Notes on Krusell and Smith (1998)

Johannes Wieland

May 29, 2018

1 Introduction

- Agents face idiosyncratic and aggregate risk.
- Markets are incomplete markets.
  - Only asset is physical capital.
  - Negative capital positions are ruled out by a borrowing constraint
- For comparison:
  - Bewely/Huggett: No aggregate risk, only asset is one-period bond in zero net supply.
  - Aiyagari: No aggregate risk, only asset is capital. Aggregate stock of capital is always at steady-state.
- Key challenge: how to solve a model with both idiosyncratic and aggregate risk, where the distribution of wealth is an endogenous state variable.

2 Model

2.1 Households

- Continuum of consumers $j \in [0, 1]$
- Preferences

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t \frac{c_{jt}^{1-\sigma}-1}{1-\sigma}$$

- Wage earnings w_t if employed, 0 if unemployed.

- Budget constraint:

$$c_{jt} + k_{jt} = w_t \tilde{l} \varepsilon_{it} + (1 + r_t - \delta) k_{j,t-1}$$

- Borrowing constraint:

$$k_{jt} \geq \underline{\mathrm{k}} = 0$$

- Note: if $\min\{w_t \tilde{l} \varepsilon_{it}\} = 0$, then 0 is a natural borrowing limit in the sense that there exists a path where agents can never pay back a loan of any size.

2.2 Firms

- Competitive firms produce output

$$Y_t = e^{z_t} K_{t-1}^{\alpha} L_t^{1-\alpha}$$

- FOC for factor prices

$$\begin{aligned} w_t &= (1-\alpha) e^{z_t} K_{t-1}^{\alpha} L_t^{-\alpha} \\ r_t &= \alpha e^{z_t} K_{t-1}^{\alpha-1} L_t^{1-\alpha} \end{aligned}$$

2.3 Shocks

- Aggregate productivity is $z_t \in \{z^h, z^l\}$. Follows two-state Markov process with transition probability $\pi_{ss'}$.
- Exogenous labor supply ε_{jt}, where $\varepsilon_{jt} \in \{0, 1\}$.
- Transition probabilities are correlated. Number of unemployed in good times is u^g and number of unemployed in bad times is u^b. Transition matrix:

$$\begin{pmatrix} \pi_{gg,11} & \pi_{gg,10} & \pi_{gb,11} & \pi_{gb,10} \\ \pi_{gg,01} & \pi_{gg,00} & \pi_{gb,01} & \pi_{gb,00} \\ \pi_{bg,11} & \pi_{bg,10} & \pi_{bb,11} & \pi_{bb,10} \\ \pi_{bg,01} & \pi_{bg,00} & \pi_{bb,01} & \pi_{bb,00} \end{pmatrix}$$

- Aggregate employment is only a function of the aggregate state

$$\begin{aligned} L_b &= \tilde{l}(1 - u_b) \\ L_g &= \tilde{l}(1 - u_g) \end{aligned}$$

The state of the economy is $s_t = (z_t, \mu_t)$ where μ_t is the distribution of households over (ε, k).

2.4 Recursive formulation

$$
\begin{aligned}
v(k,\varepsilon;z,\mu) &= \max_{c,k'} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \beta \mathbb{E}[v(k',\varepsilon';z',\mu')|z,\varepsilon] \right\} \\
s.t. \qquad c + k' &= r(z,\mu)k + w(z,\mu)l\varepsilon + (1-\delta)k \\
\mu'(z,\mu) &= H(\mu,z,z') \\
k' &\geq 0
\end{aligned}
$$

2.5 Equilibrium A recursive competitive equilibrium is a list of functions $k'(k,\varepsilon;z,\mu), r(z,\mu), w(z,\mu)$, and $\mu'(z,\mu)$ such that

- Taking $r(z,\mu), w(z,\mu)$, and $\mu'(z,\mu)$ as given, $k'(k,\varepsilon;z,\mu)$ satisfies

$$
c(k,\varepsilon;z,\mu)^{-\sigma} \geq \beta \mathbb{E}\left[(1 + r(z',\mu'(z,\mu)))\, c(k',\varepsilon';z',\mu'(z,\mu))^{-\sigma} | \varepsilon, z, \mu \right]
$$

  with equality if $k'(k,\varepsilon;z,\mu) > \underline{\mathrm{k}}$, where

$$
c(k,\varepsilon;z,\mu) = w(z,\mu)\tilde{l}\varepsilon + (1 + r(z,\mu) - \delta)k - k'(k,\varepsilon;z,\mu)
$$

