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BARGAINING WITHOUT A COMMON PRIOR—AN IMMEDIATE AGREEMENT THEOREM

BY MUHAMET YILDIZ¹

In sequential bargaining models without outside options, each player's bargaining power is ultimately determined by which player will make an offer and when. This paper analyzes a sequential bargaining model in which players may hold different beliefs about which player will make an offer and when. Excessive optimism about making offers in the future can cause delays in agreement. The main result states that, despite this, if players will remain sufficiently optimistic for a sufficiently long future, then in equilibrium they will agree immediately. This result is also extended to other canonical models of optimism.

KEYWORDS: Bargaining, misperception, optimism, delay, common-prior assumption.

1. INTRODUCTION

CONSIDERING A STRICT PROCEDURE that determines which player will make an offer and when, Stahl (1972) and Rubinstein (1982) show that, when delay is costly, two players will reach an agreement immediately in equilibrium. Nevertheless, when there are no such strict procedures, the players may hold any beliefs about the negotiation process, and may thereby hold any beliefs about what each player will get in case of a delay. In particular, each player may be so optimistic about what he will get in case of a delay that the players may not reach an agreement at the beginning.

This paper analyzes the problem of reaching an agreement in a model that extends the Rubinstein-Stahl framework, where players hold subjective and possibly optimistic beliefs about *the recognition process* determining which player will make an offer when. The players' beliefs are assumed to be common knowledge; hence we simply drop the common prior assumption in the Rubinstein-Stahl framework. (See the Agreeing to Disagree Theorem by Aumann (1976). For plausibility of the common prior assumption, see Milgrom and Stokey (1982), Kreps (1996), Aumann (1998) and Gul (1998).)

In our model as well as in the Rubinstein-Stahl framework, the recognition process is the only source of bargaining power. In equilibrium, the recognized player at a given date extracts a (noninformational) rent, as he makes an offer

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that can be rejected only by destroying some of the pie. These rents constitute the bargaining power: a player's continuation value is the present value of the rents he expects to extract when he is recognized in the future. Therefore, our analysis will help us to explore the broader question of when two rational individuals can reach an agreement despite holding incompatible beliefs about each player's "bargaining power."

In our model, we have optimism about a date t whenever each player thinks that the probability that he will be recognized (and hence will make an offer) at t is higher than that assessed by the other player. We measure the level of optimism about t by $y_t = p_t^1 + p_t^2 - 1$, where p_t^i is the probability player i assigns to the event that i is recognized at t .

EXAMPLE 0 (Excessive Optimism Can Cause a Delay): Consider the case where two risk-neutral players are trying to divide a dollar, which is worth 1 at $t = 0$, $\delta \in (1/2, 1)$ at $t = 1$, and zero afterwards. It is also common knowledge that each player believes with probability 1 that he will be recognized (and hence will make an offer) at $t = 1$ independent of recognition at $t = 0$. Since the dollar is worth zero afterwards, at $t = 1$, each player is willing to accept any division; hence the recognized player takes the whole dollar. Anticipating this, at $t = 0$, each player expects to take the whole dollar the next day, which is worth δ . Thus, they can agree on a division at $t = 0$ only if each gets at least δ , requiring a minimum total amount of $2\delta > 1$. Since they have only one dollar, they cannot reach an agreement at $t = 0$. Therefore, in equilibrium, it is common knowledge at the beginning that players will not reach an agreement before $t = 1$.

This example illustrates the two-period model of Landes (1971) and Posner (1972). In this model, two parties are to decide whether to settle (in the first period) or to go to a costly trial by a judge or an arbitrator (the second period). The parties have possibly optimistic beliefs about the judgement, typically a wage or an award that is to be paid to the plaintiff by the defendant. As in Example 0, if the parties are excessively optimistic about the judgement, there will be no settlement that satisfies both parties' expectations. Using such reasoning, many authors proposed excessive optimism as a major cause of delay and impasse in bargaining.² Variants of this model have been used to analyze important problems, such as the selection of disputes for litigation (Priest and Klein (1984)), and to test whether optimism is a major cause for bargaining delays (e.g., Farber and Bazerman (1989), who concluded that optimism cannot explain the data about settlement rates before the conventional and the final-offer arbitration).³

² See Hicks (1932), Neale and Bazerman (1985), Babcock et al. (1995), and Babcock and Loewenstein (1997). Some other terms, such as over-confidence or self-serving biases, are also used for what can be called optimism in our context.

³ Of course, the reality is more complicated. In many cases several parts of the case are litigated in various courts. Even when there is only one final judgement, uncertainty about the judgement is *gradually* resolved throughout the process, while the parties can continue bargaining on the side. See Mnookin and Wilson (1989) for such a case study and Yildiz (2002a) for a model in which the cost of delay is the lost contracting opportunities due to the resolution of uncertainty.

In contrast with this literature and Example 0, for the general case we obtain an immediate agreement theorem. It states that, if it is common knowledge that the players will remain *sufficiently optimistic* for a sufficiently long while, then in equilibrium they will reach an agreement *immediately*. That is, in our model, excessive optimism alone cannot be a reason for a delay in agreement.

These two seemingly conflicting results share a common intuition. As in Example 0, players will delay the agreement if they are very optimistic about getting a very high rent in the near future. But the size of this rent depends upon players' expectations in the future. Particularly, if their expectations about the rent at $t + 1$ are high, then the scope of trade at t will be small, allowing only a small rent at t . We show that if the players will remain sufficiently optimistic for a sufficiently long while, then in equilibrium the rents in the near future will be so small that each player will prefer to agree to his opponent's terms rather than waiting and getting these rents, proving our immediate agreement theorem. This is illustrated in the next example.

EXAMPLE 0 (Continued): Now consider a four-period version of the previous game. The dollar is worth 1, δ , δ^2 , and δ^3 at dates 0, 1, 2, and 3, respectively, where $\delta \in (1/2, 1/\sqrt{2})$. The dollar is worth 0 afterwards. Assume that each player is always sure that he will make all the remaining offers. As in the previous game, if the players do not agree before $t = 2$, then they will disagree at $t = 2$, too, as each player will be sure that he will get the whole dollar at date 3. Now consider date $t = 1$. At $t = 1$, each player is certain that, if they do not agree now, then they will not agree at date 2 either, and he will get the dollar only at date 3, which is worth only δ^2 , which is less than $1/2$. In that case, at $t = 1$, in equilibrium the recognized player offers δ^2 to the other player and keeps $1 - \delta^2 > \delta^2$ for himself. His offer will be accepted. Since the recognized player gets a higher share and the players are optimistic about being recognized at $t = 1$, we may again have a disagreement at date 0. It turns out that this is not possible. Since the players are very optimistic at date 1 about their future bargaining powers, the scope of trade at date 1 is very small—the difference between a player's shares at any two individually rational trades can be at most $1 - 2\delta^2$. Hence, optimism about recognition at date 1 can induce only relatively moderate optimism about the shares in an agreement at date 1. In fact, although each player is sure at $t = 0$ that he will be recognized at date 1, he expects only $1 - \delta^2$ as his share at date 1, which is worth only $\delta(1 - \delta^2) \leq 2/(3\sqrt{3}) < 1/2$ at $t = 0$. Therefore, they agree at $t = 0$.

We support our result with other immediate agreement results. First, assuming that the players do not change their beliefs as they play the game, we show that there will be an immediate agreement whenever $y_t - y_{t+1}$ is expected to remain sufficiently small for a sufficiently long while. Second, we extend our immediate agreement theorem to an example where players are optimistic about their outside options, despite the fact that excessive optimism can again cause a delay in a two-period version of this example. Finally, we obtain an immediate agreement

result in a model where the players hold optimistic beliefs about the probability of bargaining-breakdown (and about their discount rates). All these suggest that the delay result in the literature may rely on a simplifying assumption.

