$. In what follows, we discuss the other four phenomena in detail.

4.1 Probability Weighting

One of Prospect Theory's major contributions to decision theory is the notion of probability weighting (Tversky and Kahneman, 1992; Tversky and Fox, 1995). Probability weighting assumes that there exists a nonlinear function, $\pi(p)$, which maps objective probabilities into subjective decision weights. The function $\pi(\cdot)$ is normally assumed to be S-shaped. Figure 3 plots the popular one-parameter functional form $\pi(p) = p^\gamma/(p^\gamma + (1-p)^\gamma)^{1/\gamma}$ with $\gamma = 0.61$

as estimated by (Tversky and Kahneman, 1992).[18] Low probabilities are upweighted and high probabilities are downweighted. Identifying the general shape of the probability weighting function and determining its parameter values has received substantial attention both theoretically and in experiments (see e.g., Tversky and Fox, 1995; Wu and Gonzalez, 1996; Prelec, 1998; Gonzalez and Wu, 1999).

Figure 3: Standard Probability Weighting



Note: The function $\pi(p) = p^\gamma/(p^\gamma + (1-p)^\gamma)^{1/\gamma}$ is plotted with $\gamma = 0.61$ as found by Tversky and Kahneman (1992).

Experiments demonstrating an S-shaped probability weighting function elicit preferences using certainty equivalence techniques (see Tversky and Kahneman, 1992; Tversky and Fox,

[18] Other analyses with similar functional forms yield similar patterns and parameter estimates (for reviews, see Prelec, 1998; Gonzalez and Wu, 1999).

1995; Gonzalez and Wu, 1999). Individuals are asked to state a certain amount, C, that makes them indifferent to a lottery that yields X with probability p and 0 otherwise. The experiments are conducted with either hypothetical or real payments and with stakes varying from \$25 to \$400. Certain and uncertain utility are assumed identical, the utility of zero is normalized to zero, a functional form for utility is posited, and $\pi(p)$ is identified as the value that rationalizes the indifference condition

$$u(C) = \pi(p) \times u(X). \tag{1}$$

In order to estimate probability weighting from certainty equivalence responses, both the utility function and the probability weighting function are parameterized and then estimated via non-linear least squares routines. In Tversky and Kahneman (1992), $u(X) = X^\alpha$ is assumed along with the popular probability weighting function noted above and the parameters γ and α are estimated jointly.[19] They find $\gamma = 0.61$ and $\alpha = 0.88$. In Tversky and Fox (1995), the curvature parameter from Tversky and Kahneman (1992) of $\alpha = 0.88$ is assumed and the two-parameter weighting function $\pi(p) = \delta p^\gamma/(\delta p^\gamma + (1-p)^\gamma)$ is estimated. In Gonzalez and Wu (1999), both α and the two-parameter $\pi(\cdot)$ function are estimated.[20]

Tversky and Kahneman (1992) and Gonzalez and Wu (1999) both report median responses for their certainty equivalents experiments. These median cash equivalents at various outcome values are reproduced in Table 1. In order to make comparisons across these data and others, we follow the methodology of Tversky and Fox (1995). We assume $U(X) = X^\alpha$ with $\alpha = 0.88$ and estimate the probability weighting function $\pi(p) = p^\gamma/(p^\gamma + (1-p)^\gamma)^{1/\gamma}$ based on each series of responses. That is, the parameter $\hat{\gamma}$ is estimated as the value that minimizes the sum of squared residuals of the non-linear regression equation

$$C = [p^\gamma/(p^\gamma + (1-p)^\gamma)^{1/\gamma} \times X^{0.88}]^{1/0.88} + \epsilon \tag{2}$$

[19] Additional utility parameters such as the degree of loss aversion are also estimated.
[20] Gonzalez and Wu (1999) also provide non-parametric estimates.