- Prices $r(z,\mu), w(z,\mu)$ satisfy,

$$
\begin{aligned}
r(z,\mu) &= \alpha e^{z} K^{\alpha-1} L^{1-\alpha} - \delta \\
w(z,\mu) &= (1-\alpha) e^{z} K^{\alpha} L^{-\alpha}
\end{aligned}
$$

- For all measurable sets Δ_k

$$
\mu'(z,\mu)(\varepsilon,\Delta_k) = \sum_{\tilde{\varepsilon}} \pi(\varepsilon|\tilde{\varepsilon}) \int 1\{k'(k,\tilde{\varepsilon};z,\mu) \in \Delta_k\} d\mu(k,\tilde{\varepsilon})
$$

3 Computation

Computational difficulty:

- The endogenous wealth distribution is a state variable.
- Why? Since consumers have different propensities to save, need to know the distribution to forecast next periods capital stock K'. This then determines the return on capital r'. Hence μ is part of the value function.

- Problem is in fact more severe: In effect also need to forecast K'' to get r'' (and further) since consumption/savings decisions are forward looking. Hence, μ' is part of the value function next period.
- The problem is that μ is an infinite dimensional object (think moment generating function).
- Further, the transition equation $H(\bullet)$ is also infinite dimensional.

3.1 Approximate Aggregation KS solution:

- In an approximate equilibrium, agents will use a forecasting rule for aggregate capital next period:

$$
\begin{aligned}
z = z^g: &\qquad \log K' = a_0 + a_1 \log K \\
z = z^b: &\qquad \log K' = b_0 + b_1 \log K
\end{aligned}
$$

- Recursive, so also helps forecast K'' and so on.
- Interpretation: bounded rationality or best linear forecast.

Then the model simplifies to

$$
\begin{aligned}
v(k, \varepsilon; z, K) &= \max_{c,k'} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \beta \mathbb{E}[v(k', \varepsilon'; z', K') | z, \varepsilon] \right\} \\
s.t. \qquad c + k' &= r(K, L, z)k + w(K, L, z)l\varepsilon + (1-\delta)k \\
\log K' &= a_0 + a_1 \log K \text{ if } z = z^h \\
\log K' &= b_0 + b_1 \log K \text{ if } z = z^b \\
k' &\geq 0
\end{aligned}
$$

Idea is to find coefficients $\{a_0, a_1, b_0, b_1\}$ as a fixed point to this problem. So solve the value function given initial guesses. Then update the guesses based on the solution to the problem. Repeat until convergence.

3.2 Value function iteration

1. Start with initial guess of $V(k, \varepsilon, z, K)$ on a grid. Discreteness of ε, z means the dimensionality of the problem can be handled. (It's a bit like 2 state variables)
2. Forecasting rule tells us K' given K, z.

3. For each realization of ε', z' compute $V(\tilde{k}, \varepsilon', z', K')$.

4. Then compute optimal k, update value function. Repeat

Your algorithm will converge faster if you restrict choices to lie on the grid for capital that you compute. Extrapolation outside the grid can lead to non-trivial swings that dominate convergence criteria.

3.3 Policy function iteration

1. Start with guess $k'(k, \varepsilon; z, K)$.

2. Calculate $k''(k', \varepsilon'; z', K')$ for each possible realization of ε', z'. K' comes from the forecast given the current state (note it is not random since determined by today's choices).

3. From the budget constraint obtain,

$$c'(k', \varepsilon'; z', K') = r(K', z')k' + w(K', z')l\varepsilon' + (1 - \delta)k' - k''$$

4. Calculate marginal utility $MU = c'(k', \varepsilon'; z', K')^{-\sigma}$

5. Obtain today's consumption from the Euler equation,

$$c(k, \varepsilon; z, K)^{-\sigma} = \mathbb{E}\left[\beta(1 + r(K', z') - \delta)c'(k', \varepsilon'; z', K')^{-\sigma}\right]^{-\frac{1}{\sigma}}$$

6. Obtain today's capital from the budget constraint and update initial guess.

For this problem policy function iteration tends to be faster than value function iteration.