There are four points that need to be emphasized. First, this paper is not meant to deny the role of optimism in bargaining delays. It is rather meant to be a first step towards a more careful analysis of optimism and the important role it plays in negotiations. Second, the arguments in this paper are confined to the cases where optimism is about the players' shares in future agreements. Clearly, parties can be optimistic about their outside options that they expect to take in the future. Then, they may wait to take these outside options regardless of the persistency of their opponents' optimism. Third, the agreement results here require that the players do not update their beliefs about the future events drastically as they play the game. If they do, there will be a new incentive to wait, as each player expects the other player to change his mind, and this may cause long delays in equilibrium (see Yildiz (2002b)). Finally, in order to focus on optimism, we rule out asymmetric information about the players' beliefs. If the players' beliefs were not common knowledge, the players would try to form a reputation for being optimistic, as each player's equilibrium payoff is increasing in his own optimism. One would expect that this would lead to delays as in Abreu and Gul (2000).

In the next section, we lay out our model. We compute the equilibrium payoffs in Section 3, and derive our main results in Section 4. Section 5 contains counterexamples. In Section 6, we discuss how the basic intuition can be extended to the other canonical models of optimism. Section 7 concludes. A technical proof is relegated to the Appendix.

2. MODEL

Write \mathbb{R}^k for a k -dimensional Euclidean space, \mathbb{N} for the set of nonnegative integers. Take $T = \{t \in \mathbb{N} | t < \bar{t}\}$ to be the time space for some $\bar{t} \leq \infty$, $N = \{1, 2\}$ to be the set of players, and $U = \{u \in [0, 1]^2 | u^1 + u^2 \leq 1\}$ to be the set of all feasible expected utility pairs.

We will analyze the following perfect-information game. At each $t \in T$, Nature recognizes a player $i \in N$; i offers an alternative $u = (u^1, u^2) \in U$; if the other player accepts the offer, then the game ends yielding a payoff vector $\delta' u = (\delta' u^1, \delta' u^2)$ for some $\delta \in (0, 1)$; otherwise, the game proceeds to date $t + 1$, except for $t = \bar{t} - 1$, when the game ends yielding payoff vector $(0, 0)$. We write $\rho = (\rho_t)_{t \in T}$ for the recognition process, $\rho^t \in N^t$ for a generic history of the recognized players before date t (i.e., on $\{0, 1, \dots, t-1\}$), and $p_i^t(\rho^s)$ for the probability player i assigns to the event that i will be recognized at date t given the history $\rho^s \in N^s$ with $s \leq t$. By a *belief structure*, we mean any full list $p = \{p_i^t(\rho^s)\}_{i,t,\rho^s}$ of such probability assessments. Everything described in this paragraph is common knowledge.

Notice that we have two sets of beliefs, one for each player; this is our only departure from the Rubinstein-Stahl framework (e.g., Binmore (1987) and Merlo and Wilson (1995)).

Since we excluded the contingency that no player is recognized, players' probability assessments agree for any date t and at any history ρ^s if and only if $p_t^1(\rho^s) + p_t^2(\rho^s) = 1$. If $p_t^1(\rho^s) + p_t^2(\rho^s) > 1$, then each player thinks that the probability that he is going to be recognized at t is higher than what the other player assesses. As explained in the Introduction, being recognized is (weakly) good; so we say that *players are optimistic for t at ρ^s* if and only if $p_t^1(\rho^s) + p_t^2(\rho^s) \geq 1$; we say that *players are pessimistic for t at ρ^s* if and only if $p_t^1(\rho^s) + p_t^2(\rho^s) \leq 1$.⁴ We write

$$y_t(\rho^s) = p_t^1(\rho^s) + p_t^2(\rho^s) - 1$$

for the level of optimism for t at ρ^s . The common-prior assumption in our context is

$$(\text{CPA}) \quad y_t \equiv 0 \quad (\forall t \in T).$$

NOTATION: We will designate dates $t, s \in T$, a utility pair $u = (u^1, u^2)$, players $i, j, k \in N$ with $i \neq j$, and histories $\rho^t \in N^t$ and $\rho^s \in N^s$ as generic members. Also, $E^i[\cdot | \rho^t]$ denotes the conditional expectation of i at ρ^t , and $(\rho^t, i) \in N^{t+1}$ denotes the history in which i is recognized after ρ^t .

3. CONTINUATION VALUES IN EQUILIBRIUM

For each player i , write Σ_i^* for the set of strategies of i that survive iterated elimination of conditionally dominated strategies; write also $\Sigma^* = \Sigma_1^* \times \Sigma_2^*$. This procedure of elimination is equivalent to backwards induction for generic finite games, and Σ^* contains all subgame-perfect equilibria (henceforth SPE).⁵ Since our game does have a SPE,⁶ Σ^* is nonempty. Our first theorem establishes that Σ^* is very small, and all SPE are payoff-equivalent.

THEOREM 1: *Given any $(t, \rho^t, i) \in T \times N^t \times N$, there exists a unique $V_t^i(\rho^t) \in [0, 1]$ such that, for each fixed strategy of $j \neq i$ in Σ_j^* , $V_t^i(\rho^t)$ is the supremum of the present value of the payoffs of player i at the beginning of t , given ρ^t , as the strategy of player i varies. In particular, at any SPE, the continuation value of i at the beginning of t given ρ^t is $V_t^i(\rho^t)$. Moreover,*

$$(1) \quad V_t^i(\rho^t) = p_t^i(\rho^t) \max\{1 - \delta V_{t+1}^j(\rho^t, i), \delta V_{t+1}^i(\rho^t, i)\} \\ + (1 - p_t^i(\rho^t)) \delta V_{t+1}^i(\rho^t, j).$$

⁴ We define optimism of a player relative to the other player's beliefs. We could, of course, use an absolute notion of optimism, by introducing an "objective" probability distribution for ρ . Clearly, our strategic analysis would not refer to such an "objective" probability distribution, rendering the relative notion of optimism as the appropriate notion. Note that, whenever our agents are optimistic (resp., pessimistic) in our relative sense, there exists an "objective" distribution under which both agents are optimistic (resp., pessimistic) in the absolute sense; one such distribution is given by the arithmetic averages of the two players' probability assessments.

⁵ See Fudenberg and Tirole (1991) for details.

⁶ For instance the result of Harris (1985) can be extended to our case. The existence and uniqueness of SPE payoffs can also be proven using techniques similar to Merlo and Wilson (1998).

PROOF: The proof of this theorem is relegated to the Appendix. *Q.E.D.*

Theorem 1 establishes two facts. First, all the SPE are payoff-equivalent, and derived via iterative application of conditional dominance. Hence, our analysis is immune to the critique of Dekel, Fudenberg, and Levine (2002). Second, by (1), at any SPE, if recognized, a player i gets $\max\{1 - \delta V_{t+1}^j(\rho^t, i), \delta V_{t+1}^i(\rho^t, i)\}$, while he gets only $\delta V_{t+1}^i(\rho^t, j)$ if he is not recognized. Hence, when he is recognized, he uses his temporal monopoly power and extracts a (noninformational) *rent* of size $\max\{1 - \delta S_{t+1}(\rho^t, i), 0\}$, in addition to the difference $\delta V_{t+1}^i(\rho^t, i) - \delta V_{t+1}^i(\rho^t, j)$ in the discounted continuation values in the next date. This is similar to the well-known first-mover advantage in the alternating-offer bargaining. The players' SPE payoffs are the present value of all these rents (see Yildiz (2001)). The players have different priors about who is going to get these rents.

