Lecture notes for ECON-210C
Business cycles

Johannes Wieland

UC San Diego[1]

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[1]These notes build on Yuriy Gorodnichenko's notes for Econ 202B at UC Berkeley. I am grateful for his help and dedication and I aspire to meet the high standards he has set.

Contents

Chapter 1

Basic real business cycle model

We begin our analysis of business cycles by considering an elementary economy perturbed by shocks to productivity. This elementary model, constructed in the spirit of the seminal paper by Kydland and Prescott (1982), is obviously unrealistic. However, this model is a useful benchmark for more sophisticated models which we will consider later. Our objective now is to have an understanding of how variables behave in response to shocks, what basic mechanisms are involved, and how to solve/analyze dynamic models.

In this basic model, we assume that the economy is populated by a representative firm and a representative household and that factor and product markets are competitive and clear at all times.

1.1 Household

Households maximize expected utility subject to budget constraints:

$$
\begin{aligned}
\max \quad & \mathbb{E}_t \sum_s \beta^s \big(\ln C_{t+s} - \frac{L_{t+s}^{1+1/\eta}}{1+1/\eta} \big) \\
\text{s.t.} \quad & A_{t+s} + C_{t+s} = (1+R_{t+s}) A_{t+s-1} + W_{t+s} L_{t+s}
\end{aligned}
\tag{1.1}
$$

$$lim_{s\to+\infty}\Big(\prod_{k=1}^{s}(1+R_{t+k})\Big)^{-1}A_{t+s}=0 \tag{1.2}$$

where C_t is consumption, A_t is asset holding, W_t is wages, L_t is labor services supplied by the household, R_t is the return on assets. We assume log utility in consumption to simplify solution and to match the fact that labor supply was insensitive to long-term growth in income. The operator $\mathbb{E}_t$ denotes the household's expectations at time t. We will use rational expectations for most of this course, so $\mathbb{E}_t f(x_{t+1})=\int f(x_{t+1})dF(x_{t+1})$. The constraint (1.1) is the budget constraint. The constraint (1.2) is the transversality condition, also known as the "No Ponzi game" condition. It ensures that the debt of the household does not grow exponentially over time.

My timing convention is that $t+s$ dated variables are chosen at $t+s$. Thus, A_{t+s} are asset holdings chosen at time $t+s$ that are then carried into the next period.

The Lagrangian of the problem is:

$$\mathcal{L}=\mathbb{E}_t\sum_s\beta^s\Big(\ln C_{t+s}-\frac{L_{t+s}^{1+1/\eta}}{1+1/\eta}+\lambda_{t+s}(-A_{t+s}-C_{t+s}+(1+R_{t+s})A_{t+s-1}+W_{t+s}L_{t+s})\Big)$$

λ_t is the Lagrange multiplier of the budget constraint, and it corresponds to the marginal utility of wealth. The first order conditions (FOCs) are:

$$\frac{\partial\mathcal{L}}{\partial C_t} = \frac{1}{C_t}-\lambda_t=0 \tag{1.3}$$

$$\frac{\partial\mathcal{L}}{\partial L_t} = -L_t^{1/\eta}+\lambda_t W_t=0 \tag{1.4}$$

$$\frac{\partial\mathcal{L}}{\partial A_t} = -\lambda_t+\mathbb{E}_t\beta(1+R_{t+1})\lambda_{t+1}=0 \tag{1.5}$$

The labor supply equation is derived from equation (1.4):

$$L_t = (\lambda_t W_t)^\eta \tag{1.6}$$

$\frac{\partial \ln L_t}{\partial \ln W_t}\Big|_{\lambda=\text{const.}}=\eta$ is the **Frisch labor supply elasticity**. Using $\lambda_t=\frac{1}{C_t}$ we can eliminate

λ and simplify the FOCs:

$$L_t^{1/\eta} = \frac{W_t}{C_t} \tag{1.7}$$

$$\frac{1}{C_t} = \beta \mathbb{E}_t (1 + R_{t+1}) \frac{1}{C_{t+1}} \tag{1.8}$$

Equation (1.7) is an *intratemporal* condition linking consumption and labor supply. Equation (1.8) is the Euler equation. The Euler equation is an *intertemporal* condition which governs the allocation of resources over time.

1.2 Firm

Households own firms. Firms own capital. The price of the good produced by firms is normalized at 1. Firms maximize the present value of the cash flow $CF_{t+s} = Y_{t+s} - W_{t+s}L_{t+s} - I_{t+s}$

$$\max \quad \mathbb{E}_t \sum_s \left(\prod_{k=1}^{s} (1 + R_{t+k}) \right)^{-1} CF_{t+s} \tag{1.9}$$

subject to capital accumulation constraint

$$K_{t+s} = (1 - \delta) K_{t+s-1} + I_{t+s}. \tag{1.10}$$

where δ is the rate of depreciation of physical capital. The production function is:

$$Y_{t+s} = Z_{t+s} K_{t+s-1}^{\alpha} L_{t+s}^{1-\alpha} \tag{1.11}$$

Note that capital K is dated by time index $t-1$. This timing highlights that capital is a predetermined variable at time t. Recall that time t-dated variable are chosen at time t, so K_t is used in production at time $t+1$.

The Lagrangian of the problem is:

$$\mathcal{L} = \mathbb{E}_t \sum_s \big(\prod_{k=1}^{s}(1+R_{t+k})\big)^{-1}\big(Z_{t+s}K_{t+s-1}^{\alpha}L_{t+s}^{1-\alpha} - W_{t+s}L_{t+s} - I_{t+s} + q_{t+s}(-K_{t+s} + (1-\delta)K_{t+s-1} + I_{t+s})\big)$$

The costate variable q_t is nothing else but Tobin's Q, i.e. the shadow value of capital. The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial L_t} = (1-\alpha)Z_t L_t^{-\alpha} K_{t-1}^{\alpha} - W_t = 0 \tag{1.12}$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = -1 + q_t = 0 \tag{1.13}$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = -q_t + \mathbb{E}_t \frac{1}{(1+R_{t+1})}\big(\alpha Z_{t+1} L_{t+1}^{1-\alpha} K_t^{\alpha-1} + q_{t+1}(1-\delta)\big) = 0 \tag{1.14}$$

To see why q_t captures the value of capital, solve the last equation forward:

$$q_t = \sum_{s=1}^{\infty} \mathbb{E}_t \frac{(1-\delta)^{s-1}}{\prod_{k=1}^{s}(1+R_{t+k})} \alpha Z_{t+s} L_{t+s}^{1-\alpha} K_{t+s-1}^{\alpha-1} \tag{1.15}$$

This is the expected discounted extra output produced from an extra unit of capital K_t.

Note that in this model there is no capital adjustment costs and hence $q_t = 1$ at all times. (The second equation.) This is an arbitrage condition. Because the marginal cost of an extra unit of capital is always 1 unit of output, at the optimum its marginal benefit also has to be equal to 1. Hence we can combine equations (1.13) and (1.14) to get,

$$\alpha Z_{t+1} L_{t+1}^{1-\alpha} K_t^{\alpha-1} = \delta + R_{t+1} \tag{1.16}$$

(dropping the expectations operator for now.)

Note that $MPK_t = \alpha Y_t / K_{t-1} = \alpha Z_t L_t^{1-\alpha} K_{t-1}^{\alpha-1}$ is the **marginal product of capital** (MPK). Hence, equation (1.16) says that the marginal product on capital is equal to the cost of using capital $\delta + R_{t+1}$ which is the *gross* return of capital since it also includes the term due to depreciation of capital (δ).

Equation (1.12) gives the demand for labor:

$$MPL_t = (1-\alpha)Y_t/L_t = (1-\alpha)Z_t L_t^{-\alpha} K_{t-1}^{\alpha} = (1-\alpha)Y_t/L_t = W_t \tag{1.17}$$

MPL_t is the **marginal product of labor** (MPL). The firm hires workers until the MPL equals the market wage. Note that because output has decreasing returns to scale in labor, the MPL (and hence the labor demand curve) is downward sloping in labor.

1.3 Competitive Equilibrium

A competitive equilibrium is a stochastic process for quantities $\{C_t, L_t, L_t^d, L_t^s, A_t, K_t, I_t, Y_t, Z_t\}$ and prices $\{W_t, R_t, q_t\}$ such that households solve their utility-maximization problem (equations (1.7) and (1.8)); firms solve their profit-maximization problem (equations (1.14), (1.12), (1.13) and (1.16)); the technological constraints (1.10) and (1.11) are satisfied; and the capital, labor and goods markets clear, i.e., the following conditions are satisfied:

$$L_t^s = L_t^d = L_t \tag{1.18}$$

$$A_t = K_t \tag{1.19}$$

$$Y_t = C_t + I_t \tag{1.20}$$

Equation (1.18) is the labor market equilibrium: labor supply (L_t^s) equals labor demand (L_t^d). Equation (1.19) is the capital market equilibrium saying that the value of firms is equal to the asset holdings of households. Equation (1.20) is the goods market equilibrium. Since $I_t \equiv K_t - (1-\delta)K_{t-1}$, equilibrium in the goods market can be rewritten: $Y_t = C_t + K_t - (1-\delta)K_{t-1}$. Only two of the three conditions above are necessary according to Walras' Law. Also note that the economy is closed and there is no government spending.

Finally, we assume that technology evolves according to the following exogenous process:

$$Z_t = Z_{t-1}^{\rho} \exp(\epsilon_t), \quad \rho < 1 \tag{1.21}$$

where ϵ_t is an i.i.d. zero-mean random variable. Since our focus is business cycles, we ignore that output/productivity can grow over time.

1.4 Recap

We can now summarize the economy with the following system of equations:

$$
\begin{aligned}
C_t L_t^{1/\eta} &= (1-\alpha)Y_t/L_t = W_t \\
\frac{C_{t+1}}{C_t} &= \beta(1+R_{t+1}) = \beta(\alpha Y_{t+1}/K_t + (1-\delta)) \\
Y_t &= Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} \\
Y_t &= C_t + I_t \\
K_t &= (1-\delta)K_{t-1} + I_t \\
Z_t &= Z_{t-1}^{\rho} \exp(\epsilon_t)
\end{aligned}
$$

1.5 Steady State

To understand why the system is fluctuating, we first need to find the values the system takes when it is not perturbed with shocks. The **steady state** is the state of the economy when the economy is “still”. The steady-state value of variable X_t is denoted with $\bar{X}$. Using the description of the competitive equilibrium given above, we infer that the steady-state is characterized by the following equations:

$$
\begin{aligned}
\bar{C}\bar{L}^{1/\eta} &= (1-\alpha)\bar{Y}/\bar{L} && (1.22) \\
\frac{\bar{C}}{\bar{C}} &= \beta(\alpha\bar{Y}/\bar{K} + (1-\delta)) && (1.23) \\
\bar{Y} &= \bar{Z}\bar{K}^{\alpha}\bar{L}^{1-\alpha} && (1.24) \\
\bar{Y} &= \bar{C} + \bar{I} && (1.25) \\
\bar{K} &= (1-\delta)\bar{K} + \bar{I} && (1.26)
\end{aligned}
$$

$$\bar{Z} = \bar{Z}^{\rho} \tag{1.27}$$

Notice that $\mathbb{E}[\exp\epsilon] = 1$ because there are no shocks in steady state and $\epsilon = 0$ at all times.

We now solve for the steady-state values of the variables. Immediately, from equations (1.24), (1.26) and (1.27) we get:

$$\begin{aligned}
\bar{I} &= \delta\bar{K} \\
\bar{Z} &= 1 \\
\bar{Y} &= \bar{K}^{\alpha}\bar{L}^{1-\alpha}
\end{aligned}$$

Equation (1.23) yields:

$$\begin{aligned}
1 &= \beta(\alpha\bar{Y}/\bar{K} + (1-\delta)) \\
1 &= \beta(\alpha(\bar{K}/\bar{L})^{\alpha-1} + (1-\delta)) \\
A \equiv \bar{K}/\bar{L} &= \Big(\frac{1/\beta - (1-\delta)}{\alpha}\Big)^{\frac{1}{\alpha-1}} \\
\bar{K} &= \Big(\frac{1/\beta - (1-\delta)}{\alpha}\Big)^{\frac{1}{\alpha-1}}\bar{L}
\end{aligned}$$

where A is solely a function of the parameters of the economy. Equation (1.22) leads to:

$$\begin{aligned}
\bar{C} &= (1-\alpha)\bar{L}^{-1/\eta}(\bar{Y}/\bar{L}) \\
\bar{C} &= (1-\alpha)\bar{L}^{-1/\eta}\Big(\frac{\bar{K}}{\bar{L}}\Big)^{\alpha}
\end{aligned}$$

Next, using $\bar{Y} = \bar{C} + \delta\bar{K}$, and plugging in the expressions for $\bar{Y}$, $\bar{C}$, $\bar{K}$ as a function of $\bar{L}$, we find:

$$\Big(\frac{\bar{K}}{\bar{L}}\Big)^{\alpha}\bar{L} = (1-\alpha)\bar{L}^{-1/\eta}\Big(\frac{\bar{K}}{\bar{L}}\Big)^{\alpha} + \delta\Big(\frac{\bar{K}}{\bar{L}}\Big)\bar{L}$$

Finally, $\bar{L}$ is implicitly defined by:

$$A^\alpha \bar{L} = (1-\alpha)\bar{L}^{-1/\eta}A^\alpha + \delta A\bar{L}$$

This is a non-linear equation in $\bar{L}$ and hence it's hard to have a simple expression for $\bar{L}$. However, once $\bar{L}$ is determined, we can find steady state values for all other variables.

1.6 Log-Linearization

Non-linear systems are hard to analyze. For small perturbations, linear approximations capture the dynamics of the model well and are easy to analyze. We will linearize the system around the steady state and study its properties using the linearized version of the model in the neighborhood of the steady state. It is conventional to work with *percent* deviations from the steady state:

$$\check{X}_t = \frac{X_t - \bar{X}}{\bar{X}} = \frac{dX_t}{\bar{X}} \approx \ln(X_t/\bar{X})$$

The theory and practice of log-linearization will be covered in section. The log-linearized system of equations around the steady state will also be derived in great details in section. Let's work through two simple examples to get a flavor of it. Start with the equation defining investment:

$$\begin{aligned}
K_t &= (1-\delta)K_{t-1} + I_t \\
dK_t &= (1-\delta)dK_{t-1} + dI_t \\
\frac{dK_t}{\bar{K}} &= (1-\delta)\frac{dK_{t-1}}{\bar{K}} + \frac{dI_t}{\bar{K}} \\
\frac{dK_t}{\bar{K}} &= (1-\delta)\frac{dK_{t-1}}{\bar{K}} + \frac{\bar{I}}{\bar{K}}\frac{dI_t}{\bar{I}} \\
\check{K}_t &= (1-\delta)\check{K}_{t-1} + \delta\check{I}_t
\end{aligned}$$

where we used the steady state condition: $\frac{\bar{I}}{\bar{K}} = \delta$.

Consider now the equation describing equilibrium in the goods market:

$$\begin{aligned}
Y_t &= C_t + I_t \\
dY_t &= dC_t + dI_t \\
\frac{dY_t}{\bar{Y}} &= \frac{\bar{C}}{\bar{Y}}\frac{dC_t}{\bar{C}} + \frac{\bar{I}}{\bar{Y}}\frac{dI_t}{\bar{I}} \\
\check{Y}_t &= \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t \\
\check{Y}_t &= \left(1 - \frac{\bar{I}}{\bar{Y}}\right)\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t
\end{aligned}$$

Moreover, the ratio $\frac{\bar{I}}{\bar{Y}}$ can be determined using the steady-state equations:

$$\begin{aligned}
\frac{\bar{I}}{\bar{Y}} &= \delta\frac{\bar{K}}{\bar{Y}} \\
1 &= \beta(\alpha\bar{Y}/\bar{K} + (1-\delta))
\end{aligned}$$

Following the same strategy, the remaining equations of the log-linearized system around the steady state can be derived. The log-linearized system is:

$$\check{Y}_t = (1 - \frac{\bar{I}}{\bar{Y}})\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t \tag{1.28}$$

$$\check{K}_t = (1-\delta)\check{K}_{t-1} + \delta\check{I}_t \tag{1.29}$$

$$\check{C}_t + (1/\eta)\check{L}_t = \check{Y}_t - \check{L}_t \tag{1.30}$$

$$\mathbb{E}_t\check{C}_{t+1} - \check{C}_t = \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Y}_{t+1} - \check{K}_t) \tag{1.31}$$

$$\check{Y}_t = \check{Z}_t + \alpha\check{K}_{t-1} + (1-\alpha)\check{L}_t \tag{1.32}$$

$$\check{Z}_t = \rho\check{Z}_{t-1} + \varepsilon_t \tag{1.33}$$

Note that we have put the expectations operators back into the equations. An important property of linear rational expectations models is *certainty equivalence*. This means that even though the future is uncertain ($\varepsilon_t \neq 0$ for $t > 0$), agents behave exactly *as if* they knew with certainty that there were no shocks in the future ($\varepsilon_t = 0$ for $t > 0$). For our purposes it has two convenient properties. First, we can linearize the model with certainty and then simply add expectations operators to any $t+1$-dated variables. Second, when we want to know how the model reacts to a single shock at t_0, we can drop the expectations operators after $t > t_0$.