3.4 Full algorithm

1. Find solution for policy function given initial guess (either value function or policy function iteration).

2. Simulate economy with many agents.

3. Update the forecasting regressions using OLS.

4. Go to step 1 and repeat until convergence.

3.5 Calibration

- $\delta = 0.025$
- $\beta = 0.99$
- $\sigma = 1$
- $\tilde{l} = 0.3271$
- $z^g = 1.01$, $z^b = 0.99$
- $u^g = 0.04$, $u^b = 0.1$
- Average duration of good/bad times = 8 quarters, so $\pi_{gg} = \pi_{bb} = 1 - \frac{1}{8} = 0.875$
- Unemployment spell 1.5 quarters in good times, 2.5 quarters in bad times: $\pi_{gg,00} = 1 - \frac{1}{1.5} = 1/3$, $\pi_{bb,00} = 1 - \frac{1}{2.5} = 0.6$.
  Implies $\pi_{gg,01} = 1 - \frac{1}{1.5} = 0.875 - 1/3$, $\pi_{bb,01} = 1 - \frac{1}{1.5} = 0.875 - 0.6$
- Additional restriction:

$$\frac{\pi_{gb,00}}{\pi_{bb,00}} = 1.25\frac{\pi_{gb}}{\pi_{bb}}$$

  Implies $\pi_{gb,00} = 1.25\frac{1/8}{7/8}0.6 = 0.107$. In turn we get $\pi_{gb,01} = 0.125 - 0.107 = 0.018$

- Additional restriction:

$$\frac{\pi_{bg,00}}{\pi_{gg,00}} = 0.75\frac{\pi_{bg}}{\pi_{gg}}$$

  Implies $\pi_{bg,00} = 0.75\frac{1/8}{7/8}1/3 = 0.060$. In turn we get $\pi_{bg,01} = 0.125 - 0.060 = 0.065$

- Remaining probabilities have to be such that unemployment is always constant:

$$u_s\frac{\pi_{ss',00}}{\pi_{ss'}} + (1 - u_s)\frac{\pi_{ss',10}}{\pi_{ss'}} = u_{s'}$$

  Good state:

$$0.04\frac{7/8 - 1/3}{7/8} + 0.96\frac{\pi_{gg,10}}{7/8} = 0.04$$
$$0.04\frac{0.107}{1/8} + 0.96\frac{\pi_{gb,10}}{1/8} = 0.1$$

  Implies:

Bad state:

$$0.1\frac{0.06}{1/8} + 0.9\frac{\pi_{bg,10}}{1/8} = 0.1$$
$$0.1\frac{0.6}{7/8} + 0.9\frac{\pi_{bb,10}}{7/8} = 0.1$$

- Final matrix:

$$\begin{pmatrix} 0.875 - 0.014 & 0.014 & 0.125 - 0.009 & 0.009 \\ 0.875 - 1/3 & 1/3 & 0.018 & 0.107 \\ 0.125 - 0.007 & 0.007 & 0.875 - 0.030 & 0.031 \\ 0.125 - 0.060 & 0.060 & 0.875 - 0.6 & 0.6 \end{pmatrix}$$

4 Results

4.1 Diagnostics

- How good is the forecasting rule?

$$\log K' = 0.094199 + 0.96266 \log K, \qquad R^2 = 0.99997 \qquad \text{if } z = z^h$$
$$\log K' = 0.082411 + 0.96529 \log K, \qquad R^2 = 0.99992 \qquad \text{if } z = z^b$$

- Fit looks very good. High R^2. Adding more moments changes little.
- Why does it work? Savings functions are linear especially for high-k agents (figure 1). And these agents matter most for aggregate capital stock (figure 3). Low-k agents have non-linear savings functions, but largely irrelevant for aggregate savings.
- What does high R^2 mean? E.g. $R^2 = 0.99$ means SD of numerical error is 10% of SD of capital. So $R^2 = 0.99$ is terrible for numerical accuracy.
- Den Haan (2010) alternative:
  - Forecast K for entire simulation using only forecasting rule.
  - Then compare with the realized K you get from the stochastic simulation.
  - This tests if errors accumulate.
  - Accumulation of errors is problematic because it means the sequence of expected real interest rates is way off and thus consumption choices today are way off.
  - It turns out this is not an issue for KS (see figure 5).