Towards describing the SPE and thereby equation (1) further, let us write $S_t = V_t^1 + V_t^2$ for the “perceived size of the pie” at the beginning of any date t —as a function of ρ^t .

AGREEMENT AND DISAGREEMENT REGIMES IN EQUILIBRIUM: We have two regimes. The first one, which we call the *disagreement regime*, is characterized by the inequality $S_{t+1}(\rho^t, i) > 1/\delta$. In this regime, if they have not yet reached an agreement, the players do not reach an agreement at (ρ^t, i) , either. For, if they do not agree at (ρ^t, i) , each player k believes that he will get $V_{t+1}^k(\rho^t, i)$ at $t + 1$, which is equivalent to $\delta V_{t+1}^k(\rho^t, i)$ at t . They will then agree on $u = (u^1, u^2)$ only if $u^k \geq \delta V_{t+1}^k(\rho^t, i)$ at each $k \in N$, which requires that $u^1 + u^2 \geq \delta V_{t+1}^1(\rho^t, i) + \delta V_{t+1}^2(\rho^t, i) = \delta S_{t+1}(\rho^t, i) > 1$, showing that such u is not feasible. The second regime is called the *agreement regime*, and is characterized by the inequality $S_{t+1}(\rho^t, i) \leq 1/\delta$. In this regime, if they have not yet reached an agreement, the players immediately agree on a division that gives $1 - \delta V_{t+1}^j(\rho^t, i)$ to the recognized player i , leaving $\delta V_{t+1}^j(\rho^t, i)$ to the other player j , who barely accepts this. This is expressed in (1) in terms of player i 's expectations.

Notice also that the SPE actions are unique at each (ρ^t, i) except for the following two cases: (i) when $S_{t+1}(\rho^t, i) > 1/\delta$, the recognized player can make different offers that are meant to be rejected; (ii) when $S_{t+1}(\rho^t, i) = 1/\delta$, both players will be indifferent between agreeing at t and delaying the agreement until $t + 1$, and either behavior is consistent with equilibrium. In the sequel, we will ignore these trivial multiplicities.

4. IMMEDIATE AGREEMENT

In this section, under Assumption IND below, we will prove that in equilibrium players will reach an agreement immediately, so long as they remain optimistic for a sufficiently long future. We will then strengthen this result by showing also that there is an immediate agreement, provided that the level of optimism does not drop suddenly, i.e., $y_t - y_{t+1}$ remains small for a long while.

ASSUMPTION IND: *The players perceive the recognition process ρ to be independently distributed: $p_i(\rho^s) = p_i(\hat{\rho}^s)$ for all $(\rho^s, \hat{\rho}^s, t, i)$.*

Under this assumption, p, y, V , and S are all deterministic, hence whether we have an agreement regime at any date does not depend on the history. This yields a uniform bound on delay: In any disagreement regime, by (1), we have $V_t = \delta V_{t+1}$, and hence $S_t = \delta S_{t+1}$. Since $S_{t+1} \leq 2$, this implies that an interval of disagreement regimes can be at most as long as $\bar{L}(\delta)$ defined by $1 < 2\delta^{\bar{L}(\delta)} \leq 1/\delta$.⁷

On the other hand, in any agreement regime, the recognized player i extracts a rent $1 - \delta S_{t+1}$. Now, (1) becomes $V_t^i = p_i^i(1 - \delta S_{t+1}) + \delta V_{t+1}^i$. By adding this equation up for players, we obtain

$$(2) \quad S_t = 1 + y_t(1 - \delta S_{t+1}).$$

This implies that when $y_t(1 - \delta S_{t+1})$ is large, S_t may be larger than $1/\delta$, causing a delay at $t - 1$. That is, a high rent for the recognized player aligned with excessive optimism may prevent the players from reaching an agreement in some previous dates. We now show that there is an immediate agreement in equilibrium whenever the players are expected to remain optimistic for a sufficiently long while.

LEMMA 1: *Assume IND. Given any t with $y_t \geq 0$, if $S_{t+1} \in [1, 1/\delta]$, then $S_t \in [1, 2 - \delta] \subset [1, 1/\delta]$.*

PROOF: Assume that $S_{t+1} \in [1, 1/\delta]$. Then, we have an agreement regime at t , and hence by (2), we have $S_t = 1 + y_t(1 - \delta S_{t+1})$. Since $y_t \in [0, 1]$ and $1 - \delta S_{t+1} \in [0, 1 - \delta]$, we therefore have $S_t = 1 + y_t(1 - \delta S_{t+1}) \in [1, 2 - \delta]$. Note that $2 - \delta \leq 1/\delta$. *Q.E.D.*

Lemma 1 can be spelled out as follows. Consider a date t at which the players are expected to reach an agreement (i.e., $S_{t+1} \leq 1/\delta$). Firstly, if the players' expectations about $t + 1$ are sufficiently high at t (i.e., if $S_{t+1} \geq 1$), then the rent $1 - \delta S_{t+1}$ at t will be so low that the players will prefer agreeing at $t - 1$ to getting this rent at t , even if each player is extremely optimistic about getting the rent; i.e., $S_t \leq 1/\delta$. Second, if $y_t \geq 0$, then $S_t \geq 1$. That is, if the players are optimistic for t , then their expectations about t will be so high, and hence the rent at $t - 1$ will be so low that, whenever the players are expected to agree at $t - 1$, they are also expected to agree at $t - 2$. Lemma 1 yields the following immediate agreement theorem, which is our main result.

THEOREM 2: *Assume IND. Given any $\hat{t} \in T$, assume also that $y_t \geq 0$ for each $t \leq \hat{t}$. Then, we have an agreement regime at each $t \in T$ with $t < \hat{t} - \bar{L}(\delta) - 1$.*

⁷ **PROOF:** Given any interval $\{t, \dots, \hat{t} - 1\}$ of disagreement regimes, $S_t = \delta^{\hat{t}-t} S_{\hat{t}}$. Hence, whenever $\hat{t} - t \geq \bar{L}(\delta)$, $S_t = \delta^{\hat{t}-t} S_{\hat{t}} \leq 2\delta^{\hat{t}-t} \leq 1/\delta$, yielding an agreement regime at $t - 1$, and showing that no interval of disagreement regimes can be longer than $\bar{L}(\delta)$. On the other hand, if $S_{\hat{t}} = 2$, then for any $t \geq \hat{t} - \bar{L}(\delta) + 1$ we have $S_t = \delta^{\hat{t}-t} S_{\hat{t}} \geq 2\delta^{\bar{L}(\delta)-1} = 2\delta^{\bar{L}(\delta)}/\delta > 1/\delta$. Hence, $\{\hat{t} - \bar{L}(\delta), \dots, \hat{t}\}$ is an interval of disagreement regimes—of length $\bar{L}(\delta)$.

PROOF: Firstly, $S_i \geq 1$. For, if $S_{i+1} > 1/\delta$, then we have a disagreement regime at \hat{t} , and hence $S_i = \delta S_{i+1} > 1$; if $S_{i+1} \leq 1/\delta$, then $S_i = 1 + y_i(1 - \delta S_{i+1}) \geq 1$ as $y_i \geq 0$.