We will use this system of equations to examine how endogenous variables in this economy respond to shocks to technology and to study other properties of the model.

1.7 Calibration

In the vast majority of cases, general equilibrium models do not have simple analytical solutions. We have to rely on numerical methods to study the properties of these models. To do the numerical analysis, we need to assign parameter values such as capital share, elasticity of labor supply, volatility and persistence of technology shocks, etc. There are several strategies to assign parameter values. The first strategy is to estimate the model using some moments of the data and later use the estimated parameters to evaluate the performance of the model using some other moments of the data. Although this approach is quite intuitive, it is hard to believe that the simple model we consider is a realistic description of the world. Clearly the model is misspecified and hence estimates can be badly biased. This critique is less appealing when we consider more sophisticated models such as Smets and Wouters (2007) and Christiano, Eichenbaum, and Evans (2005) with tens of variables, shocks, and equations.

The second strategy is to calibrate parameter values, that is, assign parameter values that are plausible from the economic standpoint. This strategy is, in some sense, less rigorous but it is also more robust to misspecification. The trick in this strategy is to judiciously pick moments in the data such that we can assign a value to a given parameter without relying on estimation of the whole model. In other words, we can split the model into blocks such that each parameter governs a particular moment. Hence, even if there is a misspecification in one equation (moment) it is less likely to spill over to other parameters/moments. This was the motivation for this strategy in the seminal paper by Kydland and Prescott (1982).

More specifically, Kydland and Prescott (1982) envisioned that the researcher should use estimates from **micro**-level studies to assign parameter values (e.g., labor studies reporting estimates of labor supply elasticity based on micro-level data) and then study the **macroeconomic** properties of the model. This ideal scenario is however not always desirable or feasible. For example, the labor supply elasticity at the macro level has two margins: extensive (= how many people work) and intensive (= how many hours a person works). On the other hand, micro-level studies report typically only the intensive margin and as result can miss an important extensive margin. Hence, in some cases, we have to rely on macroeconomic data to assign parameter values.

Note that we do not want to assign parameter values using a set of moments and then evaluate the calibrated model using the same set of moments. For example, if the model is just-identified, we'll have a perfect fit of the model and it would not be a very informative test of how successful the model is at capturing the properties of the data in general. Kydland and Prescott (1982) suggested the following approach which became the standard practice in macroeconomic analysis of business cycles. The researcher uses the **first** moments of the data (i.e., means, ratios, etc.) to assign parameter values and then uses the **second** or **higher** moments of the data (e.g., variances, autocovariances, covariances, kurtosis, skewness, etc.) to evaluate the model. In general, one can evaluate the performance of the model using moments not employed in calibration. (In the estimation strategy this typically amounts to over-identification tests and out-of-sample forecasts).

Using the first order condition for labor in the firm's optimization problem, we can pin down the value of the elasticity of output with respect to capital α. The labor share in National Income and Product Accounts (NIPA) is relatively fixed over time and equals approximately:

$$1 - \alpha = \frac{WL}{Y} \approx 0.66$$

The real rate of return is somewhere between 4% and 6% (return on broad based portfolio in stock market) so an estimate of the quarterly discount factor is:

$$\beta = \Big(\frac{1}{1+\bar{R}}\Big)^{1/4} = \Big(\frac{1}{1.04}\Big)^{1/4} \approx 0.99$$

In the data (NIPA), aggregate investment to capital ratio is about 0.076. Since $\delta = \frac{\bar{I}}{\bar{K}}$, if we assume no growth, the annual depreciation rate is about 0.076. Therefore the quarterly rate is:

$$\delta = (1-0.076)^{(1/4)} \approx \frac{1}{4}(0.076) \approx 2\%$$

The labor supply elasticity η is an important parameter, but it is very hard to estimate. η has been estimated using micro-level data. There is a wide range of estimates, but studies suggest that the elasticity of labor supply is low, somewhere between 0 and 0.5. However, for macroeconomic models to generate any interesting dynamics, you need to have a high labor supply elasticity (we'll discuss later why but for now hours or work are as volatile as output and intuitively we need to have a high labor supply elasticity to capture this fact). Also this low elasticity is likely to reflect the intensive margin of the labor supply rather than the intensive + extensive margins of the labor supply.

To compute the volatility and the parameters of technology shocks, we can use a measure of Solow residual and fit an AR(1) autoregressive process . The parameters obtained are usually $\rho \approx 1$ and $\sigma_\epsilon = 0.007$. We will use $\rho = 0.95$.

Let's check the consistency of both the estimates and the model. In the steady state of the model (see equation (1.23)):

$$\beta = \frac{1}{\alpha\frac{Y}{K} + (1-\delta)}$$

We estimated $\delta = 0.076$ and $\alpha = 0.33$. In the data, the capital output ratio is about 3.32. This yields an estimate of β: $\beta \approx \frac{1}{.33\frac{1}{3.33}+(1-0.076)} = 0.976$. So the quarterly discount factor obtained is $\beta^{1/4} = 0.994$, which is consistent with the estimate obtained above using the real rate of return.

1.8 Dynamic Solution

Recall from the previous lecture that our log-linearized model is described by the system of equations (1.28)-(1.33). We will use the **method of undetermined coefficients** to solve the model. To simplify the solution, suppose that labor is supplied inelastically and assume that the labor supply is 1. We can reduce the system to a two-variable analogue (two endogenous variables):

$$
\begin{aligned}
\check{K}_t &= (1-\delta)\check{K}_{t-1} + \delta\check{I}_t \\
&= (1-\delta)\check{K}_{t-1} + \delta\frac{\bar{Y}}{\bar{I}}(\check{Y}_t - \frac{\bar{C}}{\bar{Y}}\check{C}_t) \\
&= (1-\delta)\check{K}_{t-1} + \delta\frac{\bar{Y}}{\bar{I}}(\check{Z}_t + \alpha\check{K}_{t-1} - \frac{\bar{C}}{\bar{Y}}\check{C}_t) \\
&= \Big((1-\delta) + \delta\alpha\frac{\bar{Y}}{\bar{I}}\Big)\check{K}_{t-1} + \delta\frac{\bar{Y}}{\bar{I}}\check{Z}_t - \delta\frac{\bar{C}}{\bar{I}}\check{C}_t \\
&= \Big((1-\delta) + 1/\beta - (1-\delta)\Big)\check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \delta\frac{\bar{C}}{\bar{I}}\check{C}_t \\
&= 1/\beta\check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t
\end{aligned}
$$

Next:

$$
\begin{aligned}
\mathbb{E}_t\check{C}_{t+1} - \check{C}_t &= \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Y}_{t+1} - \check{K}_t) \\
&= \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Z}_{t+1} + \alpha\check{K}_t - \check{K}_t) \\
&= \frac{\alpha\bar{Y}/\bar{K}}{\alpha\bar{Y}/\bar{K} + (1-\delta)}\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t) \\
&= \frac{1/\beta - (1-\delta)}{1/\beta - (1-\delta) + (1-\delta)}\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t) \\
&= (1-\beta(1-\delta))\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t)
\end{aligned}
$$

Thus, we obtain the following system:

$$\begin{aligned}
\check{K}_t &= 1/\beta \check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t & (1.34)\\
\mathbb{E}_t\check{C}_{t+1} - \check{C}_t &= (1-\beta(1-\delta))\mathbb{E}_t(\check{Z}_{t+1} + (\alpha-1)\check{K}_t) & (1.35)\\
\check{Z}_t &= \rho\check{Z}_{t-1} + \varepsilon_t & (1.36)
\end{aligned}$$

In this system, there are two state variables: capital K and level of technology Z. From the optimal control theory, we know that the solution should take the following form:

$$\begin{aligned}
\check{C}_t &= \nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t & (1.37)\\
\check{K}_t &= \nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t & (1.38)
\end{aligned}$$

Plug equations (1.37) and (1.38) into equation (1.34):

$$\begin{aligned}
\check{K}_t &= 1/\beta \check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Z}_t - \frac{\bar{C}}{\bar{K}}[\nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t]\\
&= \Big(1/\beta - \frac{\bar{C}}{\bar{K}}\nu_{ck}\Big)\check{K}_{t-1} + \Big(\frac{\bar{Y}}{\bar{K}} - \frac{\bar{C}}{\bar{K}}\nu_{cz}\Big)\check{Z}_t
\end{aligned}$$

By equating coefficients in both equations, we get:

$$\begin{aligned}
\nu_{kk} &= 1/\beta - \frac{\bar{C}}{\bar{K}}\nu_{ck}\\
\nu_{kz} &= \frac{\bar{Y}}{\bar{K}} - \frac{\bar{C}}{\bar{K}}\nu_{cz}
\end{aligned}$$

which yields the following equations, that we will use repeatedly:

$$\begin{aligned}
\nu_{ck} &= \frac{\bar{K}}{\bar{C}}[1/\beta - \nu_{kk}] & (1.39)\\
\nu_{cz} &= \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz} & (1.40)
\end{aligned}$$

Next we turn to equation (1.35):

$$
\begin{aligned}
0 &= \mathbb{E}_t\Big[\check{C}_t - \check{C}_{t+1} + (1-\beta(1-\delta))(\check{Z}_{t+1} + (\alpha-1)\check{K}_t)\Big] \\
&= \mathbb{E}_t\Big[\nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t - \nu_{ck}\check{K}_t - \nu_{cz}\check{Z}_{t+1} \\
&\quad +(1-\beta(1-\delta))\check{Z}_{t+1} + (1-\beta(1-\delta))(\alpha-1)(\nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t)\Big] \\
&= \mathbb{E}_t\Big[\nu_{ck}\check{K}_{t-1} + \nu_{cz}\check{Z}_t - \nu_{ck}(\nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t) - \nu_{cz}\check{Z}_{t+1} \\
&\quad +(1-\beta(1-\delta))\check{Z}_{t+1} + (1-\beta(1-\delta))(\alpha-1)(\nu_{kk}\check{K}_{t-1} + \nu_{kz}\check{Z}_t)\Big] \\
&= \Big[\nu_{ck} - \nu_{ck}\nu_{kk} + (1-\beta(1-\delta))(\alpha-1)\nu_{kk}\Big]\check{K}_{t-1} + \\
&\quad \Big[\nu_{cz} - \nu_{ck}\nu_{kz} - \nu_{cz}\rho + (1-\beta(1-\delta))\rho + (1-\beta(1-\delta))(\alpha-1)\nu_{kz}\Big]\check{Z}_t
\end{aligned}
$$

This is done for all values of K_{t-1} and Z_t. Hence it must true that:

$$0 = \nu_{ck} - \nu_{ck}\nu_{kk} + (1-\beta(1-\delta))(\alpha-1)\nu_{kk} \tag{1.41}$$

$$0 = \nu_{cz}(1-\rho) - \nu_{ck}\nu_{kz} + (1-\beta(1-\delta))\rho + (1-\beta(1-\delta))(\alpha-1)\nu_{kz} \tag{1.42}$$

We start with equation (1.41):

$$
\begin{aligned}
0 &= \nu_{ck} - \nu_{ck}\nu_{kk} + (1-\beta(1-\delta))(\alpha-1)\nu_{kk} \\
0 &= \frac{\bar{K}}{\bar{C}}[1/\beta - \nu_{kk}](1-\nu_{kk}) + (1-\beta(1-\delta))(\alpha-1)\nu_{kk} \\
0 &= 1/\beta - (1/\beta+1)\nu_{kk} + \nu_{kk}^2 + \frac{\bar{C}}{\bar{K}}(1-\beta(1-\delta))(\alpha-1)\nu_{kk} \\
0 &= 1/\beta - \Big[1/\beta + 1 - \frac{\bar{C}}{\bar{K}}(1-\beta(1-\delta))(\alpha-1)\Big]\nu_{kk} + \nu_{kk}^2 \\
0 &= 1/\beta - \gamma\nu_{kk} + \nu_{kk}^2
\end{aligned}
$$

where $\gamma \equiv [1/\beta + 1 - \frac{\bar{C}}{\bar{K}}(1-\beta(1-\delta))(\alpha-1)]$. Thus:

$$\nu_{kk} = \frac{\gamma \pm \sqrt{\gamma^2 - 4/\beta}}{2}$$

We pick the stable root (which corresponds to the saddle path):

$$\nu_{kk} = \frac{\gamma - \sqrt{\gamma^2 - 4/\beta}}{2}$$

From ν_{kk} we can compute ν_{ck}. Next, we need to find ν_{cz} and ν_{kz}. Recall from equation (1.40) that:

$$\nu_{cz} = \frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz}$$

Using equation (1.42), we have:

$$\begin{aligned} \left[\frac{\bar{Y}}{\bar{C}} - \frac{\bar{K}}{\bar{C}}\nu_{kz}\right](1-\rho) - \frac{\bar{K}}{\bar{C}}[1/\beta - \nu_{kk}]\nu_{kz} + (1-\beta(1-\delta))\rho + (1-\beta(1-\delta))(\alpha-1)\nu_{kz} = 0 \\ \frac{\bar{Y}}{\bar{C}}(1-\rho) + (1-\beta(1-\delta))\rho = \Big(\frac{\bar{K}}{\bar{C}}(1-\rho) + \frac{\bar{K}}{\bar{C}}(1/\beta - \nu_{kk}) - (1-\beta(1-\delta))(\alpha-1)\Big)\nu_{kz} \end{aligned}$$

To conclude:

$$\nu_{kz} = \frac{\frac{\bar{Y}}{\bar{C}}(1-\rho) + (1-\beta(1-\delta))\rho}{\frac{\bar{K}}{\bar{C}}(1-\rho) + \frac{\bar{K}}{\bar{C}}(1/\beta - \nu_{kk}) - (1-\beta(1-\delta))(\alpha-1)}$$

Now we have a solution to the system of rational expectations equations (REE). We can use this solution to simulate the model, generate moments, and compare the model and data moments.

We can use the same technique when we add more shocks to the system. For instance, we will be interested in fiscal shocks:

$$Y = C + I + G \Rightarrow \check{Y}_t = \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{I}}{\bar{Y}}\check{I}_t + \frac{\bar{G}}{\bar{Y}}\check{G}_t$$

where $\check{G}_t = \phi\check{G}_{t-1} + u_t$, and $u_t \sim$ i.i.d. $(0, \sigma_u^2)$ is a fiscal shock.

1.9 Blanchard and Kahn's Method

A more general approach to solving linear REE models is due to Blanchard and Kahn (1980). The idea is as follows. We can write the REE system as $X_t = AX_{t+1}$ where $X_t = [X_{1t}\ X_{2t}\ X_{3t}\ X_{4t}]$.

- X_1 are control variables
- X_2 are state variables
- X_3 are co-state variables
- X_4 are shock variables

If necessary, we can create dummy variables to write the system in the desired form. For example, consider $X_{t+1} = \alpha X_t + \beta X_{t-1}$. Let $Z_t = X_{t-1}$. Then:

$$\begin{aligned} X_{t+1} &= \alpha X_t + \beta Z_t \\ Z_{t+1} &= X_t \end{aligned}$$

Using $Y_t = [X_t\ Z_t]$, the system can be rewritten as $Y_{t+1} = AY_t$.

Let $A = PVP^{-1}$ be the eigenvalue-eigenvector decomposition of A. We will partition the matrices V into two parts, $V = [V_1, V_2]'$, where V_1 collects all eigenvalues less than 1 and V_2 collects all eigenvalues greater or equal to 1. The associates (inverse) matrix of eigenvectors is given by,

$$P^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Multiply both sides of $Y_t = AY_{t+1}$ by A^{-1} and using certainty equivalence and iterating forward:

$$\begin{bmatrix} X_{1,t+j} \\ X_{2,t+j} \end{bmatrix} = P \begin{bmatrix} V_1^j & 0 \\ 0 & V_2^j \end{bmatrix} \begin{bmatrix} P_{11}X_{1,t} + P_{12}X_{2,t} \\ P_{21}X_{1,t} + P_{22}X_{2,t} \end{bmatrix} \tag{1.43}$$

For general $X \neq 0$, the bottom part of this system will typically be explosive (or at least non-mean-reverting) since V_2 has eigenvalues greater or equal to one. This is where we use the Transversality conditions. Remember, these put an optimal limit on how fast variables can grow in the model. To ensure that our solution remains finite we need that the transversality conditions (TVCs) put n_2 restrictions on V_2, where $n_2 = \#\text{rows}(V_2)$. For instance, the transversality condition (1.2) restricts the growth rate of both assets and consumption since they are linked by the budget constraint!

