Notes on Aiyagari (1994)

1 Introduction

- Model of heterogeneous agents with borrowing constraint.
- Typical work-horse model to study inequality in wealth, consumption and income.
- Key difficulty: distribution of agents is an infinite-dimensional state-variable.
- Can still easily compute stationary equilibrium (stationary in aggregate, individuals move within the stationary distribution): r is fixed.

2 Preferences

- Utility function:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

- One asset a_{it}, earns interest rate r.
- Stochastic labor income $l_{it} \in \{l_{min}, l_{max}\}$, iid distribution $dF(l)$.
- Idiosyncratic income risk is not shared. Markets are incomplete.
- Budget constraint:

$$c_{it} + a_{i,t+1} = w_i l_{it} + (1+r) a_{it}$$

- Borrowing limit (exogenous vs natural):

$$a_{it} \geq -\phi, \qquad \phi = \min\left\{b, \frac{w l_{min}}{r}\right\}$$

- Sequence problem:

$$\begin{aligned} &\max_{\{c_{it},a_{it}\}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_{it}) \\ s.t. \qquad & c_{it} + a_{it} = wl_{it} + (1+r)a_{i,t-1} \\ & a_{it} \geq -\phi \end{aligned}$$

- FOC:

$$\begin{aligned} &\frac{1}{c_{it}} = \lambda_{it} \\ &\lambda_{it} = \beta(1+r)\mathbb{E}_t \lambda_{it+1} + \mu_{it} \end{aligned}$$

  Note: even if $\mu_{it} = 0$ today, if $\mu_{it} > 0$ is some future state with positive probability, then this will affect current consumption: precautionary savings.

- Implies that a stationary equilibrium needs $(1+r)\beta < 1$. Why? Integrate over all individuals to get,

$$\bar{\lambda} = \beta(1+r)\bar{\lambda} + \bar{\mu}$$

  where $\bar{\lambda}$ and $\bar{\mu}$ are the average values in the population (time-invariant). So long as a positive mass of individuals is constrained, then $\beta(1+r) < 1$.

  Otherwise individuals will accumulate infinitely many assets to perfectly insure against idiosyncratic risk.

  (Can formally prove this using that a nonnegative supermartingale u(c) will converge almost surely to a finite limit.)

- To write down the recursive problem, define cash-on-hand as $z_{it} = w_i l_{it} + (1+r)a_{i,t-1}$,

$$V(z) = \max_{a \in [-\phi, z]} \left\{ u(z-a) + \beta \int V[wl' + (1+r)a] dF(l') \right\}$$

  Solve using your favorite value function iteration. (Alternative: policy function iteration)

- Characteristics of the solution:

  - Value function $V(z)$ is continuous and strictly concave (see Stokey-Lucas).

- Consumption is strictly increasing in c: From the envelope condition,

$$V_z(z) = u_c[c(z)]$$
$$\Rightarrow c_z(z) = \frac{V_{zz}(z)}{u_{cc}[c(z)]} > 0$$

- Whenever the borrowing constraint does not bind, then $a_z(z) > 0$:

$$u_c[c(z)] = \beta(1+r)\int V_z[(1+r)a(z) + wl']dF(l')$$
$$\Rightarrow u_{cc}[c(z)]c_z(z) = \beta(1+r)\int [V_{zz}(z')dF(l')]a_z(z)$$
$$\Rightarrow a_z(z) = \frac{u_{cc}[c(z)]c_z(z)}{\beta(1+r)\int[V_{zz}(z')dF(l')]} > 0$$