We have two cases. First consider the case $S_i \leq 1/\delta$. Then, $S_i \in [1, 1/\delta]$. Hence, using Lemma 1 inductively, we conclude that, at each $t \leq \hat{t} - 1$, we have $S_{t+1} \in [1, 1/\delta]$, and hence an agreement regime. Now consider the case that $S_i > 1/\delta$. In that case, there is an interval of disagreement regimes of length $L(S_i, \delta) \leq \bar{L}(\delta)$ that ends at $\hat{t} - 1$. Now, assuming that \hat{t} is sufficiently large, consider the last date with an agreement regime before $\hat{t} - 1$, namely $\tilde{t} = \hat{t} - 1 - L(S_i, \delta)$. By definition, we have $S_{\tilde{t}+1} \leq 1/\delta$ and $S_{\tilde{t}+2} > 1/\delta$. By the latter inequality, $S_{\tilde{t}+1} = \delta S_{\tilde{t}+2} > 1$, i.e., $S_{\tilde{t}+1} \in [1, 1/\delta]$. Once again, using Lemma 1 inductively, we conclude that $S_{t+1} \in [1, 1/\delta]$ at each $t \leq \tilde{t}$, showing that we have an agreement regime at each $t \leq \tilde{t}$. Q.E.D.

Some authors (e.g., Babcock et al. (1995)) present optimism as an alternative explanation to asymmetric information. They also argue that optimism is very common and the parties do not recognize their own biases, although they do recognize the other parties' biases. It seems to be appropriate to model such a world with persistent optimism and without asymmetric information. In such a model, it must be common knowledge that the players will remain optimistic for a long while. But Theorem 2 states for the present canonical model that in that case the players will reach an agreement immediately. It thus suggests that the previous authors' arguments rely on the specific assumptions they (implicitly) make.

The proof of Theorem 2 is based on two pieces of intuition. First, if excessive optimism about one's bargaining power causes disagreement at some date \hat{t} , then the players know at date $\hat{t} - 1$ that they will not agree at \hat{t} and hence $1 - \delta$ units of surplus will be destroyed in case of a disagreement at $\hat{t} - 1$. Hence, the effect of optimism is dampened by one period of disagreement. This dampening continues until we reach a date \tilde{t} at which, given the fact that $\hat{t} - \tilde{t}$ periods of delay will result from an inability to reach an agreement today, the players prefer to agree immediately. Second, at date \tilde{t} , the players must be relatively optimistic about their future bargaining power, because their optimism yields disagreement at $\tilde{t} + 1$. Such high level of optimism leaves relatively small room for trade. Hence, their optimism about the recognition at \tilde{t} cannot yield excessive optimism about their shares at \tilde{t} . They thus agree at $\tilde{t} - 1$. Lemma 1 establishes that if the players remain optimistic throughout $\{0, 1, \dots, \tilde{t}\}$, then their optimism about their shares at any date $t \in \{0, 1, \dots, \tilde{t}\}$ will be so low that there will be an immediate agreement. (See the paragraph after the proof of Lemma 1.)

Theorem 2 has the following immediate corollary for the infinite-horizon case, generalizing the immediate agreement result in the Rubinstein-Stahl framework to ours.

COROLLARY 1: *Let $\bar{t} = \infty$. Assume IND and that $y_t \geq 0$ for each $t \in T$. Then, we have an agreement regime at each $t \in T$.*

Theorem 2 provides a sufficient condition for immediate agreement: that the players will remain optimistic for a long while. We know another sufficient condition from the Rubinstein-Stahl framework (by continuity): that the players will never be excessively optimistic. The next theorem provides a third sufficient condition that unifies these two distinct conditions, qualitatively. It states that there is an immediate agreement provided that the level of optimism will not drop suddenly in the near future, i.e., $y_t - y_{t+1}$ remains small for a long while. (Clearly, if the players are always optimistic, or never excessively optimistic, then $y_t - y_{t+1}$ cannot be very large.)

THEOREM 3: *Assume IND. Given any $\hat{t} \in T$, assume also that $y_t - y_{t+1} \leq (1 - \delta)/\delta$ at $t \leq \hat{t}$. Then, we have an agreement regime at each $t \in T$ with $t \leq \hat{t} - \bar{L}(\delta) - 1$.*

Towards proving this theorem, the next lemma states that if a disagreement regime precedes an agreement regime, there must be a substantial drop in y .

LEMMA 2: *Assume IND. Given any $t < \bar{t} - 1$, if $S_t > 1/\delta$ and $S_{t+1} \leq 1/\delta$, then $y_t - y_{t+1} > (1 - \delta)/\delta$.*

PROOF: Take t as in the hypothesis. One can show that $y_t > 0$, $y_{t+1} < 0$, and we have agreement regimes at t and $t + 1$. Hence, by applying (2) inductively at t and $t + 1$, and writing $\Delta = y_t - y_{t+1}$, we obtain $1/\delta < S_t = 1 + y_t - \delta y_t(1 + y_t) + \delta y_t \Delta + \delta^2 y_t y_{t+1} S_{t+2}$. Since $y_t y_{t+1} S_{t+2} \leq 0$, this gives us $\Delta > (1/(\delta y_t))(\frac{1}{\delta} - 1) - (\frac{1}{\delta} - 1) + y_t$. The last expression is minimized at $y_t^* = (1/\sqrt{\delta}) \sqrt{1/\delta - 1}$, taking the value of $2\sqrt{(1/\delta)(1/\delta - 1)} - (1/\delta - 1) > 1/\delta - 1$. Q.E.D.

PROOF OF THEOREM 3: Take any $\hat{t} \in T$ and any $t < \hat{t} - \bar{L}(\delta) - 1$. Since delays are uniformly bounded by $\bar{L}(\delta)$, there exists a date $\tilde{t} \in \{t, \dots, \hat{t} - 1\}$ at which there is an agreement regime. Given any $s \in \{t, \dots, \tilde{t}\}$, since $y_s - y_{s+1} \leq (1 - \delta)/\delta$ and $s < \bar{t} - 1$, Lemma 2 implies that if there is an agreement regime at s , there is also an agreement regime at $s - 1$. By induction, there must be an agreement regime at t . Q.E.D.

There is a close relation between the loss of optimism and the deadlines. By (1), it does not matter in equilibrium whether $y_t = -1$ (when no player expects to make an offer at t) or the players will not negotiate at t . Hence, a finite-horizon game of length \bar{t} is isomorphic to an infinite-horizon game in which $y_t = -1$ at each $t \geq \bar{t}$. When $y_{\bar{t}-1}$ is large, there will be a sudden drop in y at \bar{t} . This may yield disagreement near the end of the game. Theorem 3 implies that, if the transition to -1 is smooth enough, there will be no disagreement regime near the end of the game, either. Likewise, if $y_t = -1$ at each $t > t^*$ for some t^* , then in equilibrium the players will act as if the game ends at t^* . In that case, if $y_{t^*} = 1$ and $0 < t^* \leq \bar{L}(\delta)$, then the players will reach an agreement only at t^* . That is, agreement might be delayed when the players are expected to lose their

optimism in the near future. Therefore, the assumptions about the persistency of optimism in Theorems 2 and 3 are not superfluous. The other assumptions are not superfluous either—as we show next.

5. COUNTEREXAMPLES

Now, we present counterexamples to show that Assumption IND and the assumption that the beliefs are common knowledge are not superfluous in previous results. We also discuss an extension of Theorem 2. The first example shows that Assumption IND is not superfluous for Theorem 2, and for the result that the delays are uniformly bounded.