Finally, note that the TVCs restrict choice variables: For instance, a consumer cannot consume so much and save so little that debt accumulates too fast. Thus another way of stating the condition that we need as many choice ("jump") variables as there are eigenvalues greater or equal to 1. This is known as the Blanchard-Kahn condition. Implicit here is that the TVC restricts these choices.

Our transversality conditions therefore imply that $X_{2,t+j} \to 0$ as $j \to +\infty$ given the initial condition $X \geq 0$. Since $V_2 > I_{n_2}$, the term multiplying V_2 must be zero. We must have:

$$\begin{aligned} 0 &= P_{21}X_{1,t} + P_{22}X_{2,t} \\ X_{2,t} &= -P_{22}^{-1}P_{21}X_{1,t} \end{aligned}$$

Turning to equation (1.43) again:

$$\begin{aligned} \begin{bmatrix} X_{1,t+j} \\ X_{2,t+j} \end{bmatrix} &= P \begin{bmatrix} V_1^j & 0 \\ 0 & V_2^j \end{bmatrix} \begin{bmatrix} P_{11}X_{1,t} - P_{12}P_{22}^{-1}P_{21}X_{1,t} \\ 0 \end{bmatrix} \\ &= P \begin{bmatrix} V_1^j(P_{11}X_{1,t} - P_{12}P_{22}^{-1}P_{21}X_{1,t}) \\ 0 \end{bmatrix} \end{aligned}$$

$$= P \begin{bmatrix} V_1^j (P_{11} - P_{12} P_{22}^{-1} P_{21}) \\ 0 \end{bmatrix} \begin{bmatrix} X_{1,t} \\ 0 \end{bmatrix}$$

Given initial conditions on $X_{1,t}$ we can now solve the system. This implies that we need $n_1 = \#\text{rows}(V_1)$ initial conditions, or (equivalently) n_1 state variables. Thus, we can also state the Blanchard-Kahn conditions as requiring the number of state variables equal to number of eigenvalues less than 1.

An important result from this solution method is that the system converges at rate $V_1 < I_{n_1}$ back to the steady-state (zero). For instance, in our simply model $V_1 = \frac{\gamma - \sqrt{\gamma^2 - 4/\beta}}{2} < 1$ (see previous section). This implies that given some initial capital stock $\check{K}_{t-1} \neq 0$, the economy will close $1 - V_1$ of the gap, $\check{K}_t - \check{K}_{t-1} = (V_1 - 1)\check{K}_{t-1}$. Thus, the larger the initial distance from the steady-state, the greater the decline in the gap, and the larger (in an absolute sense) is investment. This formed the basis for our assertion that investment is (in an absolute sense) monotonically declining as we converge to the steady-state.

Once we have solved the autonomous system, we can add forcing variables and shocks. (Again we use the certainty-equivalence property.)

$$X_t = AX_{t+1} + BV_{t+1}$$

with $\mathbb{E}_t V_{t+1} = 0$. This can be rewritten as:

$$\begin{aligned} A^{-1}X_t &= X_{t+1} + A^{-1}BV_{t+1} \\ X_{t+1} &= A^{-1}X_t - A^{-1}BV_{t+1} \end{aligned}$$

Chapter 2

Comparative Statics of Equilibrium in RBC

In the previous chapter, we specified, calibrated, log-linearized, and solved the basic real business cycle model. We are almost ready to assess how it fits the economy. However, before we do this, it will be useful to get some intuition about what the model does. Also we need to specify how we are going to assess the model. Finally, we need to make predictions about how the model responds to various shocks so that we can use these predictions to test the model.

Recall the FOCs of the RBC:

$$\frac{\partial \mathcal{L}}{\partial L_t} = -L_t^{1/\eta} + \lambda_t W_t = 0 \tag{2.1}$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t} - \lambda_t = 0 \tag{2.2}$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = -\lambda_t + \beta(1 + R_{t+1})\lambda_{t+1} = 0 \tag{2.3}$$

λ_t is the Lagrange multiplier of the budget constraint. It corresponds to the marginal utility of wealth. It is the shadow price of one unit of capital/wealth. Equation (2.2) is the FOC determining consumption. Equation (2.1) is the FOC determining labor supply. Equation

(2.3) is the Euler equation.

We start with the Euler equation (equation 2.3). Using repeated substitution, we find:

$$\begin{aligned}\lambda_t &= \mathbb{E}_t\beta(1+R_{t+1})\lambda_{t+1} = \mathbb{E}_t\beta(1+R_{t+1})(\beta(1+R_{t+2})\lambda_{t+2}) = \mathbb{E}_t\beta(1+R_{t+1})(\beta(1+R_{t+2})(\beta(1+R_{t+3})\lambda_{t+}\\ &= \lim_{s\to+\infty}\beta^s\mathbb{E}_t\Big(\Big[\prod_{i=1}^{s}(1+R_{t+i})\Big]\ \lambda_{t+s}\Big)\end{aligned}$$

Clearly λ_t is a forward looking variable. Also because λ_t depends on R_{t+i} and λ_{t+s} many periods into the future, it is reasonable to assume that in the short run, the variability of λ due to **temporary** shocks is small. In other words, because λ depends on the future stream of marginal utilities, the variation in λ can mostly depend on the distant future when the economy is close to the steady state provided that the discount factor is close to one. Hence, if shocks in our model are transitory, we can assume that λ_t is relatively stable. Obviously, when shocks are permanent, the steady state will move in response to the shocks and hence λ_t can move significantly too.

This is essentially the permanent income hypothesis. Since consumption is inversely proportional to λ_t, all our statements about λ_t directly map to consumption. Thus, consumption will move relatively little in response to temporary shocks but it can move a lot in response to permanent shock.

Now let's go back to the FOC for labor supply (equation 2.1):

$$L_t = \lambda_t^\eta W_t^\eta \tag{2.4}$$

The labor supply curve L^s is an upward-sloping function of the wage W, as defined in equation (2.4). Movements in λ_t can be interpreted as wealth shocks. Since we can treat λ_t as relatively constant with transitory shocks, it means that L^s is relatively fixed and most of the time, temporary shocks will give us movements along the labor supply curve rather shifts of the labor supply curve.

There is something important about this relationship. Barro and King (1984) used $L_t =$

$\lambda_t^\eta W_t^\eta$ to make an important assessment of how the model is going to fit the data. Note that the FOC for labor can be written more generally as:

$$U'(C_t)W_t = V'(L_t) \tag{2.5}$$

where $V'(L_t)$ is the marginal disutility of labor (in our specification $V(L_t) = \frac{L_t^{1+1/\eta}}{1+1/\eta}$ and hence $V'(L_t) = L_t^{1/\eta}$). Equation (2.5) is the intratemporal condition linking consumption to supply of labor, and it should hold in every point in time. We know that in the business cycle, consumption and labor supply comove. $C \nearrow$ means that $U'(C) \searrow$. $L \nearrow$ means that $V(L) \nearrow$. Therefore W **has to be very procyclical**. This is because consumption and leisure are complementary goods, which means that consumption and labor supply are substitutes. Thus, consumption and leisure will typically move in the same direction (and consumption and labor supply in opposite directions), unless the price of leisure (the real wage) moves a lot.

In contrast, wages in the data are only weakly procyclical. Hence, in this bare-bones formulation of a closed economy, the fit to the data is likely to have problems with matching weak procyclicality of wages and we need to think more carefully about potential ways to alleviate this problem. There are a few caveats however as this result relies on:

1. time separability of utility function;

2. "representative customer" paradigm;

3. free choice of consumption and labor during the cycle.

From firms' optimization, we had:

$$W_t = MPL_t = (1-\alpha)Z_t L_t^{-\alpha} K_{t-1}^{\alpha} = (1-\alpha)Z_t\Big(\frac{K_{t-1}}{L_t}\Big)^{\alpha} \tag{2.6}$$

Equation (2.6) defines a downward-sloping labor-demand curve L^d. Labor market equilibrium implies that:

$$(1-\alpha)Z_t\Big(\frac{K_{t-1}}{L_t}\Big)^{\alpha} = W_t = \frac{L_t^{1/\eta}}{\lambda_t} = L_t^{1/\eta}C_t \tag{2.7}$$

Equation (2.7) implies that holding K and Z constant, there is a negative relationship between employment and consumption. That is, $C \nearrow$ when $L \searrow$.

Equilibrium in the goods market implies:

$$C + I + G = Y = Z_t L_t \Big(\frac{K_{t-1}}{L_t}\Big)^{\alpha}$$

This condition will help us to sign changes in variables in response to shocks.

Finally, we can observe a useful relationship between wages and interest rate from the first order conditions for inputs:

$$\begin{aligned} W_t &= MPL_t = (1-\alpha)Z_t\Big(\frac{K_{t-1}}{L_t}\Big)^{\alpha} \\ R_t + \delta &= MPK_t = \alpha Z_t\Big(\frac{K_{t-1}}{L_t}\Big)^{(\alpha-1)} \end{aligned}$$

Note that the interest rate and wages are pinned down by two variables: the level of technology Z_t and the ratio of capital to labor. An increase in the capital-to-labor ratio will increases wages and reduces interest. This trade-off is called the factor price possibility frontier (FPPF). Changes in the capital-to-labor ratio correspond to movements along the frontier. Changes in technology shift the frontier. For example, an increase in Z_t allows firms to pay higher wages and interest rates which corresponds to an upward shift of the frontier.

FPPF can provide further insight about the behavior of wages and interest rate in the long run. We know from the Euler equation that $C_{t+1}/C_t = \beta(1 + R_{t+1})$. Since consumption is fixed in the steady state, the left hand side of this equation is equal to one in the steady state and therefore the steady-state interest rate is effectively set by the discount factor β. It follows that while shifts of the FPPF due to changes in technology can result in temporary changes of the interest rate, these shifts have no effect the interest rate in the steady state. Furthermore, any changes in technology will be absorbed by wages.

2.1 Derivation of Phase Diagram

Our model is in discrete time which is not very convenient for phase diagrams but it is convenient later when we solve the model and simulate it on a computer. Another issue with the discrete time formulation is that we can have a few timing inconsistencies which do not arise in continuous time. Since the purpose of this exercise is to get intuition rather than get quantitative answers, we will ignore these issues. We will also focus on the linearized version of the model. The advantage of working with the linearized version is that we can locally (that is, in the neighborhood of the steady state) determine the signs of various loci and arms.

Relative to our baseline model, we introduce at this point more explicitly government expenditures G_t. Similar to previous instances of having G_t in the model, we will assume that consumers do not derive the utility from G_t (for example, G_t cannot be a public good such as national defense, radio broadcasting, navigation systems, etc.) and that G_t does not have any productive use (e.g., it does not raise the level of technology via research and development such as the Strategic Defence Initiative or Manhattan Project).

Since the phase diagram will have only two variables, we need to reduce the system of equations to a two-equation system. Using capital accumulation equation and goods market equilibrium, we can eliminate investment from the model and express the law of motion for capital as a function of output, consumption, government expenditures:

$$K_t = (1-\delta)K_{t-1} + I_t \Rightarrow \check{K}_t = (1-\delta)\check{K}_{t-1} + \delta\check{I}_t \tag{2.8}$$

Likewise,

$$
\begin{aligned}
Y_t &= I_t + C_t + G_t \Rightarrow \\
\check{Y}_t &= \frac{\bar{I}}{\bar{Y}}\check{I}_t + \frac{\bar{C}}{\bar{Y}}\check{C}_t + \frac{\bar{G}}{\bar{Y}}\check{G}_t \Rightarrow \\
\check{I}_t &= \frac{\bar{Y}}{\bar{I}}\check{Y}_t - \frac{\bar{C}}{\bar{I}}\check{C}_t - \frac{\bar{G}}{\bar{I}}\check{G}_t \Rightarrow \\
\check{I}_t &= \frac{1}{\delta}\frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{1}{\delta}\frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{1}{\delta}\frac{\bar{G}}{\bar{K}}\check{G}_t \qquad (2.9)
\end{aligned}
$$

Plugging equation (2.9) into equation (2.8), we obtain:

$$
\begin{aligned}
\check{K}_t &= (1-\delta)\check{K}_{t-1} + \delta\big(\frac{1}{\delta}\frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{1}{\delta}\frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{1}{\delta}\frac{\bar{G}}{\bar{K}}\check{G}_t\big) \\
&= (1-\delta)\check{K}_{t-1} + \frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t \qquad (2.10)
\end{aligned}
$$

Now combine equations to get:

$$
\begin{aligned}
\Delta\check{K}_t &\equiv \check{K}_t - \check{K}_{t-1} = \frac{\bar{Y}}{\bar{K}}\check{Y}_t - \frac{\bar{C}}{\bar{K}}\check{C}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t - \delta\check{K}_{t-1} \\
&= \frac{\bar{Y}}{\bar{K}}(\check{Z}_t + (1-\alpha)\check{L}_t + \alpha\check{K}_{t-1}) + \frac{\bar{C}}{\bar{K}}\check{\lambda}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t - \delta\check{K}_{t-1} \\
&= \frac{\bar{Y}}{\bar{K}}\check{Z}_t + \frac{\bar{Y}}{\bar{K}}\frac{(1-\alpha)}{\alpha+1/\eta}\big(\check{\lambda}_t + \check{Z}_t + \alpha\check{K}_{t-1}\big) + \alpha\frac{\bar{Y}}{\bar{K}}\check{K}_{t-1} + \frac{\bar{C}}{\bar{K}}\check{\lambda}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t - \delta\check{K}_{t-1}
\end{aligned}
$$

The last line makes use of the labor market equilibrium. Using FOC for labor supply and the labor market equilibrium condition that wages are equal to the marginal product of labor :

$$
L_t^{1/\eta} = \lambda_t W_t \;\Rightarrow\; (1/\eta)\check{L}_t = \check{\lambda}_t + \check{W}_t \qquad (2.11)
$$

$$
\check{W}_t = M\check{P}L_t \;\Rightarrow\; \check{W}_t = \check{Y}_t - \check{L}_t = \check{Z}_t + \alpha\check{K}_{t-1} - \alpha\check{L}_t \qquad (2.12)
$$

The second line comes from $MPL_t = \alpha\frac{Y_t}{L_t} \Rightarrow M\check{P}L_t = \check{Y}_t - \check{L}_t$ and $\check{Y}_t = \check{Z}_t + \alpha\check{K}_{t-1} + (1-\alpha)\check{L}_t$. To eliminate labor, we do the following steps:

$$
\check{W}_t = \check{Z}_t + \alpha\check{K}_{t-1} - \alpha\check{L}_t \qquad (2.13)
$$

$$\eta^{-1}\check{L}_t = \check{\lambda}_t + \check{W}_t \tag{2.14}$$

$$\eta^{-1}\check{L}_t = \check{\lambda}_t + \check{Z}_t + \alpha\check{K}_{t-1} - \alpha\check{L}_t \tag{2.15}$$

$$\check{L}_t = \frac{1}{1/\eta + \alpha}(\check{\lambda}_t + \check{Z}_t + \alpha\check{K}_{t-1}) \tag{2.16}$$

Hence, the change in the capital stock is described by

$$\begin{aligned}\Delta\check{K}_t &= \frac{\bar{Y}}{\bar{K}}\Big(1 + \frac{1-\alpha}{\alpha + 1/\eta}\Big)\check{Z}_t + \Big(\frac{\alpha(1-\alpha)}{\alpha + 1/\eta}\frac{\bar{Y}}{\bar{K}} + \alpha\frac{\bar{Y}}{\bar{K}} - \delta\Big)\check{K}_{t-1} \\ &\quad + \Big(\frac{\bar{C}}{\bar{K}} + \frac{\bar{Y}}{\bar{K}}\frac{(1-\alpha)}{\alpha + 1/\eta}\Big)\check{\lambda}_t - \frac{\bar{G}}{\bar{K}}\check{G}_t \end{aligned} \tag{2.17}$$

In the steady state, the change in the capital stock is zero and we can use this fact to find the properties of the locus of points such that the capital stock is not changing. Also note that we have eliminated many endogenous variables and now the change in the capital stock is only a function of past capital stock, marginal utility of wealth λ_t, and exogenous forcing variables Z_t and G_t.