- There exists $\bar{z}$, s.t. $a(z) = -\phi$ for all $z \leq \bar{z}$, and $a_z(z) = 0$. The existence follows from optimality and a finite borrowing limit. If you are at the borrowing constraint, then MU is high, so rather consume than save. Next, suppose that $a(\tilde{z}) > -\phi$ for some $\tilde{z} < \bar{z}$. We get

$$u_c[c(\bar{z})] \geq \beta(1+r)\int V_z[(1+r)a(\bar{z}) + wl']dF(l')$$
$$u_c[c(\tilde{z})] = \beta(1+r)\int V_z[(1+r)a(\tilde{z}) + wl']dF(l')$$
$$\Rightarrow u_c[c(\bar{z})] \geq \beta(1+r)\int V_z[-(1+r)\phi + wl']dF(l') > \beta(1+r)\int V_z[(1+r)a(\tilde{z}) + wl']dF(l')$$
$$= u_c[c(\tilde{z})]$$
$$\Rightarrow c(\bar{z}) < c(\tilde{z})$$

which is a contradiction.

- Finally, from the budget constraint,

$$c_z(z) + a_z(z) = 1$$

So mpc is 1 when constraint and less than 1 otherwise.

- Figure 1
- Stationary distribution $\Phi(z)$ will only have positive mass over interval $[z_{min}, z_{max}]$.
- At $z_{max} = wl_{max} + (1+r)a_{max}$ individuals will chose $a' = a_{max}$ and $c_{max} = wl_{max} + ra_{max}$. So the mpc out of wealth is r similar to a complete markets model. In that sense individuals with high wealth are fully self-insured against

hitting the borrowing constraint.

– Any individual that starts with wealth $a > a_{max}$ will pick $a' < a$ until convergence to a_{max}. This convergence happens gradually and occurs because $\beta(1+r) < 1$. Because these individuals are essentially fully insured c' is essentially constant, which implies $c' < c$ given $\beta(1+r) < 1$.

– Note: within $\Phi(z)$ individuals constantly move around. But they effectively trade places so the aggregate distribution $\Phi(z)$ does not change. In that sense there is idiosyncratic risk (my place in the distribution) but no aggregate risk (the distribution does not change).

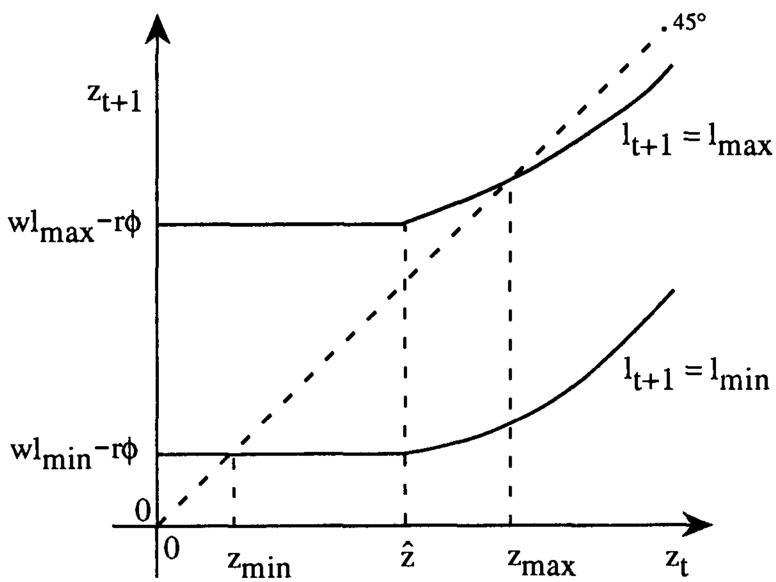

Figure 1 – Cash-on-hand function in Aiyagari

3 Technology

- Output is produced using CD production,

$$y_t = k_{t-1}^{\alpha} l_t^{1-\alpha}$$

- Capital follows a standard accumulation equation

$$k_t = k_{t-1}(1-\delta) + i_t$$

- Normalize total labor supply to $l = pl_{min} + (1-p)l_{max} = 1$ and use stationarity,

$$y = k^{\alpha}$$
$$i = \delta k$$

- Firms are perfectly competitive so

$$r = \alpha k^{\alpha-1} - \delta$$
$$w = (1-\alpha)k^{\alpha}$$

4 Steady-state Equilibrium

- Asset market clearing:

$$k = \int a(z, r, w, \phi) d\Phi(z)$$

  where $d\Phi$ is the distribution of assets.