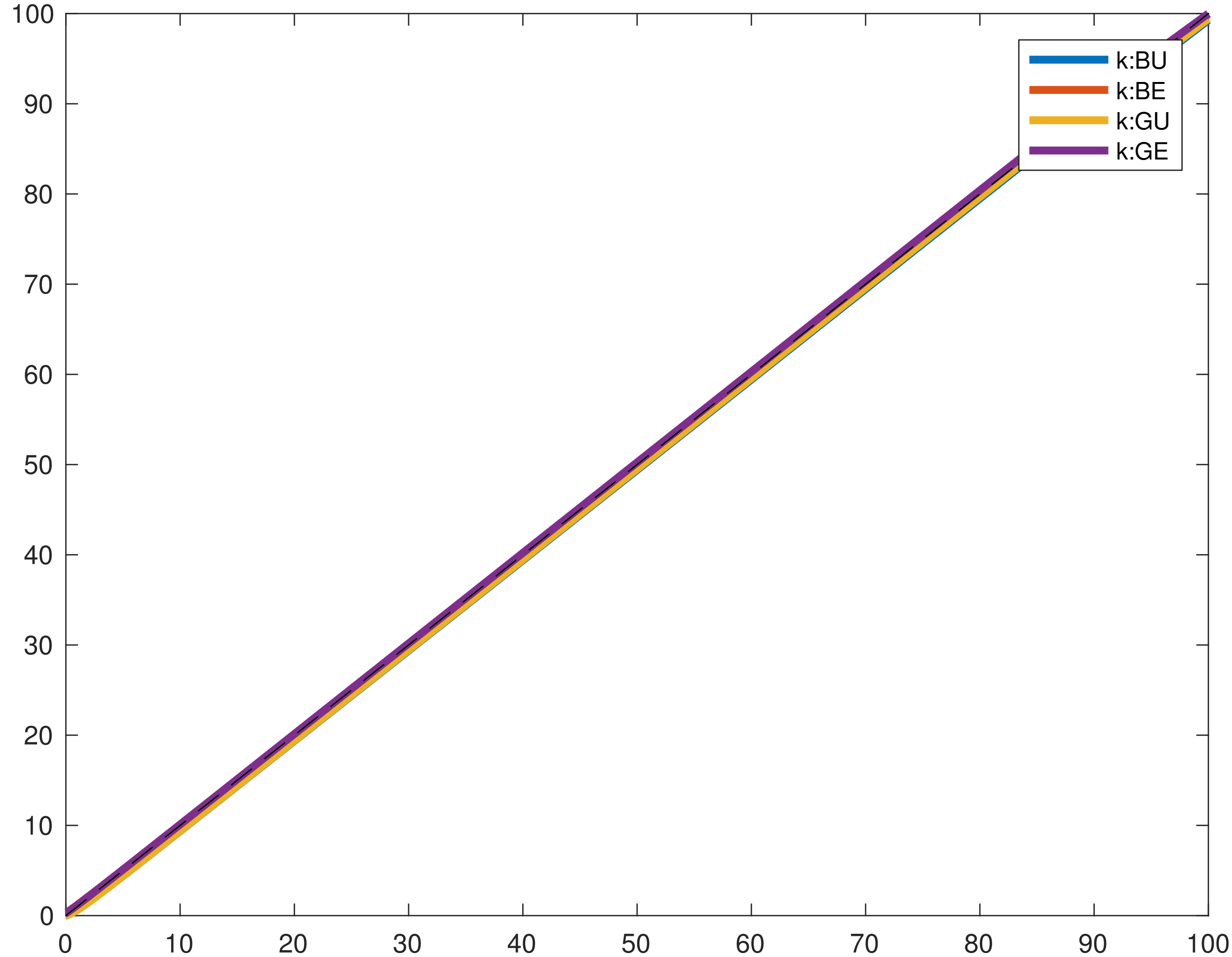

Figure 1 – Policy functions in baseline KS model (medium K)

- Maximum error = 0.37536% of steady-state capital.
  Average error = 0.087292% of steady-state capital.

4.2 Wealth distribution

- Difficult in this model to get agents hold very little wealth and a lot of wealth. "Too much" mean-reversion. See figure 3 for typical distribution.

- Intuitively, being at borrowing constraint is very costly, so workers will save more to make these states very unlikely (self-insurance). Also, not enough serial correlation in shocks to get agents stuck at the bottom or the top.

- Stochastic beta model also has three idiosyncratic patience states.

- High-beta agents keep saving even with lots of capital because their perceived return to saving $\beta_i(1 + r_t - \delta)$ is high.