We can do a similar trick with the marginal utility of wealth. Using $\frac{1}{C_t} = \lambda_t \Rightarrow \check{\lambda}_t = -\check{C}_t$, we have from the Euler equation

$$\begin{aligned}\Delta\check{C}_{t+1} &\equiv \check{C}_{t+1} - \check{C}_t = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + 1 - \delta}(\check{Y}_{t+1} - \check{K}_t) \\ &= \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + 1 - \delta}M\check{P}K_{t+1} \Rightarrow \\ \Delta\check{\lambda}_{t+1} &= -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + 1 - \delta}M\check{P}K_{t+1}\end{aligned}$$

MPK is a function of capital/labor ratio and we need to eliminate labor as before.

$$\begin{aligned}\Delta\check{\lambda}_{t+1} &= -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + 1 - \delta}\Big(\check{Z}_{t+1} + (\alpha - 1)\check{K}_t + (1-\alpha)\check{L}_{t+1}\Big) \qquad (2.18) \\ &= -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}} + 1 - \delta}\Big(\check{Z}_{t+1} + (\alpha - 1)\check{K}_t + (1-\alpha)\frac{1}{1/\eta + \alpha}(\check{\lambda}_{t+1} + \check{Z}_{t+1} + \alpha\check{K}_t)\Big)\end{aligned}$$

Finally we have:

$$\Delta\check{\lambda}_{t+1} \quad = \quad -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}\Big[\big(1+\frac{1-\alpha}{1/\eta+\alpha}\big)\check{Z}_{t+1}+(1-\alpha)\Big(\frac{\alpha}{1/\eta+\alpha}-1\Big)\check{K}_t+\frac{1-\alpha}{1/\eta+\alpha}\check{\lambda}_{t+1}\Big]$$

which is the law of motion for the change in the marginal utility of wealth. Note that the marginal utility of wealth does not depend directly on government expenditures (why?). Now by setting $\Delta\check{\lambda}_t = 0$, we have the locus of points where the marginal utility of wealth is not changing. We need to determine the slope of the loci such that capital stock and marginal utility of wealth are not changing. The intersection of these loci will give us the steady state.

Locus $\Delta\check{K}_t = 0$

$$\check{\lambda}_t = -\frac{1}{\Big(\frac{\bar{C}}{\bar{K}}+\frac{\bar{Y}}{\bar{K}}\frac{(1-\alpha)}{\alpha+1/\eta}\Big)}\Big[\frac{\bar{Y}}{\bar{K}}\Big(1+\frac{1-\alpha}{\alpha+1/\eta}\Big)\check{Z}_t+\Big(\alpha\frac{\bar{Y}}{\bar{K}}\big(\frac{1-\alpha}{\alpha+1/\eta}+1\big)-\delta\Big)\check{K}_{t-1}-\frac{\bar{G}}{\bar{K}}\check{G}_t\Big]$$

Because $\alpha\frac{\bar{Y}}{\bar{K}}(\frac{1-\alpha}{\alpha+1/\eta}+1)-\delta > 0$, the locus is **downward sloping**. On this locus, consumption, labor and investment choices are such that the current capital stock can be maintained. It is downward sloping because when less capital and thus less output is available, consumption must fall (and employment will rise) in order to maintain the same capital stock.

Locus $\Delta\check{\lambda}_t = 0$

$$\check{\lambda}_{t+1} \quad = \quad -\frac{1/\eta+\alpha}{1-\alpha}\Big\{\big(1+\frac{1-\alpha}{1/\eta+\alpha}\big)\check{Z}_{t+1}+(1-\alpha)\Big(\frac{\alpha}{1/\eta+\alpha}-1\Big)\check{K}_t\Big\}$$

This locus is **upward sloping** because $(1-\alpha)\Big(\frac{\alpha}{1/\eta+\alpha}-1\Big) < 0$ since $1-\alpha > 0$ and $\frac{\alpha}{1/\eta+\alpha} < 1$. On this locus, consumption and labor choices are such that the real interest rate and the MPK are equal to their steady-state values. Recall our Euler equation states that the growth rate of consumption is proportional to the real interest rate, $C_{t+1}/C_t = \beta(1+R_t)$. Thus, for consumption and marginal utility not to change we must have $1 = \beta(1+R_t) = \beta(1+MPK_{t+1}-\delta)$. The locus is upward-sloping because a higher capital stock lowers the

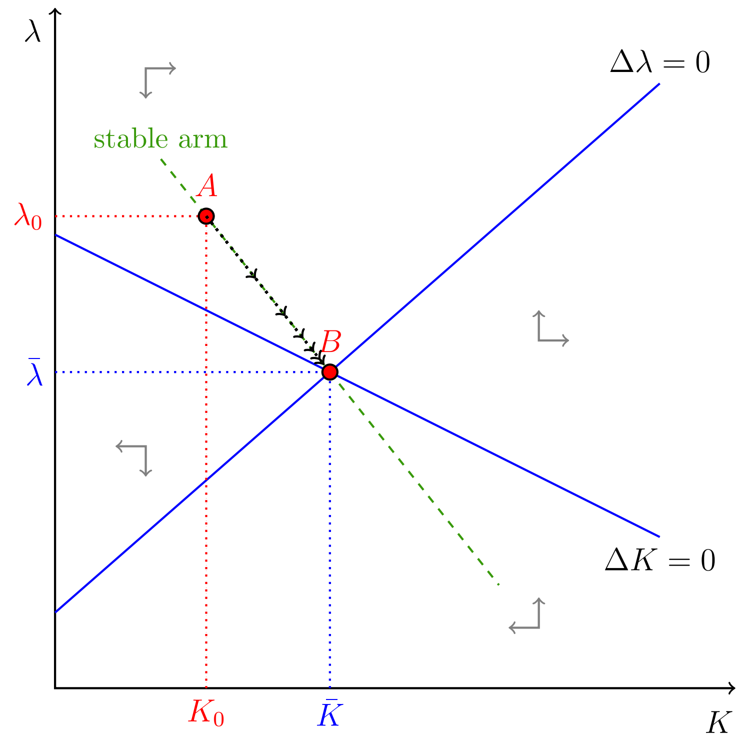


Figure 2.1: Model behavior when capital is below the steady-state at K_0. The economy begins at point A and converges to point B.

MPK and a higher marginal utility of consumption raises labor supply and thus the MPK. On the locus these two forces exactly balance.

2.1.1 Variables in phase diagram

K is given by history (it cannot jump on impact at the time of the shock) while λ is pointed to the future (it can jump on impact at the time of the shock). This means that the phase diagram will have two arms: one stable and one unstable. Here, the stable arm goes from north-west above both $\Delta\check{\lambda}_t = 0$ and $\Delta\check{K}_t = 0$ loci to the steady state and from south-east below both $\Delta\check{\lambda}_t = 0$ and $\Delta\check{K}_t = 0$ loci to the steady state. The unstable arm goes from north-west above $\Delta\check{\lambda}_t = 0$ and below $\Delta\check{K}_t = 0$ loci to the steady state and from south-east below $\Delta\check{\lambda}_t = 0$ and above $\Delta\check{K}_t = 0$ loci to the steady state. This is shown in Figure 2.1.

The stable arm depicts the optimal choice of $\check{\lambda}$ for a given level of capital. All other choices for $\check{\lambda}$ that are either above or below the stable arm produce explosive behavior that eventually converges to the unstable arm. For instance, if $\check{\lambda}$ is too low (consumption is too high), then the economy will converge towards the bottom left, eventually running out of capital and thus any potential for consumption. Above the stable arm the household consumes too little ($\check{\lambda}$ is too high) and keeps accumulating an infinite amount of capital. This violates the transversality condition. This consumer would on average have higher welfare by consuming more today.

Along the stable path the economy gradually converges towards the steady-state. The convergence will be at the rate of the stable eigenvalue in the economy. Based on the arguments in section 1.9, we know that along the stable arm the economy will converge at a rate $\mu_2 < 1$ to the steady-state, where μ_2 is the stable eigenvalue. Thus, in each period the distance to the steady-state gets reduced by a fraction $1 - \mu_2$. In turn, this implies that investment is highest (in an absolute sense) the further away the economy is from the steady state.

2.1.2 Shifts of loci after shocks

Using this linearized system of equations, we can easily get a sense of what should happen with the steady state when there is a permanent shock to technology or government expenditures. An increase in technology always shifts down the $\Delta\check{\lambda} = 0$ and $\Delta\check{K} = 0$ schedules. Therefore $\bar{\lambda}$ unambiguously falls. What happens to $\bar{K}$ depends on which locus shifts more.

When government expenditure G increases, only the locus $\Delta\check{K} = 0$ shifts up. Thus both steady state values of $\bar{\lambda}$ and $\bar{K}$ increase. These shifts are illustrated on Figure 2.2. In what follows, we will examine in detail the responses of the variables at the time of the shock ("impact"), on the transitional path to a new (or the old) steady state ("transition"), and at the time when the system converges to the new (or old) steady state ("steady state").

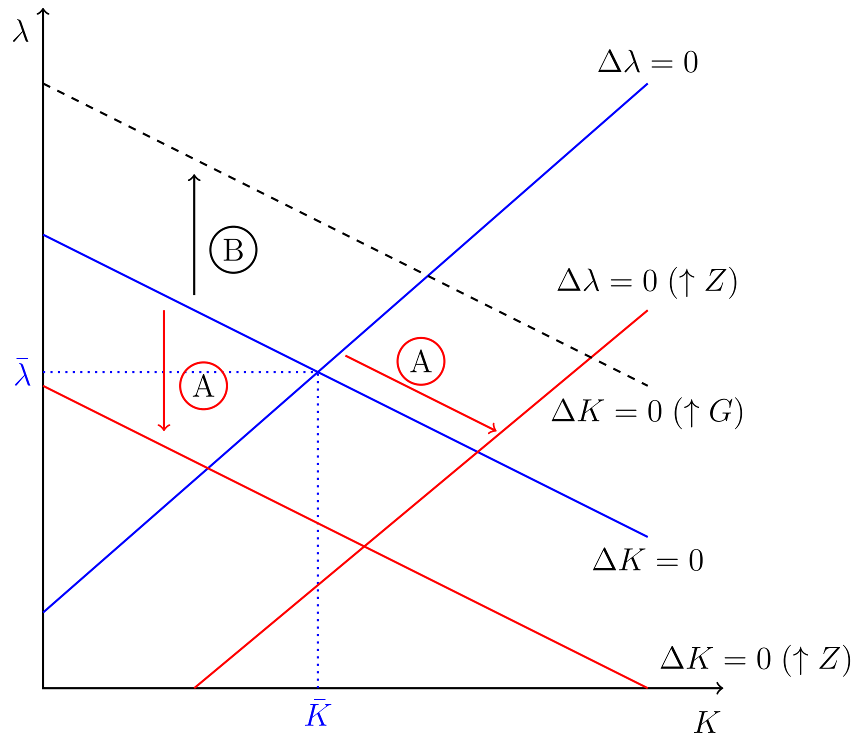


Figure 2.2: Response to technology shocks (A) and government-spending shocks (B).

2.2 Response to a Permanent Government-Spending Shock

We start our analysis of shocks with an easy case: a permanent, unanticipated increase in government spending. We assume that the shock occurs at time t_0. Remember again what this shock entails: the government is effectively throwing away output in this economy and taxing you (lump-sum) to pay for it. Think of wars on foreign territory. It is not investment in roads or schools! That we would have to model very differently. To organize our qualitative predictions about the behavior of the variables, we put results in Table 2.1.

We know from the phase diagram that $\Delta K = 0$ locus shifts up. The new steady state has a higher value of λ and a higher level of capital K (figure 2.3). Since capital is a predetermined variable and it does not move at the time of the shock (hence we put 0 in the second row of the first column in Table 2.1), at the time of the shock ($t = t_0$) the only

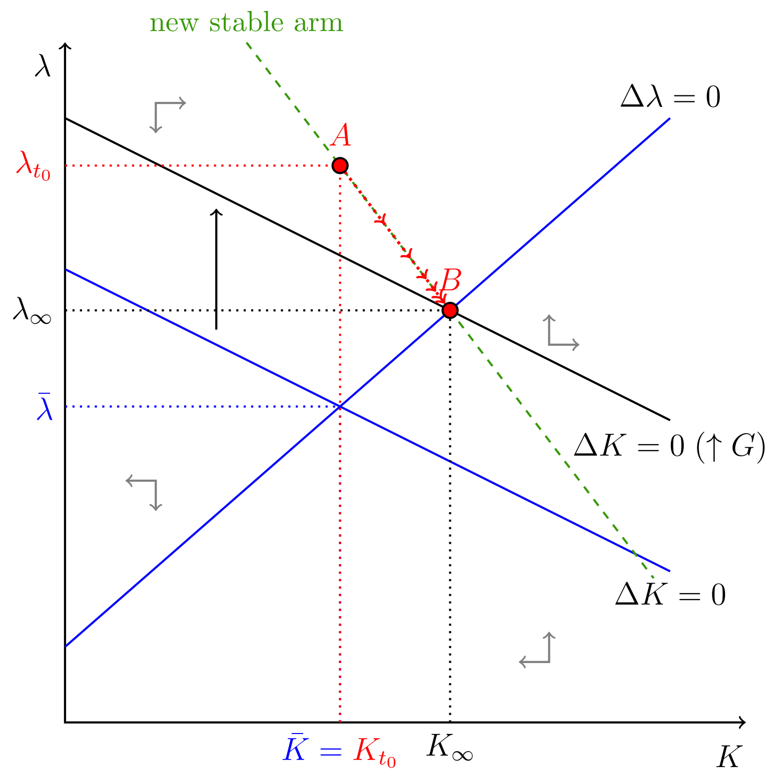


Figure 2.3: Response to permanent government-spending shocks.

variable in the diagram that can react is λ. To put economy on the stable arm (so that the economy converges to the new steady state), λ has to jump up to the new stable arm. We record this movement with $\uparrow$in the first row of the first column of Table 2.1.

From the first order condition for consumption, we know that consumption and λ are inversely related. Therefore, consumption has to fall on impact ($\downarrow$).

The impact effect of a permanent government-spending shock in the labor market is described in Figure 2.4. On impact, K is fixed. There is no change in technology and as a result there is no change in the marginal product of labor: L^d does not move. We also know from the first order condition for labor supply that for a given wage an increase in λ makes the household supply more labor as an additional unit of work is more valuable. Hence, labor supply L^s shifts down. As a result, the labor market clears with $W \searrow$ and $L \nearrow$.

What happens with output at $t = t_0$? Using the production function, we find that output has to increase because technology and capital are fixed by labor increases unambiguously.

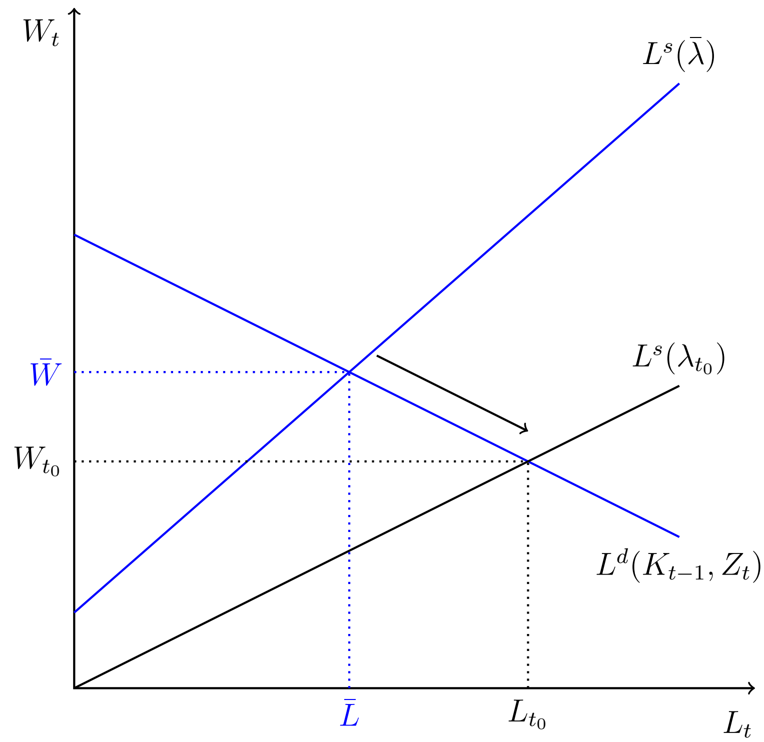


Figure 2.4: Response to permanent government-spending shocks in the labor market in the short run.

So now we have that consumption falls while output and government spending increase. Unfortunately, this combination of signs means that we can't use the goods market equilibrium ($Y = C + G + I$) to sign investment because we do not know if the change in output is large or small relative to $C + G$. To answer what happens with investment, we have to look elsewhere: the phase diagram. Specifically, we know that while capital is predetermined, investment is not. We also know that the change in investment has to be the largest at the time of the shock since as we move along the stable arm increments in capital are increasingly smaller and therefore investment changes by smaller amounts as we approach the steady state. Therefore, investment has to jump up on impact ($t = t_0$) and then gradually decrease.