EXAMPLE 1: Letting $\bar{t} = \infty$, consider the following belief structure where we distinguish the “alternating” history in which no player is recognized twice in a row: Given any “alternating” history until any date t , each player thinks that he will be recognized at t with some probability $\bar{p} > 1/2$. In any other history, the players commonly believe that the player who has been recognized twice in a row for the first time in the history will be recognized with probability \bar{p} . For high values of δ , this will yield disagreement at any alternating history. (See Yildiz (2001) for details.)

Under Assumption IND and the assumption of transferable utility (i.e., $U = \{u \in [0, 1]^2 | u^1 + u^2 \leq 1\}$), Theorem 2 established that, if the level y of optimism stays *nonnegative* for a sufficiently long future, the players will reach an agreement immediately. Example 1 has shown that Assumption IND is not superfluous. Yildiz (2001) also provides a counterexample showing that the assumption of transferable utility is not superfluous either. Nevertheless, *qualitatively*, this theorem can be extended beyond these assumptions: if the level y of optimism stays *sufficiently high* for a sufficiently long future, the players will reach an agreement immediately (see Yildiz (2001)). While a weaker form of Assumption IND remains through the requirement that y stays sufficiently high, the assumption of transferable utility is dropped entirely.

The next example shows that Assumption IND is not superfluous in Theorem 3. Note that in the specific model analyzed by Yildiz (2002b), where Assumption IND does not hold, a sufficient condition for agreement at $t - 1$ is $y_t - y_{t+1} < (1 - \delta)/\delta$. Hence, Assumption IND is not necessary for Theorem 3, either.

EXAMPLE 2: Take $\bar{t} = \infty$, $\delta^2 > 1/2$, and $p_0^1 = p_0^2 = p_1^1 = p_1^2 = 0$. Assume that the players agree that, if a player is recognized at $t = 1$, he will not be recognized again. Note that $y = (-1, -1, 0, 0, 0, \dots)$. For $j \neq i$, we have $V_2^j(\rho^1, i) = 1$, hence the recognized player i at date 1 will offer δ to the other player j and keep $1 - \delta$ for himself. Hence, $V_1^1 = V_1^2 = \delta$, yielding $\delta S_1 = 2\delta^2 > 1$, and causing a disagreement at $t = 0$.⁸

⁸ In this example, optimism for $t = 1$ is equivalent to pessimism for the rest of the game; hence the players do lose their optimism about the rest of the game at $t = 1$, despite the rise in y . It remains an

A major assumption in our model is that the players' beliefs are common knowledge, as we want to focus on the implications of differing beliefs. Without this assumption, there might be delays (and multiple equilibria) as in usual bargaining models with asymmetric information. Firstly, as already mentioned in the Introduction, the players might delay the agreement in order to form a reputation for being optimistic. Second, the players might try to separate their opponents using delay—as illustrated in the next example (which was suggested by Eddie Dekel).

EXAMPLE 3: Consider the following belief structure with asymmetric information. Player 1 is always sure that he will make all the remaining offers, i.e., $p_t^1 = 1$ for all t . Player 2 has two types: type 1 shares the beliefs of player 1 completely, while type 2 differs from them by believing with certainty that player 2 will make an offer at $t = 1$. Type 2 is also sure that player 1 will be recognized at any $t \neq 1$. Player 1 assigns probability $\pi > 1 - \delta(1 - \delta)$ to type 1, and everything described up to here is common knowledge. There will be delay whenever player 2 is of type 2. To see this, note that, at $t = 0$, player 2 of type 1 will accept any offer, while player 2 of type 2 will accept an offer (u^1, u^2) if and only if $u^2 \geq \delta(1 - \delta)$, as the latter is sure that player 2 will be recognized at $t = 1$ and offer $(\delta, 1 - \delta)$, which will be accepted. Now, player 1 must offer either $(1, 0)$, which will be accepted with probability π , or $(1 - \delta(1 - \delta), \delta(1 - \delta))$, which will be accepted with probability 1. Since $\pi > 1 - \delta(1 - \delta)$, in equilibrium he will offer $(1, 0)$.

6. ROBUSTNESS TO ALTERNATIVE MODELS OF OPTIMISM

Considering a sequential bargaining model in which the players can be optimistic about who will make an offer and when, we have shown that, even though excessive optimism may cause a delay in agreement, if the optimism is sufficiently persistent, the players will reach an agreement immediately. We have chosen to analyze this model, because in the most widely used bargaining model (namely the Rubinstein-Stahl framework), the players' bargaining power is ultimately determined by who will make an offer and when. Of course, bargaining power can be modeled through other aspects in the bargaining, such as the outside options or the discount rates, and the players may have optimistic views about these aspects. In this section, we will illustrate that our conclusion holds in a broader world, by examining two such examples.⁹

open question whether Theorem 3 can be extended to all games with affiliated recognition processes, where the recognition at any date (weakly) increases the probability of recognition in the remaining game.

⁹ There are other plausible models in which we have long delays even under the common-prior assumption, e.g., models with option values as in Avery and Zemsky (1994) or with the possibility of retracting the offers as in Muthoo (1995). Clearly, allowing differing beliefs would not prevent the bargaining delays in these models, and hence our results cannot be extended to these models. Incidentally, Muthoo (1995) also allows differing beliefs.

6.1. Optimism about the Outside Options

We now consider an example in which we allow our players to opt out and take their outside options, about which they have differing opinions. The outside options here are as in Shaked and Sutton (1984). We illustrate that all our results can be extended to this example: even though excessive optimism can cause a delay in a short game, there is an immediate agreement when the game is sufficiently long.

EXAMPLE WITH OUTSIDE OPTIONS: Consider the original model with the following modifications. At any t , the recognized player either opts out or makes an offer. If he makes an offer, the other player either accepts the offer, rejects it, or opts out. If the offer is accepted or rejected, then the game proceeds as in the original model. If any player opts out, the bargaining ends and each player i gets b_i^t . Players commonly believe that $b_i^t \equiv 0$, but they have differing beliefs about b_i^t ; player 1 is sure that $b_1^t = B \in (0, 1)$ while player 2 thinks that $b_1^t = 0$. The process (b_0^1, b_1^1, \dots) is perceived to be independently distributed and b_i^t becomes common knowledge at the beginning of t . We take $p_1^1 + p_2^1 = 1$, so that the outside options are the only source of optimism.

A SPECIAL CASE (When the Game is Short, Optimism Can Cause a Delay): Take $\bar{t} = 2$. At $t = 1$, they will agree on $(1, 0)$ if player 1 is recognized, and on $(b_1^1, 1 - b_1^1)$ if player 2 is recognized. Player 1 is sure that $b_1^1 = B$, hence $V_1^1 = p_1^1 + p_2^1 B$. On the other hand, player 2 is sure that $b_1^1 = 0$, hence $V_1^2 = p_1^2$. Therefore, $S_1 = 1 + p_1^2 B$. When δ and B are sufficiently large, we can choose p_1^2 so that $S_1 > 1/\delta$ and $\delta V_1^1 > B$ (i.e., $(1/\delta - 1)/B < p_1^2 < (1 - B/\delta)/(1 - B)$). In that case, if player 1 is recognized at $t = 0$, the agreement will be delayed. (At $t = 0$, player 2 will accept an offer u if $u^2 \geq \delta p_1^2$, and reject it otherwise. Since $\delta V_1^1 > \max\{B, 1 - \delta p_1^2\}$, player 1 must make an offer that will be rejected.)