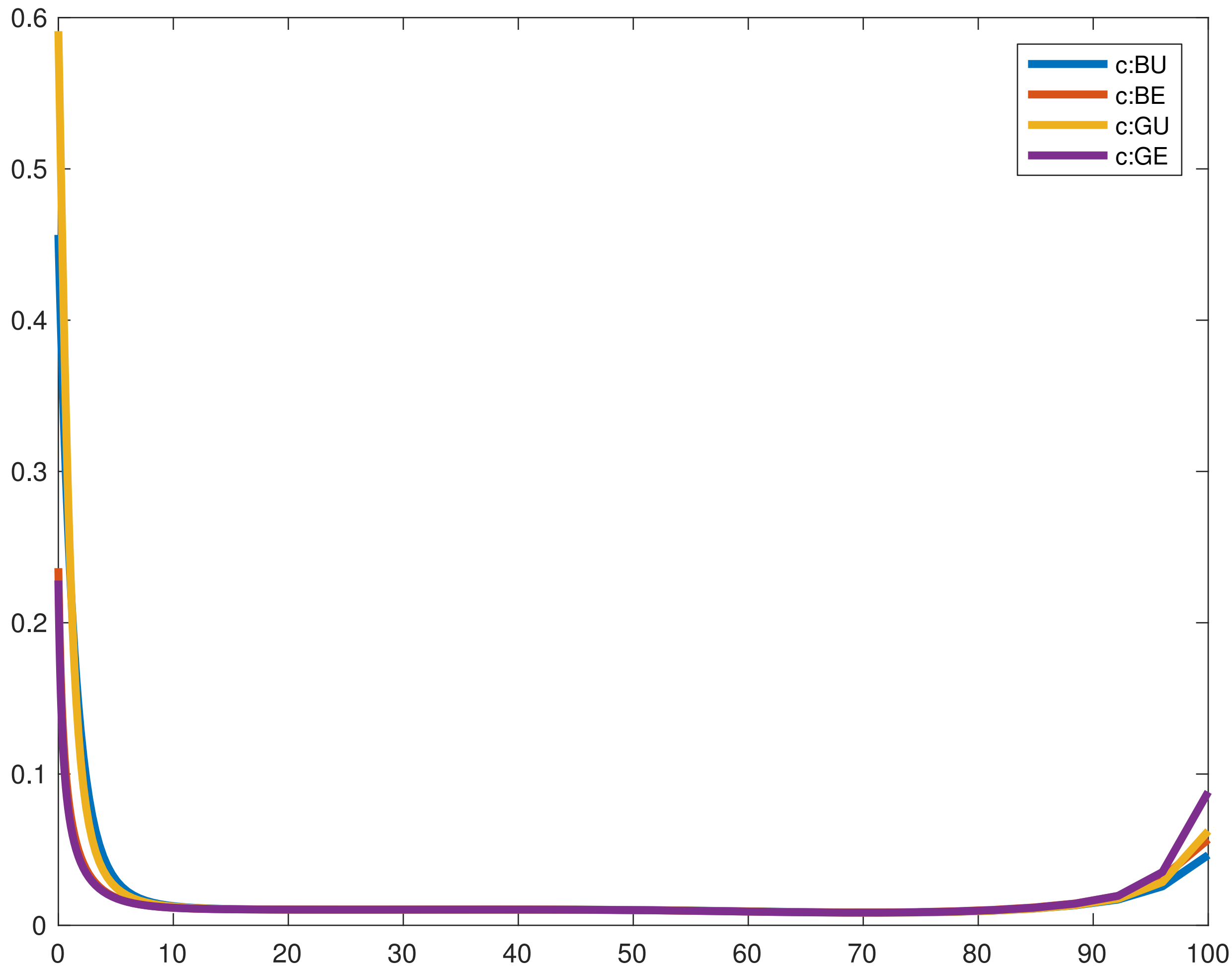

Figure 2 – MPCs in baseline KS model (medium K). (Note numerical errors at high K.)