Finally, we can sign the reaction of the interest rate using the factor price possibility frontier. With no change in technology, any changes in wages and interest have to occur along the FPPF. Since we know wages fall, it means that the interest rate has to rise. This

	Impact $t = t_0$	Transition $t \in (t_0, +\infty)$	Steady State $t = +\infty$
λ	$\uparrow$	$\downarrow$	$\uparrow$
K	0	$\uparrow$	$\uparrow$
C	$\downarrow$	$\uparrow$	$\downarrow$
L	$\uparrow$	?	$\uparrow$
Y	$\uparrow$	?	$\uparrow$
I	$\uparrow$	$\downarrow$	$\uparrow$
W	$\downarrow$	$\uparrow$	0
R	$\uparrow$	$\downarrow$	0

Table 2.1: RESPONSE TO A PERMANENT GOVERNMENT-SPENDING SHOCK

completes our analysis of how variables react to the shock on impact.

In the next step, we study how variables behave in transition to the new steady state. In this analysis, we again sign changes in variables but these changes are relative to the position of variables right after the shock; that is, relative to values variable took at $t = t_0$. With this "base" for comparison, note that a "decrease" does not mean that a variable is below the value of the variable in the initial steady state if this variable jumps up at $t = t_0$. We record the movements of the variation in the transition stage in the second column of 2.1.

Using the phase diagram, we can easily find that λ falls while capital increases. Given the first order condition for consumption, a fall in λ means a rise in consumption. We have already discussed that investment falls on the transition path. In the labor market, a decrease in λ shift the labor supply curve up while an increase in capital shifts the labor demand up. As a result, wages increase unambiguously. From the FPPF, it follows that the interest rate has to fall (factor prices change along the curve). What happens to output and labor is ambiguous. From the labor market, we know that both labor demand and supply shift up and the movement of labor depends on the relative strength of these shifts. Because the dynamics of labor is ambiguous, we can't use the production function to sign output. Likewise, we cannot use the goods market equilibrium to sign output because consumption increases while investment falls. Thus, we put "?" in the Y and L rows of the second column.

In the final step, we determine how variables behave in the new steady state relative to the initial steady state. From the phase diagram we know that capital is higher, lambda is higher and consequently consumption is lower. From the capital accumulation constraint, we know that in the steady state $\bar{I} = \delta\bar{K}$. Since capital is higher in the new steady state, investment is higher too. The factor price possibility frontier does not move and we know from our previous discussion that the steady-state interest rate is pinned down by the discount factor β. Therefore, nothing happens to the interest rate and wages. However, since capital is higher in the new steady state, we can have the same wages and interest in the new steady state only if labor increased to keep the capital-to-labor ratio constant. Thus, labor is higher in the new steady state. From the production function, it follows that output is higher in the new steady state.

Why is the household behave this way? Remember the government is effectively throwing away resources in this economy and taxing the household for the remainder of time. So the household is worse off as it cannot maintain its original consumption and leisure choices. There are three ways for it to respond and still satisfy the budget constraint (or resource constraint): reduce consumption, increase labor supply, and/or reduce investment (draw on savings). All these margins are costly. In our case, the household decides to use the first two margins. How much of each will depend on parameters: If increasing labor supply is not very costly — the Frisch elasticity is high — then the household will primarily use that margin. Conversely, if raising labor supply is very costly, then the household will lower consumption more.

We also see from the phase diagram that the economy in fact accumulates capital. Why is the household not operating on the third margin? The reason is that the long-run increase in labor supply makes capital much more valuable. For a given level of capital, higher labor usage raises the marginal product of capital and thus the return on saving and investing. Consider what would happen if the household decided to maintain capital at a constant level instead (the argument is more extreme for eating into it): This would raise the real interest rate in the economy permanently, so $\beta(1+\tilde{R}) > 1$, where the $\tilde{X}$ denotes the variable X on this alternative path. Then from the Euler equation, $\tilde{C}_t = [\beta(1+\tilde{R})]^{-1}\tilde{C}_{t+1} = [\beta(1+\tilde{R})]^{-2}\tilde{C}_{t+2} =$

$\lim_{T\to\infty}[\beta(1+\tilde{R})]^{-T}\tilde{C}_{t+T}$. If long-run consumption is finite, then todays consumption is zero since $\lim_{T\to\infty}[\beta(1+\tilde{R})]^{-T} = 0$. The incentives to save are just too strong in this case. This contradicts our premise that todays investment is not increasing: if consumption is zero then income has to be saved for investment. Consider next the case where long-run consumption is infinite. With a constant, finite capital stock (remember we did not invest in this thought experiment), infinite consumption is infeasible unless labor supply is infinite. But that is also not optimal because disutility of labor rises at a faster rate than the utility of consumption. In short, the rise in labor supply raises the marginal product of capital, which provides incentives for this household to accumulate more capital.

Note that incentive to save is particularly strong on impact: At t_0 the marginal product of capital, $MPK_{t+1} = \alpha Z_{t+1}K_t^{\alpha-1}L_{t+1}^{1-\alpha}$, is very high because labor supply has increased a lot and capital is fixed at that instant. From the Euler equation, $\check{C}_{t+1} - \check{C}_t = \frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}M\check{P}K_{t+1}$, we know that the higher the interest rate, the greater the increments to consumption. Thus consumption and marginal utility will converge towards the steady-state at a pace commensurate to the real interest rate. As the real interest rate declines this pace slows. From the phase diagram it then follows that the investment rate is also slowing as capital and consumption gradually approach the steady-state. Intuitively, investment is most valuable when the marginal product of capital is highest, which occurs at t_0, so the investment will be highest then. As capital accumulates, the real interest rate declines and so do the incentives to save and invest. Thus, the consumer reduces her investments during (t_1, ∞) relative to t_0.

These qualitative predictions about the dynamics of the variables lead us to an important conclusion. In this basic RBC model, consumption and output move in opposite direction in response to permanent government spending shocks. This pattern contradicts a key empirical regularity that consumption and output comove over the business cycle. As a result, one may conclude that shocks similar to government spending could not be a key driving force of business cycle (if there were, we should have observed a negative correlation between output and consumption).

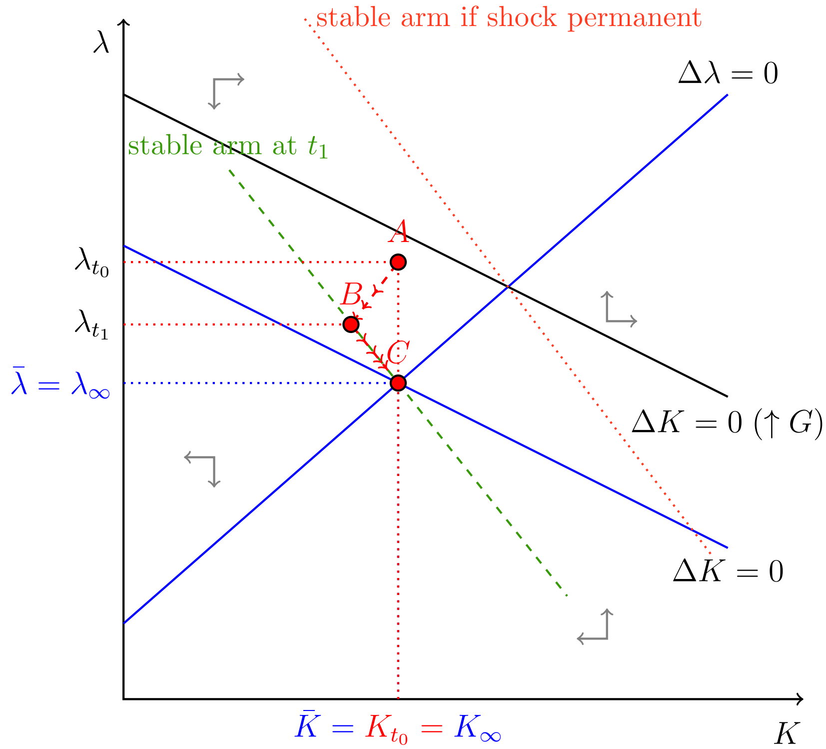


Figure 2.5: Response to transitory government-spending shocks.

2.3 Response to a Transitory Government-Spending Shock

In the previous section we found that permanent government spending shocks appear to lead to results inconsistent with the comovement of macroeconomic variables observed in the data. In this section, we explore if unanticipated, *transitory* shocks to government spending may match data better. We assume that G increases at time t_0 and stays high until t_1 when it returns to the initial level. All agents in the economy know that government spending returns to the initial level at time t_1. So the change in government spending at time t_1 is perfectly anticipated. Similar to the case of permanent government spending shocks, we will summarize the dynamics of the variables in Table 2.2.

Since the shock is temporary, we know that the economy reverts to the initial steady state and therefore there is no change in the position of the loci $\Delta\check{\lambda} = 0$ and $\Delta\check{K} = 0$ in the long run. As a result, we can immediately put "0" in the last column of Table 2.2. However, for $t \in [t_0, t_1)$ the economy is governed by $\Delta \check{K} = 0$ locus shifted up. We know from the previous case that λ jumps up at the time of a permanent, unanticipated increase in government spending to put the economy on the stable arm of the phase diagram. In the case of a transitory shock, λ also jumps up but not as much as in the case of a permanent shock. Intuitively, the economy has to be on the stable arm only at time t_1 when the shock dissipates and between t_0 and t_1 the economy can be on an explosive path. However, this explosive path has to be such that precisely at time t_1 the economy hits the stable arm. This explosive path will be below the stable arm of the economy with a permanent increase in government spending and hence an increase in λ for a temporary shock is smaller. For now we consider the case where λ jumps to a value *below* the new $\Delta K = 0$ locus. As we argue below, this will be the case when the shock to government spending is relatively transitory. (Remember we showed earlier that transitory shocks will affect marginal utility relatively less, because it is forward looking.) Note that similar to the case of a permanent shock, λ is falling for any $t \in (t_0, +\infty)$ and so the qualitative dynamics for λ is similar for permanent and transitory shocks.

In contrast, the dynamics of capital is different. In the case of a permanent shock to government spending, the economy accumulates more capital to support increased spending while in the case of a transitory shock, capital falls for $t \in (t_0, t_1)$ and rises back to the initial level of capital for $t \in (t_1, +\infty)$. Thus, the economy "eats" capital (this is the only form of saving in the economy) during the time of increased government spending. Note that at time t_1 there is no jump in the stock of capital because it is a predetermined variable. We can use the dynamics of capital to infer the dynamics of investment since investment is effectively a time derivative of capital. On impact ($t = t_0$) investment jumps down since capital starts to decrease. What happens on the explosive path may depend on parameter values (how persistent the shock is), but in this case investment will keep declining: over (t_0, t_1) consumption rises while output falls and G_t is constant. Then the only way the goods market equilibrium ($Y_t = C_t + I_t + G_t$) can hold is if investment declines further over (t_0, t_1).

That is, over (t_0, t_1), the economy eats increasingly more of its capital stock.

At time $t = t_1$, there is a change in the direction of change for capital: it starts to rise. Hence, investment has to jump up ($\uparrow$in column 3). When the economy is on the stable arm, accumulation of capital slows down as the economy approaches the steady state and hence investment should fall relative to it's position at time t_1.

Now we can turn to labor markets. On impact, labor supply curve shift to the right while labor demand curve does not move (neither capital nor technology change). Therefore, employment increases while wages fall. Since employment increases, we can use the production function to predict that output increases at t_0.

Between t_0 and t_1, λ is falling and hence the labor supply curve is shifting up. At the same time, capital is generally falling over the same period and hence the labor demand curve shifts down. This combination of shifts means that labor unambiguously decreases relative to its level at $t = t_0$ but we cannot sign the effect on wages from the labor market alone. Let us check if using FPPF can help us to sign the change in wages. First, both capital and labor decrease and thus we do not know what happens with capital-to-labor ratio. So this avenue to use FPPF is not fruitful. Second, we know from the Euler equation that the change in consumption is proportional to the interest rate. We know from the phase diagram that consumption is rising monotonically between t_0 and t_1. Hence, the interest rate has to be above steady state. But to sign the change in the interest rate we need to know what happens with the change of changes (i.e, acceleration, second time derivative) in consumption. Because the economy is on an explosive path, the acceleration depends potentially on many factors and we cannot sign it. Thus, using FPPF can't help and we have ambiguous predictions about the direction of change for the interest rate and wages (we put "?" for both of these variables in the second column).

At $t = t_1$, λ continues to fall which pushes labor supply further but given no discontinuity in λ we will record movements in λ and consumption as "fixed" to highlight that none of these variables jumps at this time. Capital is fixed (this is the turning point for capital) and hence labor demand curve is fixed. Since both the supply and demand labor curves are fixed, there is not change in wages, employment, output (from the production function), and interest rate (from the FPPF). Note that at $t = t_1$, the government spending shock dissipates and thus from the goods market equilibrium one may expect a decrease in output too. However, a fall in government spending is "compensated" by a jump up in investment so that we do not observe a change in output.

When the economy reaches the stable arm and stays on it ($t \in (t_1, +\infty)$), consumption keeps rising as λ is falling. In the labor market, labor supply curve keeps shifting up (λ is falling) and labor demand curve starts shifting up as the economy starts to accumulate capital. Consequently, wages rise unambiguously in this period but we cannot sign the change in employment over this period because the outcome depends on the relative strength of the shifts. From the FPPF, a rise in wages entails a decrease in the interest rate. An ambiguous dynamics of labor means that we cannot use the production function to sign the change in output. We can try to use the goods market equilibrium to sign output. After t_1, consumption is rising but investment is falling and so this does not help to sign output either. Output's dynamics is ambiguous over $t \in (t_1, +\infty)$.

We note that on some dimensions the model behaves very similarly following a permanent shock. In particular, consumption falls, labor supply rises, and output rises. Again the consumer responds to being poorer by reducing consumption and raising labor supply. But the behavior of investment and capital are very different: while investment and capital rose following a permanent shock, they declined following a temporary shock. Why is this the case? We noted earlier that a permanent shock raises the value of capital, because it caused a permanent rise in labor supply. But with a temporary shock, there is no longer a long-run benefit from raising the capital stock, because in the long-run labor supply returns to normal. Thus, the returns to investment rise less when the shock is temporary than when it is permanent all else equal. We therefore know that investment must rise less than in the permanent case.

But in our example investment not only rose less; in fact, it declined. Note again from the Euler equation that today's consumption response is given by,

$$\check{C}_t = -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}M\check{P}K_{t+1} + \check{C}_{t+1} = -\frac{\alpha\frac{\bar{Y}}{\bar{K}}}{\alpha\frac{\bar{Y}}{\bar{K}}+1-\delta}\sum_{s=0}^{\infty}M\check{P}K_{t+1+s} + \underbrace{\lim_{s\to\infty}\check{C}_{t+s}}_{=0 \text{ for temporary shock}}.$$

If the marginal product of capital (the interest rate) does not rise very much, then consumption will fall relatively little. From the resource constraint, $Y_t = C_t + I_t + G_t$, the household then uses investment to smooth out the government spending shock. Intuitively, for a transitory shock that goes away quickly, it is not optimal for the household to give up consumption and accumulate long-lasting capital. That capital will not be valuable for very long! It can instead achieve a higher consumption path by eating into the capital stock while the shock lasts and then rebuilding it once the shock is over. (You can also check that our earlier proof by contradiction when the shock is permanent no longer works when the shock is temporary.)

This argument also suggests that for a shock that is temporary but very persistent, it may still make sense for the consumer to accumulate some capital (although not as much as with a permanent shock). The more persistent the shock, the longer is labor supply elevated, and the longer is the marginal product high. In fact that is exactly what will happen when the shock is so persistent that λ_{t_0} is located above the new $\Delta K = 0$ locus. This is what happens in figure 2.6. The main changes will be to the behavior of investment on impact and the transition for investment and capital. Note that as the shock becomes increasingly more persistent the behavior of the economy will approach that of the permanent shock. Note though, that the consumer would never optimally chose to be above the stable arm for the permanent shock! (Recall that temporary shocks have smaller effects on marginal utility than temporary shocks.)

However, even in this case where government spending shocks are very persistent, we still observe opposing movements in consumption and output. So our conclusion regarding business cycle facts are unchanged.