GENERAL CASE: The equilibrium behavior is as follows. If $\delta V_{t+1}^1 \geq b_1^t$ (i.e., if $b_1^t = 0$ or if $\delta V_{t+1}^1 \geq B$), then the equilibrium behavior at t is as before: if $\delta S_{t+1} > 1$, they do not agree at t ; otherwise, they agree on the division that gives $1 - \delta V_{t+1}^j$ to the recognized player i and δV_{t+1}^j to $j \neq i$. Assume that $b_1^t = B > \delta V_{t+1}^1$. Then, player 1 accepts an offer u if $u^1 \geq B$, and opts out otherwise. Hence, if recognized, player 2 offers $(B, 1 - B)$. When player 1 is recognized, player 2 accepts an offer u if $u^2 \geq \delta V_{t+1}^2$ and rejects it otherwise. In that case, player 1 offers $(1 - \delta V_{t+1}^2, \delta V_{t+1}^2)$ if $B \leq 1 - \delta V_{t+1}^2$, and opts out otherwise.

PROPOSITION 1: Define $\hat{\delta} \equiv \delta + B(1 - \delta)$ and $\hat{L} \equiv \min\{L | 2\hat{\delta}^L \leq 1/\delta\}$. In the example of this subsection, whenever $\bar{t} > \hat{L} + 1$, the bargaining ends at $t = 0$.

PROOF: Using the equilibrium behavior above, we derive the difference equations for V_t and S_t , and tabulate them in Table I. If $S_{t+1} > 1/\delta$, then $S_t =$

TABLE I
THE DIFFERENCE EQUATIONS WITH OPTIMISM ABOUT THE OUTSIDE OPTIONS

Case ^a	V_t^1	V_t^2	S_t
$\delta S_{t+1} > 1$			
$\delta V_{t+1}^1 \geq B$	δV_{t+1}^1	δV_{t+1}^2	δS_{t+1}
$\delta V_{t+1}^1 < B$	B	δV_{t+1}^2	$B + \delta V_{t+1}^2$
$\delta S_{t+1} \leq 1$			
$\delta V_{t+1}^1 \geq B$	$p_t^1(1 - \delta V_{t+1}^2) + p_t^2 \delta V_{t+1}^1$	$p_t^1 \delta V_{t+1}^2 + p_t^2(1 - \delta V_{t+1}^1)$	1
$\delta V_{t+1}^1 < B \leq 1 - \delta V_{t+1}^2$	$p_t^1(1 - \delta V_{t+1}^2) + B$	$p_t^1 \delta V_{t+1}^2 + p_t^2(1 - \delta V_{t+1}^1)$	$1 + p_t^2(B - \delta V_{t+1}^1)$
$B > 1 - \delta V_{t+1}^2$	B	$p_t^1 \delta V_{t+1}^2 + p_t^2(1 - \delta V_{t+1}^1)$	$V_t^1 + V_t^2$

^a The case in which the equation is valid, e.g., when $\delta S_{t+1} > 1$ and $\delta V_{t+1}^1 < B$, $V_t^1 = B$ and $V_t^2 = \delta V_{t+1}^2$.

$\max\{\delta S_{t+1}, B + \delta V_{t+1}^2\} \leq \hat{\delta} S_{t+1}$.¹⁰ Hence, no interval with $S_{t+1} > 1/\delta$ can be longer than \hat{L} . Thus, if the game is sufficiently long, we will have $S_{\tilde{t}+1} \leq 1/\delta$ for some $\tilde{t} < \bar{t} - 1$. On the other hand, at any $t < \tilde{t} - 1$, since player 1 thinks that he has an option to get B at $t + 1$, we have $V_{t+1}^1 \geq B$. Using this, we check from Table I that, if $S_{t+1} \leq 1/\delta$, then $S_t \leq 1/\delta$,¹¹ showing by induction that $S_{t+1} \leq 1/\delta$ at each $t \leq \tilde{t}$. But, when $S_{t+1} \leq 1/\delta$, the agreement is not delayed—either player 1 opts out or they reach an agreement. *Q.E.D.*

In summary, even though optimism about the outside options may cause a delay in agreement in general, if the game is sufficiently long and the players perceive the process (b_0^1, b_1^1, \dots) to be independently distributed, then bargaining will end immediately.

It is crucial that, as in the original model, the uncertainty about t is resolved at the beginning of t , when the players learn the true value of b_t . If they did not learn until the offer is accepted or rejected, then the equilibrium outcome would be as if the players commonly believed that $b_t^1 = B < 1$. Then, we would have an immediate agreement even in a short game.¹²

REMARK: Excessive optimism can make the future outside options appear so large that we may always have delays. For example, consider the case that player 1 is sure that $b_1 = (B, 0)$, while player 2 is sure that $b_1 = (0, B)$ for some $B \in (1/2\delta, 1)$. If $b_0 = 0$, there will be a delay—independent of the length of the

¹⁰ To see this, assume $S_{t+1} > 1/\delta$. Note that $V_{t+1}^1 \geq B$, hence $B - \delta V_{t+1}^1 \leq (1 - \delta)B < (1 - \delta)BS_{t+1}$. Now, if $\delta V_{t+1}^1 < B$, then (by Table I) $S_t = B + \delta V_{t+1}^2 = B - \delta V_{t+1}^1 + \delta S_{t+1} < (1 - \delta)BS_{t+1} + \delta S_{t+1} = \hat{\delta} S_{t+1}$. If $\delta V_{t+1}^1 \geq B$, then $S_t = \delta S_{t+1} < \hat{\delta} S_{t+1}$.

¹¹ When $\delta V_{t+1}^1 < B \leq 1 - \delta V_{t+1}^2$, we have $S_t = 1 + p_t^2(B - \delta V_{t+1}^1) \leq 1 + p_t^2 B(1 - \delta) < 2 - \delta \leq 1/\delta$. Since $1 - \delta V_{t+1}^1 \geq \delta V_{t+1}^2$, when $B > 1 - \delta V_{t+1}^2$, we have $S_t \leq B + 1 - \delta V_{t+1}^1 \leq 1 + B(1 - \delta) < 1/\delta$.

¹² If they learned after the offer is made but before the reply, then there might have been long delays even under the common-prior assumption (see Avery and Zemsky (1994)). Also, if the recognized player were allowed to opt out and take his outside option if his offer is rejected, then there might have been multiple equilibria and long delays even under the common-prior assumption (see Ponsati and Sakovics (1998)).

game or of the time when they learn b_i . This is because each player expects to have an outside option of B at $t = 1$, yielding $S_1 \geq 2B > 1/\delta$, and causing a disagreement at $t = 0$. Note that, when optimism is persistent, the players here expect to take their outside options, and hence their optimism is not about their share in a future agreement. To see this, assume (i) \bar{t} is finite but large, (ii) the players perceive $(b_1, b_2, \dots, b_{\bar{t}-1})$ to be independently and identically distributed, and (iii) b_0 and b_1 are as above. One can easily check that if the players learn the value of their outside options only at the end of the period, then in equilibrium they will take outside options at each $t \geq 1$. Now consider the case that they learn b_i at the beginning of t . For any $t < \bar{t} - 1$, independent of b_i , each player expects to have an outside option of B at $t + 1$, and hence $V_{t+1}^i \geq B$, requiring at least $2\delta B$ for an agreement at t . Therefore, the players cannot agree at t . In fact, at any $t \geq 1$ with $\delta^{t-1} < B$, the players take their outside options.

6.2. Disagreement on the Probability of Breakdown and the Discount Rates

A player's bargaining power is usually measured by his discount rate, and a typical interpretation of discounting is the probability of bargaining-breakdown, as in Binmore, Rubinstein, and Wolinsky (1986). Now, we will allow players to hold differing beliefs about the likelihood of breakdown. We will show that such disagreement in beliefs cannot cause a delay.