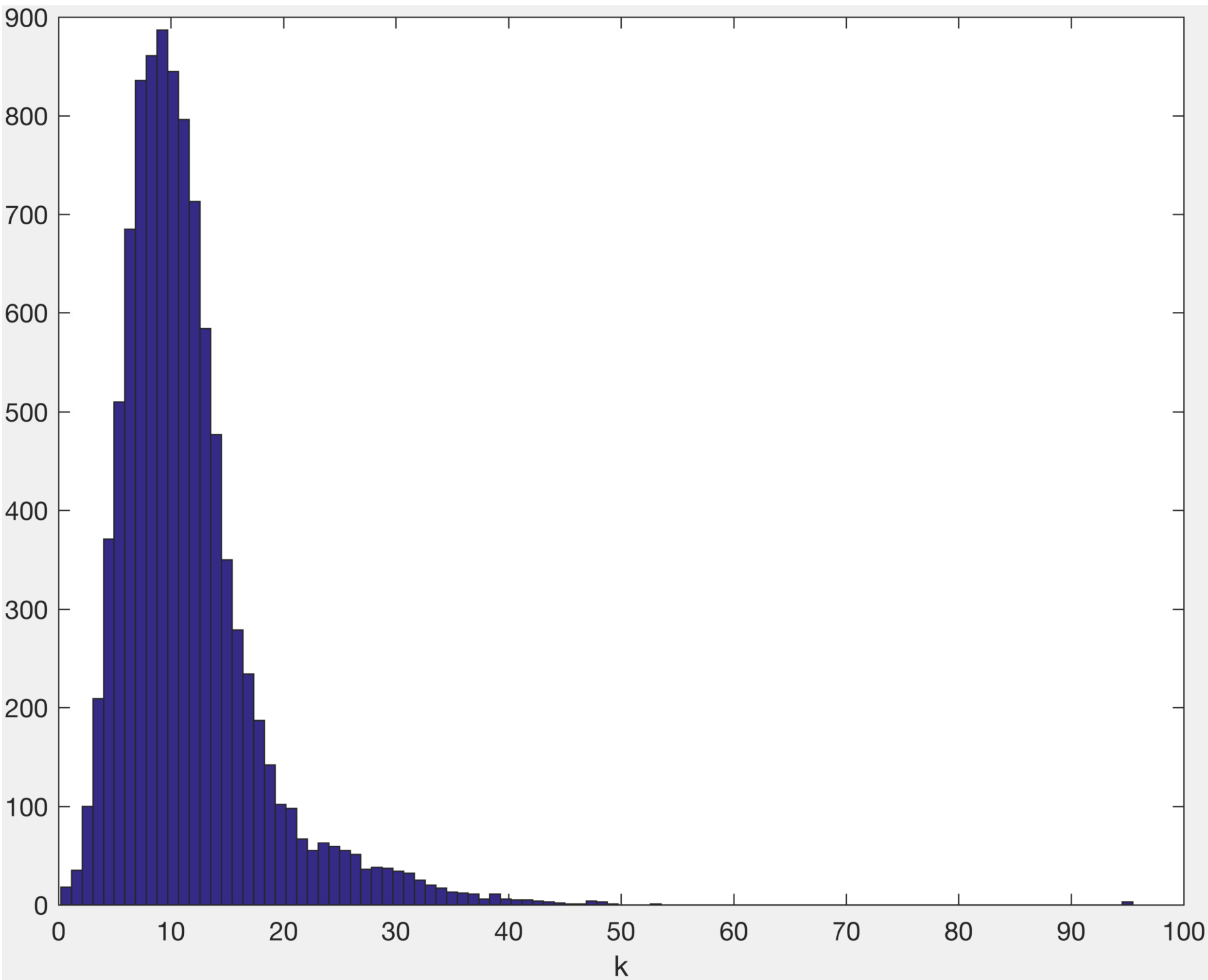

Figure 3 – Typical wealth distribution in baseline KS model

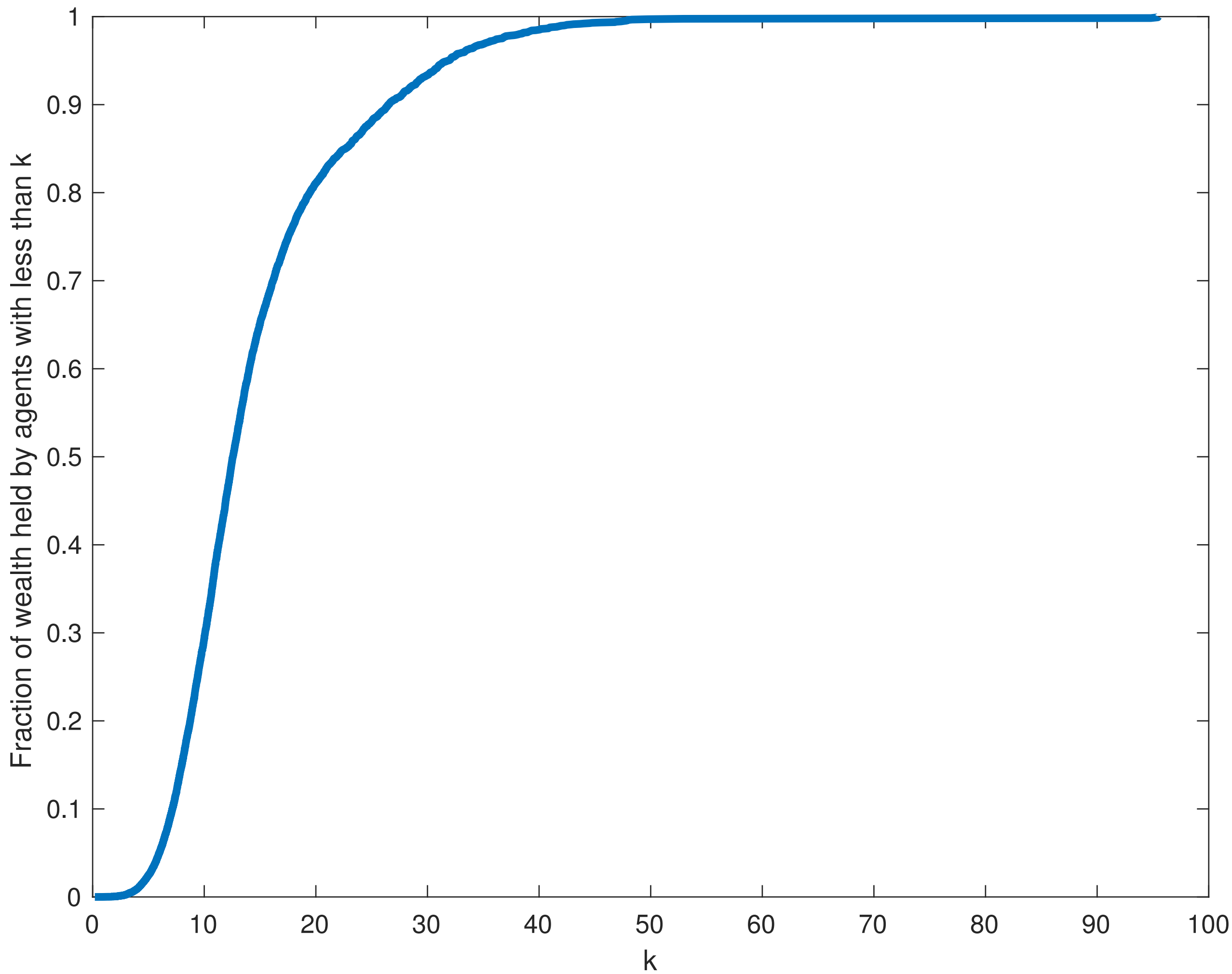

Figure 4 – Typical cumulative wealth distribution in baseline KS model

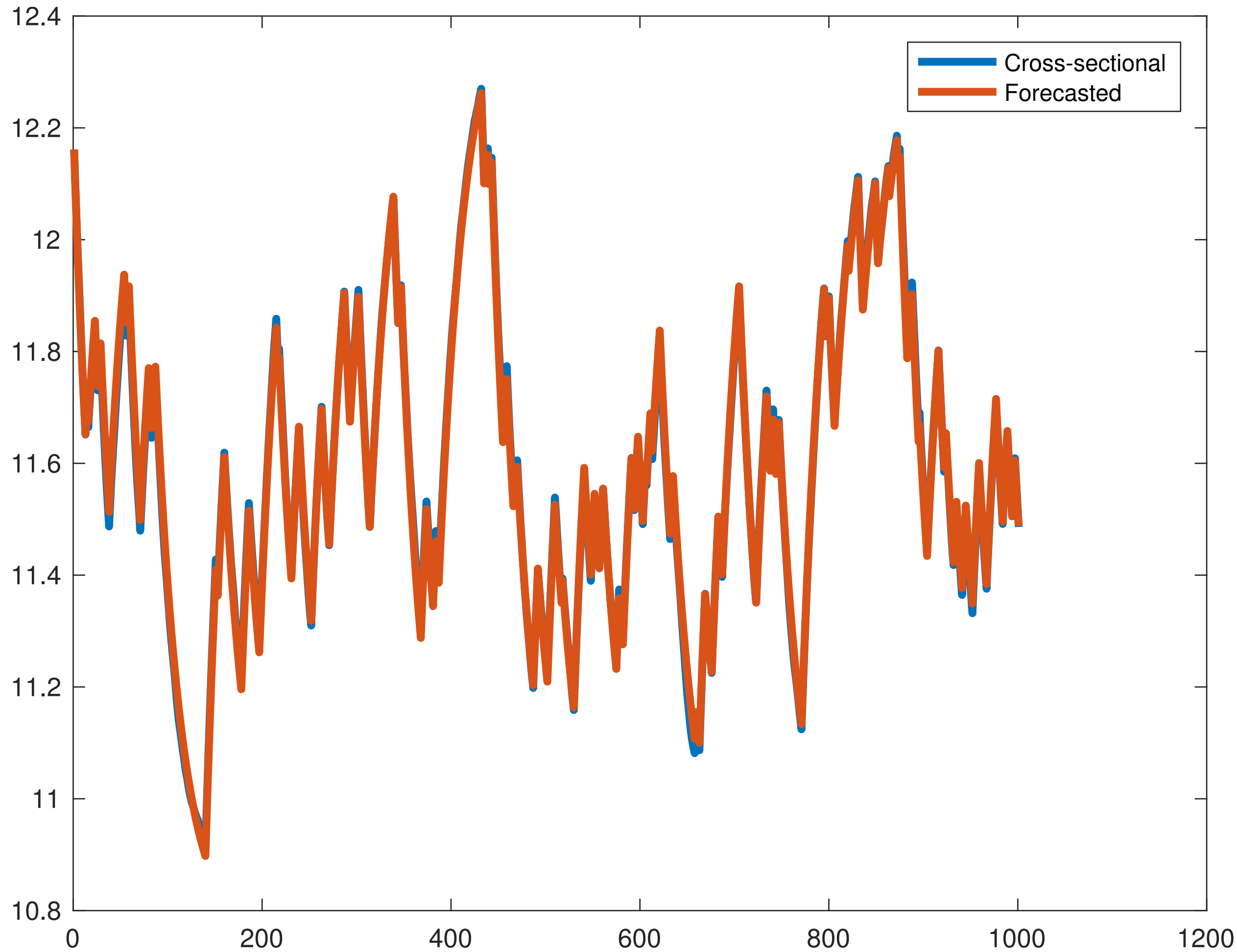

Figure 5 – Den Haan Diagnostic in baseline KS model

- Low-beta agents keep dissaving because their perceived return to saving is low.
- But is this a satisfactory micro-foundation?

4.3 Time series results:

Table 1 – Aggregate time series statistics

	mean(K)	Corr(y,c)	stdev(i)	corr(y,L4.y)
Baseline	11.5925	0.67289	0.030419	0.38158
Complete Markets	11.5211	0.65789	0.030796	0.38461
Stochastic beta	11.9219	0.78444	0.027837	0.37021

- Time-series property of baseline incomplete markets model very similar to complete markets.
- Higher mean capital stock due to self-insurance.
- Much higher $corr(c, y)$ in stochastic beta model. Low-beta agents have lower wealth and thus higher MPCs.