To see why investment must switch sign on an optimal path, note that this consumer

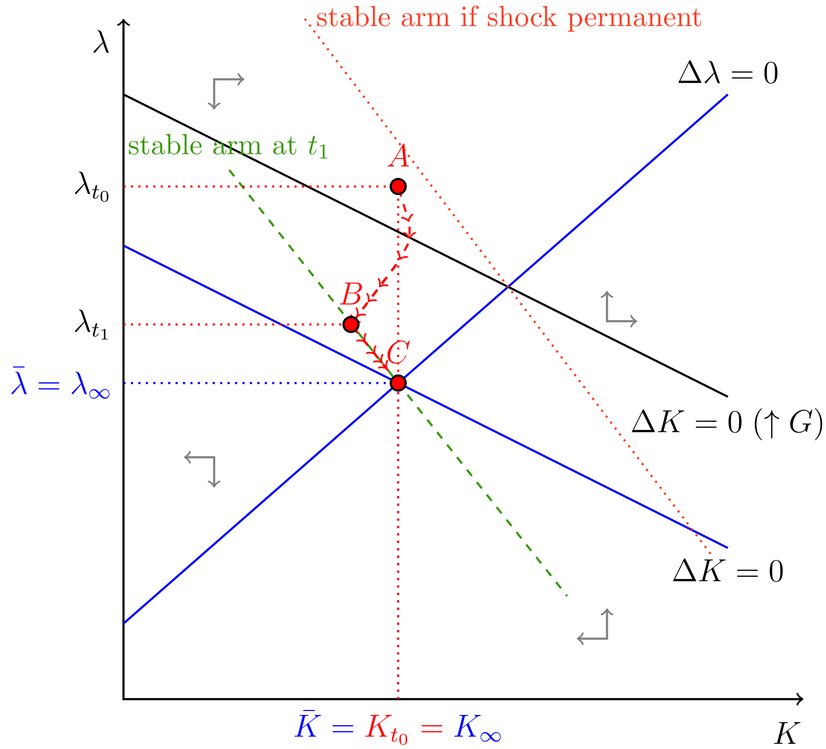


Figure 2.6: Response to a transitory but very persistent government-spending shocks.

desires smooth consumption and smooth labor supply. If those variables follow "continuous" paths (as they would at the optimum), then predetermined capital also implies that output, $Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}$, is continuous. From the resource constraint in the economy, $Y_t = C_t + I_t + G_t$, the only variable that can then exhibit a discontinuity at t_1, is investment. It exactly compensates for the decline in G_{t_1} at that instant.

In summary, using temporary spending shocks as a source of business cycles is also problematic. Consumption and now investment move in directions opposite to where employment and output move which is inconsistent with the data. As a result, temporary government spending shocks cannot be the key source of business cycles in this class of models. This analysis also highlights the importance of capital for smoothing out and propagating shocks. Households use capital (their only form of saving) to smooth a temporary shock.

	Impact $t = t_0$	Transition I $t \in (t_0, t_1)$	Inflection $t = t_1$	Transition II $t \in (t_1, +\infty)$	Steady State $t = +\infty$
λ	$\uparrow$	$\downarrow$	0	$\downarrow$	0
K	0	$\downarrow$	0	$\uparrow$	0
C	$\downarrow$	$\uparrow$	0	$\uparrow$	0
L	$\uparrow$	$\downarrow$	0	?	0
Y	$\uparrow$	$\downarrow$	0	?	0
I	$\downarrow$	$\downarrow$	$\uparrow$	$\downarrow$	0
W	$\downarrow$	?	0	$\uparrow$	0
R	$\uparrow$	?	0	$\downarrow$	0

Table 2.2: RESPONSE TO A TRANSITORY GOVERNMENT-SPENDING SHOCK

2.4 Response to a Permanent Technology Shock

Now we turn to another source of fluctuations in the model: technology shocks. To understand the reaction of the variables to these shocks, we will again use the phase diagram, labor market equilibrium, and FPPF. We start with a permanent, unanticipated increase in technology at time t_0. Since two loci shift in response to the shock, there are a few possibilities for where the new steady state is located. We focus here on the case in which $\bar{\lambda}$ **is lower** and $\bar{K}$ **is higher** after the shock, and in which the new stable arm in the phase diagram is **below the original steady state**. We draw the phase diagram in figure 2.7 collect the responses of key macroeconomic variables in Table 2.3.

Given our assumption about the position of the new steady state relative to the initial steady state, we know that, on impact ($t = t_0$), λ has to jump down to put the economy on the stable arm. There is no change in capital at the time of the shock because capital is predetermined but investment jumps up as the economy starts to accumulate more capital. A decrease in λ means an increase in consumption (first order condition for consumption). Since capital is higher in the new steady state, investment has to jump up on impact.

In the labor market, labor demand shifts up (technology raises MPL) and labor supply shifts to the left (workers fill richer as λ goes down), see Figure 2.8. As a result, wages increase unambiguously but we can't sign the change in employment. Because of this ambiguity for

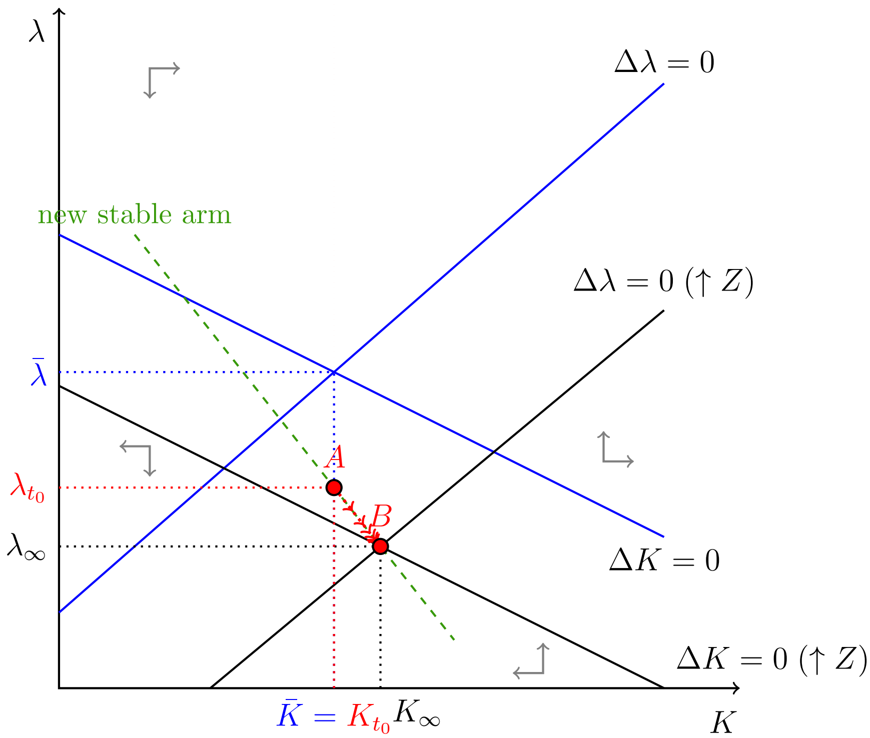


Figure 2.7: Response to permanent technology shock.

employment, we can't use the production function to sign the change in output. However, we can use the goods market equilibrium to do so. We know that both consumption and investment increase at $t = t_0$ and therefore output increases on impact. We have not signed the change of the interest rate yet but we will do it later when we understand the dynamics of other variables.

On transition path, λ falls, K increases, and consumption rises (recall that we do signs of changes relative to the position of the variables at t_0). Because capital increases at decreasing rate, we know that investment has to fall relative to its level at $t = t_0$. Falling λ means that labor supply curve keeps shifting up. Increasing capital stock means that labor demand keeps shifting up to. Hence wages rise but again we can't sign changes in employment. Because investment is falling and consumption is rising we cannot use the goods market equilibrium to sign the change in output. Likewise, using the production function is not helpful because labor dynamics is ambiguous. Hence, in general we can't sign the dynamics of output during this period but in plausible parameterizations output increases while the economy is in transition.

In the steady state, capital is higher and λ is lower (by assumptions we made earlier) than in the initial steady state. Hence, consumption is higher and investment is higher (recall that $\bar{I} = \delta\bar{K}$). Wages are higher because labor demand and supply are higher. Since both consumption and investment are higher, we know that output is higher in the new steady state too.

Now we turn to the interest rate. An increase in wages does not mean that the interest rate has to fall because the FPPF shifts up so that both higher wages and interest rate can be sustained. However, we can use "backward induction" to sign the change in the interest rate at the time of the shock. We know that the steady-state interest rate is fixed by the discount factor and hence even if the FPPF shifts we have the same interest rate in steady state. The economy converges to the new steady state monotonically after the shock and jumps at t_0. We also know that wages jump up at t_0 and continue to rise afterwards. This means that the interest rate must approach its steady state level from from above. Therefore, the interest rate jumps up on impact and then falls while in transition to the new steady state.

In contrast to spending shocks, technology shocks generate comovement of macroeconomic variables similar to what we see in the data. Thus, at least in the context of this basic model, we can make this model consistent with the data only if technology shocks is the primary force of fluctuations.

2.5 Response to a Transitory Technology Shock

In our analysis of temporary vs. permanent shocks to government spending, we saw important differences in the dynamics of variables. In this section, we explore how temporary shocks for technology affect variables in our basic model. Similar to the case of temporary government spending shocks, we assume that there is an unanticipated, positive shock to

	Impact $t = t_0$	Transition $t \in (t_0, +\infty)$	Steady State $t = +\infty$
λ	$\downarrow$	$\downarrow$	$\downarrow$
K	0	$\uparrow$	$\uparrow$
C	$\uparrow$	$\uparrow$	$\uparrow$
L	?	?	?
Y	$\uparrow$	$\uparrow$(generally)	$\uparrow$
I	$\uparrow$	$\downarrow$	$\uparrow$
W	$\uparrow$	$\uparrow$	$\uparrow$
R	$\uparrow$	$\downarrow$	0

Table 2.3: RESPONSE TO A PERMANENT TECHNOLOGY SHOCK

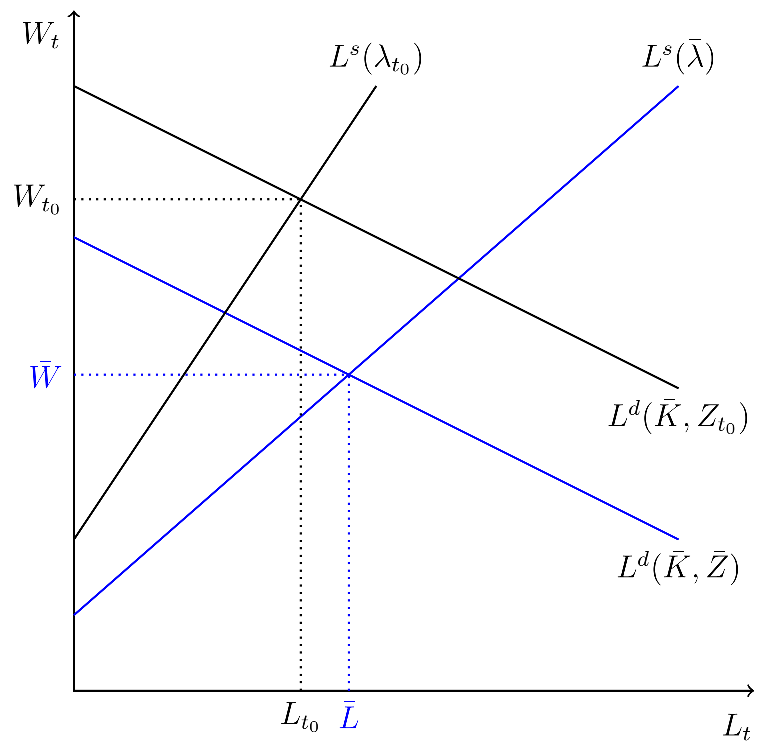


Figure 2.8: Response to permanent technology shocks in the labor market in the short run.

technology at time t_0 which lasts until t_1 and then technology reverts to its initial level. The reversal to the initial level at t_1 is anticipated by all agents in the economy.

To make this analysis comparable to the case of a permanent technology shock we have considered above, we assume that the size of the shock and shifts in the loci are such that

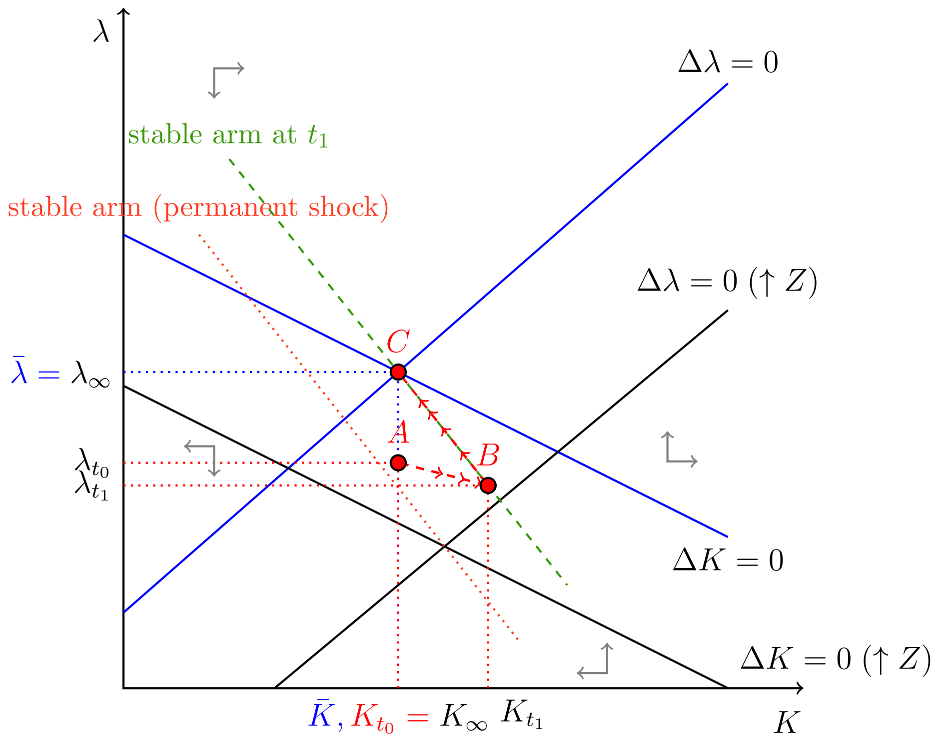


Figure 2.9: Response to transitory technology shock.

if the shock stayed permanent it would have resulted in the new steady state with a lower λ and a higher K as in figure 2.9. So for $t \in [t_0, t_1)$ the economy is governed by the same phase diagram we had for the permanent technology shock, and for $t \in [t_1, +)$ the economy is governed by the initial phase diagram. The effects of a transitory technology shock are summarized in Table 2.4. Because the shock is temporary, there is no change in the steady state and hence we put zeros in the last column of the table.

On impact ($t = t_0$), we know (given our assumptions) that λ falls. But it does not fall as much as in the case of a permanent shock because the economy has to be on the stable arm of the initial phase diagram at time t_1 and therefore the economy has to be on an explosive path between t_0 and t_1 governed by the new phase diagram. Since λ falls, consumption rises. Capital is a predetermined variables and so it does not move at the time of the shock. However, investment jumps up as the economy starts to accumulate capital. Since investment and consumption increase, we know that output increases on impact.

At the time of the shock, the labor supply curve shifts up because households feel richer (λ falls). At the same time, labor demand curve shifts up too because technology improves. Hence, wages rise unambiguously but we can't sign the change in employment. Even though wages unambiguously rise, we cannot use the FPPF because it shifts out and so higher wages *and* interest rate can be sustained. We will sign the change in the interest rate later when we understand the full dynamics of other variables.

On transition path $t \in (t_0, t_1)$, λ will (generally) fall and hence consumption will continue to rise (remember these statements are relative to the values variables take at time $t = t_0$, that is, at the time of the shock). While capital stock increases during this period, the economy is on an explosive path and hence we do not know if investment falls or rises relative to its value at $t = t_0$ (that is, accumulation of capital can accelerate or decelerate). Falling λ means that the labor supply curve will continue to shift up and continued accumulation of capital means that the labor demand will continue to shift up. Therefore, we observe rising wages but again we do not know what happens with employment as the reaction depends on the relative strength of shifts in the supply and demand curves. Since labor dynamics is ambiguous and we can't sign changes in investment, using either production function or goods market equilibrium can't help us to sign changes in output.

At $t = t_1$, technology reverts to its original level. At this time, there is not jump in λ (and hence consumption) or capital (but they both have infliction points at this time). Other variables, however, jump. Labor demand curve shift down because technology falls while there is no movement in the labor supply curve (λ is fixed). Hence, both wages and employment fall at $t = t_1$. Employment and technology fall, hence production function suggests that output has to decrease at this time too. The economy starts to "spend" capital and therefore investment has to go down.

In the second phase of the transition ($t \in (t_1, +\infty)$), λ is rising back to the initial steady state level and hence consumption falls. Capital is falling towards its initial level. Since economy is on the stable arm, we know that accumulation of capital decelerates and therefore investment has to be rising (i.e., it is negative but it becomes less negative). Decreasing capital shock and rising λ shift labor demand and supply curves down and therefore wages decrease unambiguously and, again, we can't say anything about what happens to employment. Since investment is rising and consumption is falling, we can't sign output from the goods market equilibrium. We also cannot use the production function because the dynamics of employment is ambiguous.