MODEL: We assume that the players do not discount the future payoffs but after any offer is rejected the bargaining may break down, yielding payoff vector $(0, 0)$. At $\bar{t} - 1$, the bargaining breaks down for sure. Each player i thinks that the probability of breakdown at $t < \bar{t} - 1$ is $1 - \delta_t^i \in (\epsilon, 1)$ for some $\epsilon > 0$. We take $p_i^1 + p_i^2 = 1$, and write $K_t = \delta_t^1 V_{t+1}^1 + \delta_t^2 V_{t+1}^2$.

We now show that in this model there will be immediate agreement *even if the game is short*. We first present our main step:

LEMMA 3: *At any $t \in T$, if $K_t \leq 1$, then $S_t \equiv V_t^1 + V_t^2 = 1$, and therefore $K_{t-1} < 1$.*

PROOF: If $K_t \leq 1$, at t , they agree to give $1 - \delta_t^j V_{t+1}^j$ to the recognized player i and $\delta_t^j V_{t+1}^j$ to j . Hence, $V_t^i = p_t^i(1 - \delta_t^j V_{t+1}^j) + (1 - p_t^i)\delta_t^i V_{t+1}^i = p_t^i(1 - K_t) + \delta_t^i V_{t+1}^i$. Therefore, $S_t = 1 - K_t + \delta_t^1 V_{t+1}^1 + \delta_t^2 V_{t+1}^2 = 1$. Q.E.D.

Lemma 3 implies that, if $S_{\hat{t}} = 1$ at any \hat{t} , then $S_t = 1$ at each $t \leq \hat{t}$, yielding an agreement regime at each $t < \hat{t}$.

PROPOSITION 2: *In the model of this subsection, we have an agreement regime at each $t \in T$.*

PROOF: When $\bar{t} < \infty$, $S_{\bar{t}-1} = p_{\bar{t}-1}^1 + p_{\bar{t}-1}^2 = 1$; hence the Proposition follows from Lemma 3. Assume $\bar{t} = \infty$. By Lemma 3, if $S_t \neq 1$, then $K_s > 1$ at each $s \geq t$. But then, at each $s \geq t$, there is disagreement and $V_s^i = \delta_s^i V_{s+1}^i$, yielding $K_{s-1} < (1 - \epsilon)K_s$. Since $K_s \leq 2$ at each s , this is a contradiction. Q.E.D.

We have an immediate agreement independent of the level and the duration of optimism about the likelihood of the bargaining breakdown.

Instead of bargaining breakdown, some other factors, such as the interest rates, may generate the discounting. Now, the players may have differing beliefs about the discount rates, as they may have differing beliefs about these factors. As in the case of bargaining breakdown, this will not yield any delay in equilibrium, provided that the players do not update their beliefs about the future discount rates as they play the game.¹³ This is simply because, when the beliefs are common knowledge and fixed, a player's beliefs about the other players' (time) preferences are irrelevant to equilibrium. (That is, the SPE will be the same in the following two cases: (i) the case in which each player i thinks that the discount rate of player k at t is δ_{ik}^i ; (ii) the Rubinstein-Stahl framework in which each player j has discount rate δ_{ij}^j at each t .)

6.3. General Intuition

In the model of Subsection 6.1 (as in the original model), under the independence assumption, excessive optimism can cause a delay in a short game, but not in a long one; whereas in the model of Subsection 6.2 it cannot cause any delay independent of the level and the duration of optimism. Now we try to explain that in general, whether optimism can cause a delay depends on when the uncertainty is resolved.

Abstracting from the details of the specific models, consider a bargaining situation between two players. For each t , let x_t be a random variable that remains relevant to the players' decision until the end of period t . Here, x_t can be the identity of the player who makes an offer at t , the interest rate a player faces at t , or the dividends paid by the players' assets at t , etc. The terms of trade at t depend on the players' beliefs about $(x_t, x_{t+1}, x_{t+2}, \dots)$.

There are three cases we consider. In the first case, the players learn x_t at the beginning of t and do not get any information before—as in Section 4 and Subsection 6.1. In that case, if the game is short, optimism can cause a delay. As in Example 0, it may be that, if they wait for a short while, the terms of trade will be greatly affected by the realization of one random variable, such as who makes an offer at the last round. In that case, the players' optimism about that event may lead them to have very high expectations about the terms of the trade that will take place in the near future if they wait. There may not be any trade that satisfies such expectations at the beginning.

But optimism seems to be less likely to cause a delay if the game is sufficiently long and the players are expected to remain optimistic for a long while. For, if t is not very large, then the players' decision at t is likely to depend on their beliefs about many random variables x_s with $s > t$. If the players remain optimistic about these random variables, the scope for trade at t is small, and thus x_t has little

¹³ This qualification is not needed for Proposition 2, because the game ends whenever the bargaining breaks down.

impact on the players' decision at t . Hence, the players will have little incentive to wait to learn x_t . On the other hand, if t is very large, then the players will not find it worthwhile delaying the agreement until such a distant future (namely, t) to realize the gain they expect at t .

This argument clearly assumes that optimism is about the players' shares in a future agreement. The players may also be optimistic about the outside options that they plan to take in the future. In that case, they do not need the other parties' consent, hence their optimism is not lowered by the persistency of the other players' optimism. Then, bargaining may result in a possibly delayed disagreement. (See the remark at the end of Subsection 6.1.)

Now, consider the case that any uncertainty about x_t will be resolved only after the end of period t , when x_t becomes irrelevant to the players' decision—as in Subsection 6.2. Now, the differences in beliefs will play only a minimal role in equilibrium, as the players' decisions depend only on the expected value of x_t —not x_t itself.¹⁴

Finally, consider the case that a substantial uncertainty about many random variables x_s is resolved at an early date t , e.g., when the independence assumption fails. Such a resolution of uncertainty may have a substantial impact on the players' decision at t . If the players are optimistic, then each expects the uncertainty to be resolved to his own advantage, and may decide to wait—as in Yildiz (2002b). Note that this case excludes the possibility that the players will remain excessively optimistic for a long while.

7. CONCLUSION

As suggested by Edgeworth (1881), with complete information, in equilibrium, bargaining results in an optimal outcome. Given that the delays are costly, this also implies that the agreement is reached immediately. Yet, the agreement is delayed as a rule in real life. A prominently proposed explanation for the delay is parties' excessive optimism, since at least Hicks (1932). Using simple models that are similar to Example 0, many authors (such as Landes (1971), Posner (1972), Gould (1973), Farber and Bazerman (1989), and Babcock and Loewenstein (1997)) discussed the role of excessive optimism in bargaining delays.

In this paper, we analyzed the problem of reaching an agreement in a model where players are possibly optimistic about the recognition process, the ultimate source of bargaining power in the standard model. In order to see the pure impact of optimism, we relaxed only the common prior assumption of complete information, and adhered to iterated conditional dominance.

We showed that, in this framework, excessive optimism alone cannot be a reason for delay. When the players are sufficiently optimistic for a long while, they will not settle for little. Recognizing this, each player will lower his expectations

¹⁴ In that case, we can simply replace each x_t with two variables, x_t^1 and x_t^2 , about which the players have a common prior. For instance, in Subsection 6.2, we can take δ_t^1 and δ_t^2 as the exogenously given discount rates of players, rather than their probability assessments about the continuation of the game at t .

about the future to the extent that they will reach an agreement immediately. In other words, presence of persistent optimism also includes the bad news for each player that his opponent will remain optimistic. This induces a form of pessimism that moderates players' optimism and leads them to an immediate agreement.