5 Conclusion

- Major technical advance.
- Approximate aggregation seems to work well in many applications (though not all—you need to check).
- MPCs too low in baseline model relative to studies of temporary tax cuts. Micro evidence suggests average MPCs near 0.25-0.3.
- Whether incomplete markets matter depends on the set-up. In many environments they do little. This means you should either:
  - Use incomplete markets.
  - Look at different (more interesting) environments.

6 Further Reading

- Young (2010) for algorithmic improvement using non-stochastic simulation.
- Den Haan, Judd, and Juillard (2010) and Maliar, Maliar, and Valli (2010) for discussion and codes.
- Werning (2015) and Chipeniuk, Katz, and Walker (2016) for some theory behind approximate aggregation.

References

Chipeniuk, Karsten, Nets Hawk Katz, and Todd Walker, "Approximate aggregation in the neoclassical growth model with ideosyncratic shocks," 2016.

Haan, Wouter J Den, "Assessing the accuracy of the aggregate law of motion in models with heterogeneous agents," *Journal of Economic Dynamics and Control*, 2010, *34* (1), 79–99.

__, Kenneth L Judd, and Michel Juillard, "Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty," *Journal of Economic Dynamics and Control*, 2010, *34* (1), 1–3.

Krusell, Per and Anthony A Smith Jr, "Income and wealth heterogeneity in the macroeconomy," *Journal of political Economy*, 1998, *106* (5), 867–896.

Maliar, Lilia, Serguei Maliar, and Fernando Valli, "Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm," *Journal of Economic Dynamics and Control*, 2010, *34* (1), 42–49.

Werning, Iván, "Incomplete markets and aggregate demand," Technical Report, National Bureau of Economic Research 2015.

Young, Eric R, "Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm and non-stochastic simulations," *Journal of Economic Dynamics and Control*, 2010, *34* (1), 36–41.