Now we will try to sign the dynamics of the interest rate. At time t_0 we know that consumption will be growing in the future. We can use therefore use Euler equation, $C_{t+1}/C_t = \beta(1 + R_{t+1})$, to infer that interest rates must have risen at t_0. At the time of the shock the FPPF shifts out and for $t \in [t_0, t_1)$ it is governed by this new FPPF. During this period, wages are rising and hence the interest rate has to be falling relative to where it was at $t = t_0$ (keep in mind that given the shift, it does not mean it's lower relative to the initial steady-state level of the interest rate). At time $t = t_1$, wages fall but it does not necessarily mean that interest rate falls. To sign the change at t_1, we again use the Euler equation: $C_{t+1}/C_t = \beta(1 + R_{t+1})$. There is no discontinuity in consumption at t_1, but we know that consumption changes from rising to falling and therefore C_{t+1}/C_t changes from being positive (generally) to being negative. This means that interest rate has to go down at t_1. For $t \in (t_1, +\infty)$, wages fall and thus as the economy moves along the FPPF, interest rate has to rise. Since the economy converges monotonically the the new steady state, we know that the interest rate approaches the steady state level from below. Thus, we can sign all dynamics of the interest rate.

In summary, the dynamics of the economy in response to a temporary shock is similar to the reaction in response to a permanent shock. These dynamics also highlights the importance of capital in smoothing consumption over time as the representative household uses capital to "save" temporarily high productivity and therefore spread it over longer period of time.

	Impact $t = t_0$	Transition I $t \in (t_0, t_1)$	Inflection $t = t_1$	Transition II $t \in (t_1, +\infty)$	Steady State $t = +\infty$
λ	$\downarrow$	$\downarrow$	0	$\uparrow$	0
K	0	$\uparrow$	0	$\downarrow$	0
C	$\uparrow$	$\uparrow$	0	$\downarrow$	0
L	?	?	$\downarrow$	?	0
Y	$\uparrow$	?	$\downarrow$	?	0
I	$\uparrow$	?	$\downarrow$	$\uparrow$	0
W	$\uparrow$	$\uparrow$	$\downarrow$	$\downarrow$	0
R	$\uparrow$	$\downarrow$	$\downarrow$	$\uparrow$	0

Table 2.4: RESPONSE TO A TRANSITORY TECHNOLOGY SHOCK

	$\rho(Y, L)$	$\rho(Y, C)$	$\rho(Y, I)$	$\rho(Y, W)$
Z shocks	?	+	+	+
G shocks	+	-	+ (permanent shock) - (transitory shock)	-
data	+	+	+	+, ≈ 0

Table 2.5: Actual and Predicted Correlations

2.6 Recap

These qualitative predictions gives us a sense about what shocks should be an important source of business cycles. Table 2.5 summarized qualitative predictions about correlations of variables in the data and in the model conditional on types of shocks. Clearly, government spending shocks have trouble reproducing some key correlations observed in the data. For example, $\rho(C, Y)$ is negative conditional on this shock, but this correlation is positive in the data. Technology shocks appear to have a better match but these also have issues. For example, $\rho(L, Y)$ is potentially ambiguous and its sign depends on parameters of the model (e.g., the Frisch elasticity of labor supply is very important here). In addition, technology shocks lead to highly procyclical wages which is inconsistent with only weak cyclicality of wages in the data.

Chapter 3

Quantitative Validity of the Real Business Cycle Model

We are now ready to assess the empirical success of the theoretical model we developed so far. We will evaluate our models along many dimensions. Specifically, we will look at the following properties of the data and check whether the model can reproduce them:

- volatility
- correlations
- amplification of shocks
- propagation of shocks
  - internal dynamics
  - built-in persistence (serial correlations of exogenous forcing variables)

Ideally, we would like to have a model which does not rely heavily on the persistence and volatility of exogenous forcing variables. Instead we want to have a model which can generate ***endogenously*** persistence and volatility.

Mechanisms that increase endogenous volatility are often called **amplification mechanisms**. To get a sense of endogenous amplification, consider a simple model with inelastic labor supply where output is fully reinvested in capital and every period capital is fully depreciated (i.e., $K_t = Y_t$; in the Solow growth model this would correspond to the saving rate equal to one). The linearized production function is $\check{Y}_t = \check{Z}_t + \alpha\check{K}_{t-1}$. Note that an immediate effect of Δ change in $\check{Z}$ on output is Δ since given capital stock fixed in the short run we have $\check{Y}_t = \check{Z}_t$. Hence, a one percent increase in Z_t increases output Y_t by one percent. This change in output is determined exogenously.

Over time more capital is accumulated. In the long run, $\check{Y}_{LR} = \Delta + \alpha\check{Y}_{LR}$ and hence the long run response to a change in technology is $\check{Y}_{LR} = \frac{1}{1-\alpha}\Delta$. In other words, 1% increase in technology $\check{Z}$ raises output by $\frac{1}{1-\alpha}$%. Here, capital accumulation is the amplification mechanism which can generate an extra kick to output beyond the immediate effect. We can say that $\frac{1}{1-\alpha} - 1 > 0$ is the **endogenous amplification**.

Mechanisms that increase endogenous persistence are called **propagation** mechanisms. In our simple model with full depreciation and complete reinvestment, capital accumulation also works as a propagation mechanisms because the system reaches equilibrium gradually and adjustment is not instantaneous. If the model relies too much on properties of exogenous variables, then this model is not very satisfactory since we end up explaining the data with postulated exogenous variables.

In summary, our desiderata in modeling business cycles are strong amplification and propagation in the model.

3.1 Canonical Real Business Cycle Model

Cooley and Prescott (1995) develop and calibrate a standard ("bare-bones") model in business cycle literature. They focus only on technology shocks as the driving force of the business cycles. This class of models is often called "***real business cycle***" (RBC) models to emphasize that shocks in the model are real (e.g., technology shock) and not nominal (e.g., monetary shock). The model in Cooley and Prescott (1995) is similar to what we developed and analyzed before and their calibration is similar to ours. Hence, we will take results of their model simulations to assess the fit of the model to the data.

The numerical results from their analysis are presented in Table 3.1. The moments of the data (variances and correlations) are computed after removing the trend component in the macroeconomic series (Cooley and Prescott use the Hodrick-Prescott filter which is a way to remove the trending component in a flexible way). On the other hand, we simulate the calibrated model, HP-filter the simulated series, compute the moments of the filtered series, and then compare the moments of these series with the moments in the data. Broadly, the match is not perfect, but it is not terrible either. Some prominent facts about the business cycle are reproduced by the model. For instance: $\sigma_I > \sigma_Y > \sigma_C$. This finding could be somewhat surprising given how simple our model is and this is why we do not use any formal metric such as an over-identifying restrictions test to falsify the model. The objective here is to check whether the model can "roughly" reproduce salient features of business cycles.

Summary of findings:

1. overall the match is not terrible;
2. level of volatility is ok;
3. dynamic properties of the model are off;
4. a lot of variability is driven by "exogenous" shocks (forcing variable);
5. too much correlation with output and productivity.

While the model has a number of limitations, it also has a number of desirable properties. First, the model studies business cycles in general equilibrium and hence we take into account equilibrium responses of variables to technology shocks. Second, the framework is a Walrasian economy with no frictions and all markets clear. This is important because

Variable	Data st.dev.	Model st.dev.
Output	1.72	1.35
Consumption	Total: 1.27	0.329
	Non-durable: 0.86	
Investment	Total: 8.24	5.95
	Fixed: 5.34	
Hours	1.59	0.769
Employment	1.14	
Serial corr. in output, $\rho(Y_t, Y_{t-1})$	0.85	0.698
$\rho(Y_t, C_t)$	0.83	0.84
$\rho(Y_t, I_t)$	0.91	0.992
$\rho(Y_t, L_t)$	Hours: 0.86	0.986
	Employment: 0.85	
$\rho(Y_t, Z_t)$	0.77 (Solow resid.)	0.98
$\rho(Y_t, W_t)$	Hourly rate: 0.68	0.978
	$\frac{Compensation}{hour}$: 0.03	

Table 3.1: NUMERICAL ANALYSIS IN COOLEY AND PRESCOTT (1995)

we don't need to rely on ad hoc mechanisms of how and why the markets are not clearing. Third, the model is very parsimonious. We have only a handful of parameters. We can use over-identification tests (the model is hugely over-identified) to check the validity of the model. Fourth, we can use the same model to explain growth and fluctuations around growth trend. Fifth, we try to explain macro facts based on solid/rigorous micro-foundations where all economic agents are optimizing and therefore we can study changes in policies without being subject to the Lucas critique.

In summary, with one model we can address many things and at the same time we put enormous discipline on the model. An important point about RBC models is that fluctuations are fully optimal and, hence, there is no need for policy in these models. In other words, recessions are not terribly bad in the sense that they are not reflecting disruptions to how markets operate. If taken literally, it means that enormous unemployment during the Great Depression was a great vacation since the unemployed ***optimally*** chose not to work and enjoy leisure!

3.2 Critique of the RBC Model

1. When we look at the simulated data we see that the ***quantitative*** match is far from perfect:

   ◦ consumption is too smooth

   ◦ wages are too procyclical

   To be fair, there is only one shock in the model. By introducing more shocks, we can improve the fit. If we introduce shocks to government expenditures G, we can reduce procyclicality of wages. The intuition is that:

   ◦ $Z \nearrow \Rightarrow Y \nearrow$, $W \nearrow\ \Rightarrow$ wages are very procyclical.

   ◦ $G \nearrow \Rightarrow Y \nearrow$, $W \searrow\ \Rightarrow$ wages are countercyclical.

   Thus if you have the right mix of shocks, you can improve the fit (see e.g. Christiano and Eichenbaum (1992)). There are constraints to this kind of approach. It is true that one can keep adding shocks but it does not mean that you necessary improve the fit. For instance:

   ◦ $Z \nearrow \Rightarrow Y \nearrow$, $C \nearrow$ which fits the data.

   ◦ $G \nearrow \Rightarrow Y \nearrow$, $C \searrow\ \Rightarrow C$ is countercyclical which we do not observe in the data.

   Thus if we want to keep emphasizing shocks to G, we can get deteriorated results on consumption and improved results on wages.

2. Technology shocks can be controversial:

   ◦ What is a "decline" in technology? How is it possible to have a technological regress? We measure technology using the Solow residual:

$$\check{Z}_t = \check{Y}_t - (1-\alpha)\check{L}_t - \alpha\check{K}_{t-1}$$

α is estimated with $1 - \alpha = \frac{wL}{Y}$. It looks like the TFP fell a lot during the Great Depression in the US. But technological regress is unlikely to be the source of the Depression. At least, we are not aware of any immediately apparent technological problem leading to the Great Depression.[1] More generally, we cannot identify large movements in technology (as we see in TFP) with particular events and discoveries.

- Another problem with using the Solow residual in RBC models as a structural shock is that Z_t does not seem "exogenous". Intuitively, we want to have an exogenous shock that is not correlated with other shocks if we want to identify the effects of this shock. Otherwise, when we look at the response of the system to a shock, we don't know if we see the effect of a particular force "technology" or if we see an effect of technology being correlated with something else. Hence, we often require that $\rho(Z, G) = 0$. Indeed, there is no particular reason to believe that shocks to government expenditure cause higher levels of technology (at least in the short run). However, when we look at the Solow residual in the data we see that it is correlated with non-technology shocks:

  - Hall (1988): $\rho(Z, \text{ demand shifters}) \neq 0$
  - Evans (1992): $\rho(Z, \text{ monetary policy shocks}) \neq 0$

  This is a problem because if technology shocks are endogenous, we can have a good fit of the model but there is no causal effect of technology shocks on fluctuations. We could not say that technology drives business cycles.

- RBC models typically have only real shocks:

  - technology
  - government expenditure
  - taxes

[1] Lucas (1994) notes: "Imagine trying to rewrite the Great Contraction chapter of A Monetary History with shocks of this kind playing the role Friedman and Schwartz assign to monetary contractions. What technological or psychological events could have induced such behavior in a large, diversified economy? How could such events have gone unremarked at the time, and remain invisible even to hindsight?"

- preferences

  However, nominal/monetary shocks apparently can be very important (e.g., Friedman and Schwartz (1963)). Omitting these shocks can lead to erroneous conclusions.

3. There is a concern about the plausibility of assumed parameters in the model. For instance, for RBC models to work, one should have a high labor supply elasticity, and also a high intertemporal labor supply elasticity. However, micro-level evidence suggest that these elasticities are pretty small. When we work with aggregate model, we have a combination of two things: i) micro-level elasticities; ii) aggregation. Aggregation can be very important. One striking example is due to Houtthakker (REStud, 1955). He shows that if every firm has Leontieff-type production function and distribution of firms is Pareto then aggregate production function is Cobb-Douglas. Hence the distinction between intensive and extensive margins of adjustment could be very important.

4. RBC models lack strong propagation mechanisms (Cogley and Nason (1995)). Basically to match persistence we see in the data, we have to rely on very persistent shocks. But if we have to stick very persistent shocks into the model, what is the value of the model in explaining persistence of Y, C, I? We get the conclusion by construction. Furthermore, Rotemberg and Woodford (1996) find that the model fails to reproduce predictable movements (and comovements) of variables we observed in the data. In other words, the model has to excessively rely on the volatility of exogenous shocks to match the level of volatility of variables in the data.

3.2.1 Cogley and Nason (1995)

Cogley and Nason (1995) note that in the data output growth rate is serially correlated. That is, if output growth rate is high today, it is likely to be high in next periods. To document this fact, Cogley and Nason use a variety of techniques:

- autocorrelation function (ACF) (in the data, they found that at lags 1 and 2, the autocorrelation is positive and significant).
- spectrum and impulse response (here the idea is that you can approximate series with sine and cosine functions, and since these functions are periodic we can look at the contribution of various cycles to total variation).

Then Cogley and Nason write down a basic RBC model and note that the model has four broad mechanisms of propagating shocks.

1. built-in persistence of shocks: $Z_t = \rho Z_{t-1} + \epsilon_t$ where ρ governs the persistence of Z_t.
2. capital accumulation: when technology is high, I save my output in the form of capital and then enjoy higher output in the future by using my capital in production and having smaller investment for some time.
3. intertemporal substitution: I can have large labor supply in the time of the shock and then spread below-average labor supply over subsequent periods.
4. adjustment costs, gestation lags, etc.: I get a shock today but some of its effect on K, L, or other variables are going to realize a few periods later.

The model analyzed by Cogley and Nason has two shocks:

- technology shock Z_t: random walk (RW)
- government spending G_t: AR(1) process

Then they calibrate and simulate the model (simple model with no adjustment costs). They obtain no serial correlation in the growth rate of output in the model (see Figure 3.1).

Another metric they use is empirical impulse responses. They identify the shocks from the data:

$$\begin{bmatrix} \Delta Y_t \\ L_t \end{bmatrix} = \begin{bmatrix} a_{11}(L) & a_{12}(L) \\ a_{21}(L) & a_{22}(L) \end{bmatrix} \begin{bmatrix} \epsilon_t^S \\ \epsilon_t^D \end{bmatrix} \tag{3.1}$$

Figure 3.1: Autocorrelation function for output growth rate

Notes: This figure is taken from Cogley and Nason (1995), Figure 3. Solid lines show sample moments, and dotted lines show moments that are generated by an RBC model.

$\epsilon_t^S \sim Z_t$ and $\epsilon_t^D \sim G_t$. The identifying assumption is that only Z_t has permanent effect on Y_t.[2]

Then Cogley and Nason suppose that $\Delta Z_t = \rho \Delta Z_{t-1} + \epsilon_t$ (ARIMA(1,1,0) process). Using this process they can match the serial correlation in the growth rate of output but they get counterfactual predictions in other dimensions:

- too much serial correlation in TFP
- fail to match other facts: response of growth of output to a transitory shock in technology. They cannot generate the hump-shaped response observed in the data.

If you add capital-adjustment:

$$Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \frac{\alpha_K}{2}\left(\frac{\Delta K_t}{K_{t-1}}\right)^2$$

or gestation lags (invest now, get capital installed and ready to work in S periods), the response of growth output is not satisfying either: one can get some persistence in growth output, but not a hump-shaped response. Labor-adjustment models also have problems.

The bottom line in Cogley and Nason (1995) is that you need to build a lot of persistence to shocks because the model does not have an ability to predict very persistent responses to serially uncorrelated shocks.

[2] Obviously the question is what happens if you have a long lasting demand shock, e.g. Cold War.

3.2.2 Rotemberg and Woodford (1996)

Rotemberg and Woodford (1996) have a simple but powerful idea: in the data shocks to technology lead to large ***predictable*** movements and ***co***movements in the series. But in the basic RBC model, ***predictable*** movements are small. Furthermore, ***predictable*** comovements can in fact be of the wrong sign.