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APPENDIX: PROOF OF THEOREM 1

Write $\Sigma_{j,n}^*$ the strategies of a player j that survive the n th iteration of the elimination of conditionally dominated strategies. For any strategy $\sigma_j \in \Sigma_{j,n}^*$, any ρ' , and any consistent history h_t of play at the beginning of t , write $\tilde{V}(\sigma_j, h_t, \rho')$ for the supremum of the present value of the payoffs of i (with respect to the beliefs of i at ρ') as the strategy of i varies over all his strategies. Write $\underline{V}_{t,n}^i(\rho') = \inf_{\sigma_j, h_t} \tilde{V}(\sigma_j, h_t, \rho')$ and $\bar{V}_{t,n}^i(\rho') = \sup_{\sigma_j, h_t} \tilde{V}(\sigma_j, h_t, \rho')$. Clearly, $\underline{V}_{t,n}^i(\rho')$ is monotonically increasing with n and is in $[0, 1]$ —as each player can guarantee 0 for himself by never agreeing. Hence, it has a limit $\underline{V}_t^i(\rho') \in [0, 1]$ as $n \rightarrow \infty$. Similarly, as $n \rightarrow \infty$, $\bar{V}_{t,n}^i(\rho')$ monotonically decreases and converges to some $\bar{V}_t^i(\rho') \in [0, 1]$. Write $\Delta_t = \max_{i, \rho'} \{\bar{V}_t^i(\rho') - \underline{V}_t^i(\rho')\} \in [0, 1]$. When $\bar{t} < \infty$, we have $\bar{V}_t^i = \underline{V}_t^i = \Delta_t = 0$. Since $\Sigma^* \neq \emptyset$, $\bar{V}_t^i(\rho') \geq \sup_{\sigma_j \in \Sigma_{j,n}^*} \tilde{V}(\sigma_j, h_t, \rho') \geq \inf_{\sigma_j \in \Sigma_{j,n}^*} \tilde{V}(\sigma_j, h_t, \rho') \geq \underline{V}_t^i(\rho')$. We will show that $\Delta_t = 0$ at each t , i.e., $\bar{V}_t^i(\rho') = \underline{V}_t^i(\rho')$, establishing our claim.

First, we establish inequalities (3) and (4) below. Given any t , any (ρ', k) , and any $\epsilon > 0$, since $\underline{V}_{t+1}^i(\rho', k)$ is the least i expects to get in Σ^* , there exists an iteration n in the process of elimination such that $\underline{V}_{t+1,n}^i(\rho', k) \geq \underline{V}_{t+1}^i(\rho', k) - \epsilon/\delta$. Hence, at date t with history (ρ', k) , any strategy that gives i less than $\delta \underline{V}_{t+1}^i(\rho', k) - \epsilon$ at any history of play against any $\sigma_j \in \Sigma_{j,n}^*$ is conditionally dominated by not agreeing and following one of the remaining continuation strategies thereafter. The former strategy must have been eliminated. Since i expects to be recognized with probability $p_t^i(\rho')$, this yields

$$\underline{V}_t^i(\rho') \geq \delta E^i[\underline{V}_{t+1}^i | \rho'] \equiv p_t^i(\rho') \delta \underline{V}_{t+1}^i(\rho', i) + (1 - p_t^i(\rho')) \delta \underline{V}_{t+1}^i(\rho', j).$$

Likewise, for $j \neq i$, rejecting an offer u with $u^j > \delta \bar{V}_{t+1}^j(\rho', i)$ is conditionally dominated by accepting u . Hence, given any $\epsilon > 0$, if i is recognized at t and made an offer u with $u^j = \delta \bar{V}_{t+1}^j(\rho', i) + \epsilon$ and $u^i = 1 - \delta \bar{V}_{t+1}^j(\rho', i) - \epsilon$, it will be accepted. Therefore,

$$\underline{V}_t^i \geq p_t^i(\rho') (1 - \delta \bar{V}_{t+1}^j(\rho', i)) + (1 - p_t^i(\rho')) \delta \underline{V}_{t+1}^i(\rho', j).$$

Combining these two inequalities, we obtain

$$(3) \quad \underline{V}_t^i(\rho') \geq p_t^i(\rho') \max \{1 - \delta \bar{V}_{t+1}^j(\rho', i), \delta \underline{V}_{t+1}^i(\rho', i)\} + (1 - p_t^i(\rho')) \delta \underline{V}_{t+1}^i(\rho', j).$$

On the other hand,

$$(4) \quad \bar{V}_t^i(\rho') \leq p_t^i(\rho') \max \{1 - \delta \underline{V}_{t+1}^j(\rho', i), \delta \bar{V}_{t+1}^i(\rho', i)\} + (1 - p_t^i(\rho')) \delta \bar{V}_{t+1}^i(\rho', j).$$

To see this, first consider (ρ', j) . Player i will accept any offer u with $u^i > \delta \bar{V}_{t+1}^i(\rho', j)$. Hence, for j , offering \hat{u} with $\hat{u}^i = \delta \bar{V}_{t+1}^i(\rho', j) + \epsilon$ is conditionally dominated by offering \tilde{u} with $\tilde{u}^i = \delta \bar{V}_{t+1}^i(\rho', j) + \epsilon/2$ and $\tilde{u}^j = \hat{u}^j + \epsilon/2$. Therefore, if j is recognized, i cannot expect to get higher than $\delta \bar{V}_{t+1}^i(\rho', j)$. When i is recognized, accepting u with $u^j < \delta \underline{V}_{t+1}^j(\rho', i)$ has been eliminated for j , and hence i cannot expect to get higher than $\max \{1 - \delta \underline{V}_{t+1}^j(\rho', i), \delta \bar{V}_{t+1}^i(\rho', i)\}$. (If his offer is accepted, he gets at most $1 - \delta \underline{V}_{t+1}^j(\rho', i)$; otherwise he gets at most $\delta \bar{V}_{t+1}^i(\rho', i)$.) This proves (4).

Second, given any $s \in T$ and any $\rho^s \in N^s$, by subtracting (3) from (4), after a simplification, we obtain $\bar{V}_s^i(\rho^s) - \underline{V}_s^i(\rho^s) \leq p_s^i(\rho^s) \delta \max\{\bar{V}_{s+1}^j(\rho^s, i) - \underline{V}_{s+1}^j(\rho^s, i), \bar{V}_{s+1}^i(\rho^s, i) - \underline{V}_{s+1}^i(\rho^s, i)\} + (1 - p_s^i(\rho^s)) \delta (\bar{V}_{s+1}^i(\rho^s, j) - \underline{V}_{s+1}^i(\rho^s, j)) \leq \delta \Delta_{s+1}$. Therefore,

$$(5) \quad \Delta_s \leq \delta \Delta_{s+1}.$$

Finally, take any $t \in T$. We will show that $\Delta_t = 0$, and therefore $\bar{V}_t^i = \underline{V}_t^i$ as claimed. Now, if $\bar{t} = \infty$, given any $\epsilon > 0$, there exists $t + t' \in T$ such that $\delta^{t'} < \epsilon$, hence by (5) $\Delta_t \leq \delta^{t'} \Delta_{t+t'} < \epsilon$; i.e., $\Delta_t = 0$. If $\bar{t} < \infty$, $\Delta_{\bar{t}} = 0$, and once again $\Delta_t = 0$ by (5).

Since $\bar{V}_t^i = \underline{V}_t^i = \bar{V}_t^i$, we obtain (1), by combining (3) and (4).

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