Table 1: Median Certainty Equivalents In Probability Weighting Experiments

		Probability											
Authors	Outcomes	.01	.05	.10	.25	.40	.50	.60	.75	.90	.95	.99	$\hat{\gamma}$
Gonzalez & Wu (1999)	(0, 25)	4	4	8	9	10	9.5	12	11.5	14.5	13	19	0.51
Gonzalez & Wu (1999)	(0, 50)	6	7	8	12.5	10	14	12.5	19.5	22.5	27.5	40	0.45
Gonzalez & Wu (1999)	(0, 100)	10	10	15	21	19	23	35	31	63	58	84.5	0.48
Tversky & Kahneman (1992)	(0, 50)			9			21			37			0.61
Tversky & Kahneman (1992)	(0, 100)		14		25		36		52		78		
Tversky & Kahneman (1992)	(0, 200)	10		20			76			131		188	0.65
Tversky & Kahneman (1992)	(0, 400)	12										377	

Notes: Data obtained from Gonzalez and Wu (1999) Table 1 and Tversky and Kahneman (1992) Table 3. 'Outcomes' corresponds to the high and low value of the lottery. 'Probability' refers to the probability of the higher value. $\hat{\gamma}$ estimated from (2). Due to data limitations, Tversky and Kahneman (1992) outcomes (0, 50) are merged with (0, 100) to produce estimate as are outcomes (0, 200) and (0, 400).

for each series of median data in Table 1.[21]

In Figure 4 we graph several of these probability weighting functions along with the two parameter $\pi(\cdot)$ function obtained via the same method in Tversky and Fox (1995) with stakes of \$75 to \$150. As can be seen, there is general concordance in the estimated probability weighting functions and all follow the famous S-shaped pattern. Low probabilities are up-weighted and high probabilities are down-weighted. The probability weighting function cuts the 45 degree line at near 0.3 as is normally obtained and features more convexity close to 1 than concavity close to 0.

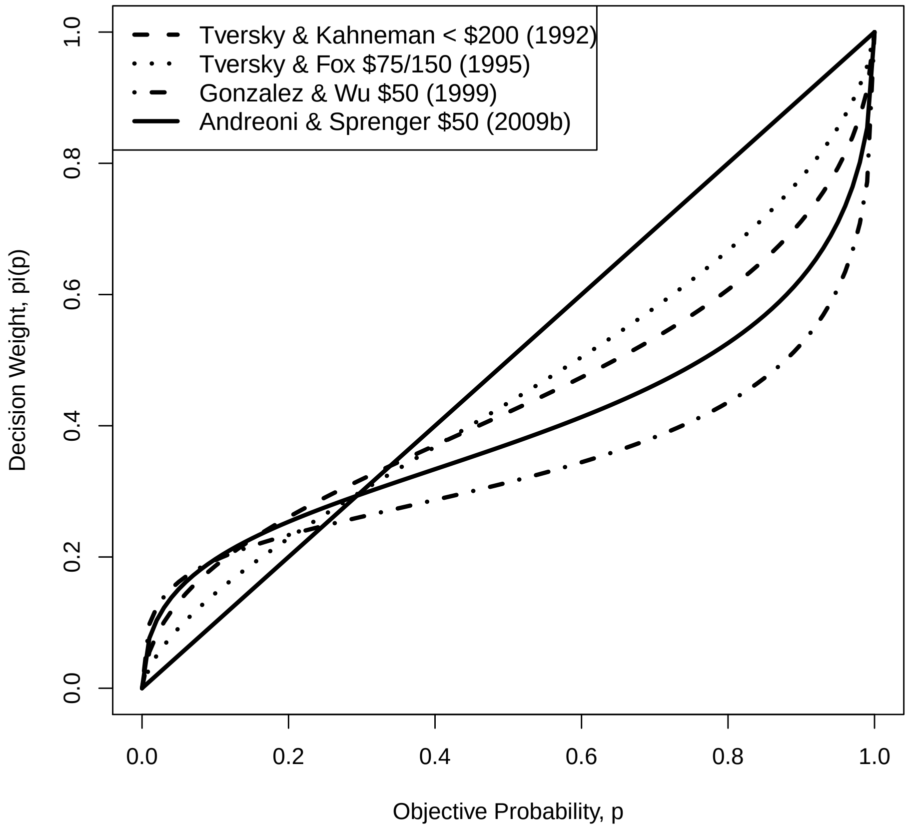
Our claim is that the strategy of identifying probability weighting from certainty equivalence responses can be problematic. If certain and uncertain utility are functionally different, instead of (1) the indifference condition is

$$v(C) = p \times u(X). \tag{3}$$

If the correct model is (3), then equation (1) would be misspecified, and the misspecification is being forced to be absorbed in part by the probability weighting parameter, γ. It is easy

[21]Because of data limitations, Tversky and Kahneman (1992) outcomes (0, 50) are merged with (0, 100) to produce estimate as are outcomes (0, 200) and (0, 400).