- A steady-state equilibrium of the model consists of the policy function $a(z, r, w, \phi)$, a steady-state distribution $d\Phi(z)$, a capital stock k and prices r, w such that

  1. The policy function is optimal given w, r.
  2. The steady-state distribution is consistent with the policy functions.
  3. Capital, labor and asset markets clear.

- Important property: Aggregates are deterministic but individual allocations are not.

5 Characteristics of the stationary equilibrium

- Market clearing picture:

  - Total asset demand converges to infinity as $1 + r \to \beta^{-1}$.
  - Converges to borrowing limit for sufficiently low (negative) interest rate.
  - Downward sloping factor price r, bounded by $-\delta$.

- Figure 2

- Inefficiently high capital accumulation relative to full-insurance because consumers want to self-ensure against idiosyncratic income risk.

- GE works against this: the equilibrium interest rate falls as more capital gets accumulated, making self-insurance more costly.

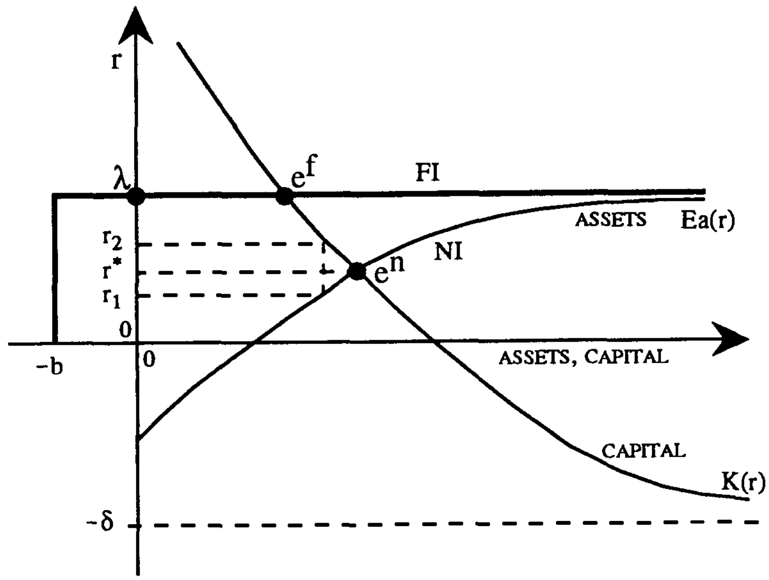

Figure 2 – Equilibrium in Aiyagari

6 Solving for the Stationary Equilibrium

1. For given k_0 get r_0 use value function iteration and compute the distribution of agents $\Phi(a_i)$. Note: need to approximate distribution fairly well to avoid Euler equation errors that accumulate over time.

2. Market clearing implies value for k_1.

3. Update guess for $r_1 = \omega r_0 + (1-\omega)(\alpha k_1^{\alpha-1} - \delta)$, where $\omega \in (0,1)$.

7 Comparative statics on stationary equilibrium

- Reduction in borrowing limit. Figure 2: asset demand curve shifts out.

- Increase in income risk (mean-preserving spread). Figure 2: asset demand curve shifts out.

- Provides micro-foundation for changes in natural rate of interest. Additional ingredients (e.g. sticky prices), then give rise to a recession.
- Key advantage of this model:
  1. Can discipline model using micro-data on income and wealth distribution.
  2. Meaningful welfare differences.
- Guerrieri and Lorenzoni (2011) show how to calculate the dynamics for these type of shocks. (Essentially shooting.)

8 Notes:

- With aggregate risk can solve using Krusell and Smith (1998). Essentially, forecasts of factor prices are only a function of $E(k)$, rather than its distribution. See Werning (2015) for a micro-foundation.

References

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