Figure 4: Comparison of Probability Weighting Estimates



to see how this misspecification might lead to downweighting of high probability events, and, in conjunction with the functional form assumption on $\pi(\cdot)$, generate an S-shaped probability weighting function. Dividing (1) by (3) we obtain $u(C)/v(C) = \pi(p)/p$. If $v(C) > u(C)$, then $\pi(p) < p$, that is, the misspecification will create downweighting of high probabilities.[22] It may also lead to upweighting of low-probability events if there is differential curvature of utility for certainty and uncertainty or if there are additional utility arguments, such as the Stone-Geary minima parameters assumed by Andreoni and Sprenger (2009b).[23]

To demonstrate the effects of misspecifying (1), we use the estimates for certain and uncer-

[22] This, of course, assumes that $u(X)$ is parameterized the same way in both cases, which may not be true if one were to estimate the data first assuming (1) and then assuming (3).

[23] The inversion between upweighting and downweighting driven by differential curvature occurs because under the CRRA utility parameterization if $C < 1,\ v(C) < u(C)$ while for $C > 1, v(C) > u(C)$.

tain utility obtained in Andreoni and Sprenger (2009b) and calculate the certainty equivalents corresponding to the $50 gambles in Table 1. Following (2), we estimate a probability weighting function and obtain an estimate for $\hat{\gamma}$ of 0.53. The estimated weighting function is also illustrated in Figure 4, where one can see it lies within the bounds of prior estimates, and shows considerable probability weighting driven solely by specification error.[24]

Using certainty equivalence techniques to identify probability weighting, by our account, suffers from an experimental flaw: the certainty effect is built into the experimental design. Allowing for a difference between certain and uncertain utility, a sharply decreasing probability weighting function and upweighting of low probabilities can be generated. Yet, our analysis shows that these hallmarks of probability weighting may actually be artifacts of both the built-in certainty effect and the restrictive functional form assumption used to estimate $\pi(p)$. Interestingly, when the certainty effect is eliminated by design, experimental data appear to reject probability weighting. Andreoni and Harbaugh (2009) eliminate any certain outcomes, ask subjects to trade probability for prize along a linear budget constraint, and find surprising support in favor of expected utility, while significantly rejecting probability weighting.

4.2 The Uncertainty Effect

The uncertainty effect of valuing a lottery lower than its worst possible outcome is a surprising and unintuitive result. The uncertainty effect violates expected utility, probability weighting and any other utility representation respecting betweenness.

The implications of the uncertainty effect are surprising, and somewhat worrying, to decision theorists. If individuals prefer a $50 gift certificate to a 50%-50% lottery paying $50 or $100 gift certificates, doesn't this imply that individuals would prefer $0 with certainty over a 50%-50% lottery paying $0 or $50? Is the uncertainty effect simply an artifact of such framing or other experimental design choices? Taking this question as valid, our analysis provides a potential resolution. Though $u(x) < v(x)$ for $x > 0$, a discontinuous CRRA utility specification implies

[24] We also conducted a similar exercise for certainty equivalents to gambles of $25 and $100 and obtained nearly identical results.

that $u(x)$ and $v(x)$ become closer in value as x approaches zero. As such, individuals might be expected to demonstrate the uncertainty effect at stakes of \$50 and \$100 but not at lower stakes.

Consider the utility parameters obtained in Andreoni and Sprenger (2009b) and the original uncertainty effect comparing a 50%-50% lottery paying \$50 or \$100 to the certainty of \$50. The utility of the lottery is given as $U_L = 0.5 \times (50 - 2.41)^{0.99-0.11} + 0.5 \times (100 - 2.41)^{0.99-0.11} = 43.13$. The utility of the certain \$50 is given as $U_C = (50 - 2.41)^{0.99} = 45.78$, demonstrating the uncertainty effect of valuing a 50%-50% lottery lower than its worst outcome. However, if the stakes were \$5 and \$55, the utility of the lottery would be $U_L = 0.5 \times (5 - 2.41)^{0.99-0.11} + 0.5 \times (55 - 2.41)^{0.99-0.11} = 17.49$ and the utility of the certain \$5 would be $U_C = (5 - 2.41)^{0.99} = 2.56$. The uncertainty effect would not be demonstrated at lower stakes.

Note should be made of three additional issues. First, there is substantial debate as to whether the uncertainty effect is robust. Though the effect is documented in Gneezy et al. (2006) and reproduced in Simonsohn (2009), other work has failed to reproduce violations of betweenness in very similar experiments (Rydval, Ortmann, Prokosheva and Hertwig, 2009). Second, the uncertainty effect seems not to be present for immediate monetary payments in certainty equivalents experiments (Birnbaum, 1992).[25] Third, the uncertainty effect has not been widely observed within individuals. Though Gneezy et al. (2006) do not find individual demonstrations of the uncertainty effect, Sonsino (2008) documents almost 30% of subjects violating betweenness in an Internet experiment. In addition to our hypothesis, future research should examine whether the uncertainty effect is consistently present within individuals and across a variety of rewards, including money, and reward values, including low stakes.

4.3 Extreme Risk Aversion

Experimentally elicited risk preferences generally yield extreme measures of risk aversion. Importantly, risk preferences are frequently elicited using certainty equivalence techniques similar

[25] Though Gneezy et al. (2006) demonstrate that it is present for monetary payments over time.

to those employed in the probability weighting literature. As such, a disproportionate preference for certainty and extreme risk aversion may be conflated.

Here we show that failing to allow for differences between certain and uncertain utility can generate false inferences of extreme risk aversion. Consider the utility parameters obtained in Andreoni and Sprenger (2009b) and the certainty equivalent of a 50%-50% lottery paying \$50 or \$0. Normalizing $u(0) = 0$, the certainty equivalent is given as $C = (0.5 \times (50 - 2.41)^{0.99-0.11})^{1/0.99} = 15.38$.

Under the assumption that $u(\cdot) = v(\cdot)$, standard practice would find the curvature parameter, a, which rationalizes $15.38^a = 0.5 \times 50^a$. The solution is $a = 0.59$. Importantly, curvature parameters obtained in low-stakes certainty equivalence studies and in auction experiments are generally between 0.5 and 0.6.[26] This suggests that part of experimentally obtained extreme risk aversion may be associated with differential assessment of certain and uncertain outcomes.[27]

Though a disproportionate preference for certainty can account for extreme experimental risk aversion, our findings will still violate standard expected utility. As Rabin (2000a) indicates, even moderate risk aversion over small experimental stakes implies unbelievable risk aversion over large stakes. Rationalizing the existence of small stakes risk aversion, however much more limited than prior estimates, therefore requires additional modeling elements. One important possibility put forward in Rabin (2000a,b) and Rabin and Thaler (2001) is that loss aversion relative to a reference point could generate small stakes risk aversion. In this vein, Kachelmeier and Shehata (1992) present evidence on both willingness to accept and willingness to pay values for lotteries. Presumably the referent differs across these contexts, as in the first case one owns the lottery and in the second one must pay for it. Though the curvature implied from willingness to pay certainty equivalents is around 0.6, the curvature from willingness to accept treatments actually suggests risk-loving behavior. This data is impressively supportive of the

[26] In the auction literature, mention is made of 'square root utility' where $\alpha \approx 0.5$. Holt and Laury (2002) discuss several relevant willingness to pay results from the auction literature in line with this value.

[27] Note, our explanation is not sufficient to produce the effect of extreme small stakes risk aversion when comparing two uncertain outcomes as in Holt and Laury (2002). This, in turn, suggests that experimental methodology may also be part of any discussion of extreme risk aversion.

claim that reference dependent preferences can help to rationalize small stakes risk aversion.

4.4 Quasi-hyperbolic Discounting

Arguments have recently been made that dynamically inconsistent preferences are generated by differential risk on sooner and later payments (for psychological evidence, see Keren and Roelofsma, 1995; Weber and Chapman, 2005). Halevy (2008) argues that differential risk leads to dynamic inconsistency because individuals have a temporally dependent probability weighting function that is convex near certainty similar to standard probability weighting. The probability of receiving payments is argued to decline through time, with present payments being certain. If individuals weight probabilities in a non-linear fashion, then apparent present bias is generated as a certainty effect.

We have demonstrated that if certain and uncertain utility are not identical, certainty effects and S-shaped probability weights can be obtained. As such, one need not call on a complex probability weighting function to explain the phenomenon. If individuals exhibit a disproportionate preference for certainty when it is available, then present, certain consumption will be disproportionately favored over future, uncertain consumption. When only uncertain, future consumption is considered, the disproportionate preference for certainty is not active, generating apparent present-biased preference reversals. In Andreoni and Sprenger (2009b), we show that apparent present bias (and future bias) can be generated experimentally via comparisons of certainty and uncertainty.

Consider an individual asked to choose between \$20 with certainty today and \$25 with uncertainty in one month. For simplicity, assume a monthly discount factor of $\delta = 1$ and let $p < 1$ be the assessed probability of receiving the \$25 in the future. Following the utility parameters of Andreoni and Sprenger (2009b), the relevant comparison is between certain utility today of $(20-2.41)^{0.99} = 17.09$ versus uncertain future utility of $p \cdot (25-2.41)^{0.99-0.11} = p \cdot 15.53 < 17.09$, and so the individual opts for the certain, sooner payment. If asked instead to choose between \$20 with uncertainty in one month and \$25 with equal uncertainty in two months, the comparison is $p \cdot (20 - 2.41)^{0.99 - 0.11} = p \cdot 12.46$ versus $p \cdot (25 - 2.41)^{0.99 - 0.11} = p \cdot 15.53$, the later payment is preferred, and a present-biased preference reversal is observed. This suggests that the discontinuity often observed in time preferences between the present and future may not always or only be due to a discontinuous present bias in discounting, but may also be due to a disproportionate preference for certainty when facing an inherently uncertain future.

5 Conclusion

We provide a simple model of discontinuous utility over certainty and uncertainty. We demonstrate that allowing for certain and uncertain utility can help to resolve five decision theory phenomena: the certainty effect, probability weighting, the uncertainty effect, extreme experimental risk aversion, and quasi-hyperbolic discounting. Moreover, the evidence taken as a whole seems to strengthen each individual argument. It is compelling that such a broad set of phenomena can be reconciled with the single simple hypothesis that there exist discontinuous preferences over certain and uncertain outcomes.

It is important, however, to consider exactly what phenomena hinge on discontinuity itself and which phenomena could be resolved with a less rigid model of preferences. Each phenomena could be resolved without relying on discontinuity. Some utility cost from moving away from certainty, continuous in probability, could potentially accommodate these results. There, of course, exist limiting cases such as probability $0.99\overline{9}$ where discontinuity alone will generate a difference between certain and uncertain behavior; however, these are likely pathological cases that Allais was attempting to avoid when vaguely describing the 'neighborhood of certainty.' Indeed, with finite data it is likely impossible to distinguish a neighborhood of certainty from certainty itself and impossible to distinguish continuous from discontinuous choice behavior. Furthermore, the neighborhood of certainty may vary with stakes and situation. The probability 0.99 may be viewed as the same as the probability 1 when the stakes are \$50, but what about when the stakes are \$50 million? These are questions one can begin to consider.

Our arguments lay the foundation for re-thinking certain and uncertain utility. It is important to remember that allowing for differences in certain and uncertain utility need not replace other interpretations of the discussed phenomena, such as a visceral present bias over primary rewards. However, accounting for a disproportionate preference for certainty, following the intuition of Allais, can potentially deepen, unify, and simplify our understanding of individual decision-making phenomena.

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