To identify technology shocks, they also use VAR with three variables growth are of output Δy_t, consumption to output ratio $c_t - y_t$, and hours h_t:

$$\begin{bmatrix} \Delta y_t \\ c_t - y_t \\ h_t \end{bmatrix} = u_t = A u_{t-1} + \epsilon_t \tag{3.2}$$

Note that variables in this system are stationary. We want to know the predictable movements in the variables. To construct these predictable moments, Rotemberg and Woodford use impulse response functions (IRFs), which is a path of a variable in response to a shock. In the empirical model considered by Rotemberg and Woodford, the impulse response for the growth rate of output is:

1. $e_1 I \epsilon_t$

2. $e_1 A \epsilon_t$

3. $e_1 A^2 \epsilon_t$

   $\vdots$

(k) $e_1 A^k \epsilon_t$

where $e_1 = [1\ 0\ 0]$ is the selection vector. Note that this response is for the first difference of output y. The cumulative response of the growth rates of output (which gives us the ***level*** response of output y) is:

$$\begin{aligned}
\Delta\hat{y}_t + \Delta\hat{y}_{t+1} + \ldots + \Delta\hat{y}_{t+k} &= \hat{y}_t - \hat{y}_{t-1} + \hat{y}_{t+1} - \hat{y}_t + \ldots + \hat{y}_{t+k} - \hat{y}_{t+k-1} \\
&= \hat{y}_{t+k} - \hat{y}_{t-1} = \hat{y}_{t+k} - y_{t-1} \\
&= e_1(I + A + A^2 + \ldots + A^k)\epsilon_t
\end{aligned}$$

where hats indicate that we look at ***predictable*** movements. We can also re-write this expression as follows:

$$\begin{aligned}
\Delta\hat{y}_t^k &\equiv \hat{y}_{t+k} - \hat{y}_t \\
&= e_1(A + A^2 + \ldots + A^k)\epsilon_t \\
\Delta\hat{y}_t^k &= B_y^k \epsilon_t
\end{aligned}$$

Here $\Delta\hat{y}_t^k$ provides an estimate (prediction) of how output is going to move between period t and $t+k$. That is, standing in time t, we can expect that on average output is going to change by $\Delta\hat{y}_t^k$ in k periods from t.

We can also consider the response of the model in the very long run by sending k to infinity:

$$\begin{aligned}
\Delta\hat{y}_t^{+\infty} &= \lim_{k\to+\infty} \hat{y}_{t+k} - \hat{y}_t \\
&= e_1(I - A)^{-1}A\epsilon_t
\end{aligned}$$

Using these predictable movements in output, we can assess how much variation in output is predictable at a given horizon k. Specifically, the predictable variation in output at horizon k due to a given shock ϵ_t is given by

$$\mathrm{cov}(\Delta\hat{y}_t^k, \Delta\hat{y}_t^k) = \mathrm{cov}(B_y^k\epsilon_t, B_y^k\epsilon_t) = B_y^k \Omega_\epsilon (B_y^k)^T$$

Likewise we can compute predictable ***co***movements across variables. For example, the predictable covariance between output and consumption is given by:

$$\text{cov}(\Delta\hat{y}_t^k, \Delta\hat{c}_t^k) = \text{cov}(B_y^k\epsilon_t, B_c^k\epsilon_t) = B_y^k\Omega_\epsilon B_c^k$$

We can then compare $\text{cov}(\Delta\hat{y}_t^k, \Delta\hat{y}_t^k)$, which are predictable movements, with $\text{cov}(\Delta y_t^k, \Delta y_t^k)$, which are total movements. Given estimates of A in the data and A implied by the model, we can look at horizon $k = 12$ quarters and get:

- $\text{cov}(\Delta\hat{y}_t^k, \Delta\hat{y}_t^k) = 0.0322$ in the data
- $\text{cov}(\Delta\hat{y}_t^k, \Delta\hat{y}_t^k) = 0.0023$ in the model
- $\text{cov}(\Delta y_t^k, \Delta y_t^k) = 0.0445$ in the data
- $\text{cov}(\Delta y_t^k, \Delta y_t^k) = 0.0212$ in the model

Importantly, predictable movements are a big factor of total output movements in the data. In the model, predictable movements are only a tiny fraction of variation in output.

One may use the following example to better understand this result. Suppose we are interested in variable X_t which follows an AR(1) process $X_t = \rho X_{t-1} + \epsilon_t$ with $\epsilon_t \sim iid(0, \sigma_\epsilon^2)$. The variance of the variable is $\sigma_\epsilon^2/(1-\rho^2)$. Thus a certain level of volatility for the variable can be achieved by increasing σ_ϵ^2 or by increasing ρ. However, these two approaches have sharply different implications for predictability of X_t. In the extreme case of $\rho = 0$, X_t is not predictable at all. In contrast with ρ close to one, predictability is high. The Rotemberg-Woodford result may be interpreted as suggesting that RBC models match volatility via σ_ϵ^2 which contrasts with the data were volatility is high via ρ.

To assess the importance of predictable comovements, we can regress one predictable movement on another. For example, if we regress the change in consumption at horizon k $\Delta\hat{c}_t^k$ on the change in output at horizon k $\Delta\hat{y}_t^k$, we get:

- 0.2-0.5 in the data

- 2.5 in the model

Likewise, regress the change in hours at horizon k $\Delta \hat{h}_t^k$ on $\Delta \hat{y}_t^k$ and get:

- 1 in the data

- -1 in the model

There are two striking differences. First, hours in the model are falling when you expect high output while in the data high hours predict high output. Second, the response of consumption in the model is too high relative to what we see in the data (10 times bigger). This means that the model predicts that the variability of predictable consumption movements should be larger than the variability of predictable output movements whereas, in practice, it is not. The model makes this counterfactual prediction because the high interest-rate changes accompanying a shortfall of capital from its steady state induce a substantial postponement of consumption. The model's high consumption coefficient implies that periods of high expected-output growth should also be periods in which the ratio of consumption to income is low. In other words, in the data $C/Y \nearrow$ means that Y has to grow in the future. In the model, for me to anticipate high output in the future, I should consume less now, i.e. C/Y is low. Intuitively, I need to postpone consumption until good times. Intertemporal substitution explains why hours h have a negative coefficient. I need to work hard now because the interest rate is high.

Figure 3.2: Evolution of output, hours, and consumption when capital starts one percent below its steady state; Rotemberg and Woodford (1996), Figure 2.

The most fundamental failing of the model can be explained simply as follows. The model generates predictable movements only as a result of departures of the current capital stock from its steady-state level. When the current capital stock is relatively low, its marginal product is relatively high so rates of return are high as well. With relatively strong intertemporal substitution, this leads individuals to enjoy both less current consumption and less current leisure than in the steady state. So, consumption is expected to rise while hours of work are expected to fall. More generally, consumption and hours are expected to move in opposite directions. This means that the regression coefficient of consumption on output and the regression coefficient of hours on output should have opposite signs. This intuition is illustrated by IRFs in a basic RBC model (see Figure 3.2). Instead, the data suggests that both of these coefficients are positive.

Why do we have a failure? Rotemberg and Woodford suggest that the model is not sufficiently rich and we miss some margins (e.g., variable factor utilization) and we need more than one shock.

Chapter 4

Extensions of the basic RBC model

In the previous lecture, we investigated the properties of the baseline real business cycle (RBC) model. Overall, the conclusion was that to a first-order approximation, the properties of the model follow from the properties of the forcing variables:

1. The model does relatively little to amplify shocks. In other words, we have to rely on very volatile exogenous shocks to match the volatility of endogenous variables.

2. The model has very weak internal propagation mechanisms. In other words, we have to rely on persistent exogenous shocks to match the persistence of endogenous variables.

3. Wages in the model are too procyclical.

We need to find mechanisms that can fix these important issues. Although there are many suggestions in the literature, we are going to focus in this lecture only on a few specific mechanisms: externalities, variable capital utilization, countercyclical markups and structure of labor markets. Each of these mechanisms are theoretically and empirically appealing. We will see that these mechanism often generate flat labor demand curves which help to reduce the volatility of wages, increase volatility of employment and more generally improve amplification and propagation properties of business cycle models.

4.1 Capital Utilization

4.1.1 Empirical regularity

We have examined cyclical properties of many variables but we did not investigate how capital utilization varies over business cycle. Capital utilization measures how intensively machines and structures are employed in the production process. There are several measures of capital utilization in the literature. Note that in the literature ***capacity*** utilization is often confused with ***capital*** utilization. The former is a measure of how intensively all resources are used while the latter refers only to capital input. In practice, however, capacity and capital utilization rates are highly correlated. Since at this point we are interested in qualitative conclusions only, we can focus only on qualitative facts about cyclical properties of variable capital utilization. Although the properties of variable capital utilization can vary from industry to industry, we are going to ignore these differences and examine only aggregate measures of capital utilization.[1] These facts are:

- capital utilization is highly procyclical
- capital utilization is a least as volatile as output at the business cycle frequencies

Note that in the basic RBC model we assume that capital utilization is the same which is sharply at odds with the data.

4.1.2 A RBC model with capital utilization

Consider the following model. The central planner solves[2]:

$$\max \qquad \mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t \left\{ \ln C_t - \frac{L_t^{1+1/\eta}}{1+1/\eta} \right\}$$

[1]Typically three types of industries are distinguished: conveyer (e.g., auto assembly line), workstation (e.g., watch factory), continuous (e.g., oil refinery or steel mill). Capital utilization is least variable for the continuous process industries since in these industries capital is used 24/7.

[2]You can show that this central planner problem leads to the same first order conditions as in the decentralized economy.

$$
\begin{aligned}
\text{s.t.} \qquad & Y_t = Z_t(u_t K_{t-1})^{\alpha} L_t^{1-\alpha} && (4.1) \\
& Y_t = C_t + I_t && (4.2) \\
& K_t = (1-\delta_t)K_{t-1} + I_t && (4.3) \\
& \delta_t = \frac{1}{\theta} u_t^{\theta} && (4.4)
\end{aligned}
$$

u_t is the capital utilization rate. Here, the cost of using capital more intensively comes from faster depreciation of capital. We assume that $\theta > \alpha$ (otherwise utilization is always full, i.e. $u = 1$).

To understand the role of capital utilization in amplifying and propagating business cycles, consider the FOC for utilization:

$$\alpha \frac{Y_t}{u_t} = u_t^{\theta-1} K_{t-1}.$$

The LHS ($\alpha \frac{Y_t}{u_t}$) is the marginal product from more intensive utilization of capital. The RHS ($u_t^{\alpha-1} K_{t-1}$) is the marginal cost of using capital more intensively. This FOC yields:

$$u_t = \left(\alpha \frac{Y_t}{K_{t-1}}\right)^{1/\theta}. \qquad (4.5)$$

Thus the optimal utilization is determined by the marginal product of capital. Here, this condition is saying that capital should be used more intensively during booms when MPK is high. From this FOC we can immediately say that u will be procyclical since capital cannot change very much over the cycle.

Now, substitute condition 4.5 into the production function to get:

$$Y_t = A Z_t^{\frac{\theta}{\theta-\alpha}} K_{t-1}^{\frac{\alpha(\theta-1)}{\theta-\alpha}} L_t^{\frac{(1-\alpha)\theta}{\theta-\alpha}} \qquad (4.6)$$

Note that $\frac{\theta}{\theta-\alpha} > 1$, so utilization directly amplifies response to Z_t. Given that the production function has constant return to scale (CRS; i.e. $\alpha + (1-\alpha) = 1$), capital utilization has no effect on return to scale:

$$\frac{\alpha(\theta-1)}{\theta-\alpha}+\frac{(1-\alpha)\theta}{\theta-\alpha}=1 \tag{4.7}$$

Now suppose that $\alpha = 0.3$ and $\theta = 1.4$. The effective capital share is:

$$\frac{\alpha(\theta-1)}{\theta-\alpha}=0.3\times\frac{1.4-1}{1.4-0.3}\approx 0.1$$

This value is much smaller than the usual estimate of 0.3. The effective elasticity of output with respect to labor is ≈ 0.9, which we often see in the data. Hence, when technology $Z \nearrow$:

- ○ direct effect from production function
- ○ "elasticity effect" resulting from the positive effect of capital utilization on labor:
  - this amplifies technology shocks as it effectively increases the responsiveness of output to shocks (capital is fixed in the short run).
  - output becomes more elastic with respect to labor

Intuitively, utilization creates intratemporal substitution away from leisure and toward consumption. Thus it generates procyclical effects on labor and consumption. When $Z \nearrow$, MPL $\nearrow$, and $L \nearrow$ and because $Y \nearrow$ so much, you can increase C and I at the same time. There is no need to sacrifice consumption today to allow for capital accumulation.

There is no need to have very high intertemporal elasticity of substitution to generate persistence and amplification (i.e. no need to let labor respond intrinsically strongly to shocks to productivity). Furthermore, variable capital utilization makes labor demand curve more elastic. The reason why it may be important is that when we have demand shocks which affect marginal utility of wealth and hence shift labor supply (e.g., government expenditures shock), we can have large moments in employment and relatively small movements in wages.

4.1.3 Results

Burnside and Eichenbaum (1996) estimate and calibrate a model similar to what is described above. They show that:

1. Once we take utilization into account, the volatility of technology shocks falls by 30%. Not surprising since $TFP = \check{Y}_t - \alpha\check{K}_{t-1} - (1-\alpha)\check{L}_t = Z_t + \alpha u_t$.

2. Variable utilization helps to match the volatility of endogenous variables in the data. This is good news: with smaller shocks we can do the job, we don't need to rely on implausibly large fluctuations in technology.

3. The impulse response functions (IRFs) to shocks are more persistent than in the baseline RBC model.

4. The model with variable utilization can match serial correlation in output without relying on counterfactual serial correlation in technology changes.

In summary, endogenously varying capital utilization can serve as a useful amplification and propagation mechanism.

4.2 Externalities

Another useful mechanism for improving RBCs is externalities. There are many ways to introduce externalities. We will work with a specific case that can be supported by a competitive equilibrium. This formulation is due to Baxter and King (1991) who introduce externalities via increasing return to scale (IRS) in production at the aggregate level and constant returns to scale (CRS) in production at the firm level.

4.2.1 Assumptions

Firms

Continuum of firms indexed by i:

$$Y_{it} = Z_t E_t K_{i,t-1}^{\alpha} L_{it}^{1-\alpha}$$

where Z_t is technology and E_t is externality. Firms treat Z_t and E_t as exogenous. Assume that $E_t = Y_t^{1-1/\gamma}$ where:

$$Y_t = \int Y_{it} di \tag{4.8}$$

and $\gamma \geq 1$. Having a continuum of firms ensures that no firm has an effect on aggregate outcomes.

Firms maximize the net present value (NPV) of profits and equalize the cost of inputs with the marginal return of inputs:

$$\begin{aligned} \text{rental rate of capital: } R_t &= \alpha \frac{Y_{it}}{K_{i,t-1}} - \delta \\ \text{wages: } W_t &= (1-\alpha)\frac{Y_{it}}{L_{it}} \end{aligned}$$

With a constant return-to-scale technology (CRS):

$$(R_t + \delta)K_{i,t-1} + W_t L_{it} = Y_{it}$$

In a symmetric equilibrium, firms are identical: $\forall i,\ K_{it} = K_t,\ Y_{it} = Y_t,\ L_{it} = L_t$. The aggregate output is:

$$\begin{aligned} Y_t &= \int Y_{it} di \\ &= \int Z_t E_t K_{i,t-1}^{\alpha} L_{it}^{1-\alpha} di \end{aligned}$$

$$
\begin{aligned}
&= Z_t E_t K_{t-1}^{\alpha} L_t^{1-\alpha} \\
&= Z_t Y_t^{1-1/\gamma} K_{t-1}^{\alpha} L_t^{1-\alpha} \\
&= (Z_t K_{t-1}^{\alpha} L_t^{1-\alpha})^{\gamma}
\end{aligned}
$$

At the aggregate level, there are increasing return-to-scale (IRS) in production. The marginal products are:

$$
\begin{aligned}
MPK_t &= \gamma\alpha \frac{Y_t}{K_{t-1}} \\
MPL_t &= \gamma(1-\alpha)\frac{Y_t}{L_t}
\end{aligned}
$$

To finish the description of the production side, we need the definition of investment and the resource constraint:

$$
\begin{aligned}
K_t &= (1-\delta)K_{t-1} + I_t \\
Y_t &= C_t + I_t + G_t
\end{aligned}
$$

Households

Households maximize the following utility:

$$\mathbb{E}\sum \beta^t \left\{\ln(C_t - \Delta_t) + \ln(1-L_t)\right\}$$

where Δ_t is a preference shock. Δ_t is introduced to generate acyclical real wages. The budget constraint is:

$$A_{t+1} + C_t = (1+r_t)A_t + W_t L_t + \Pi_t$$

A_t is the quantity of assets held in period t, $r_t = R_t - \delta$ is the interest rate, and Π_t is firms' profit, which is redistributed to households. The FOCs are:

$$\begin{aligned} \frac{1}{C_t - \Delta_t} &= \lambda_t \Rightarrow C_t = \frac{1}{\lambda_t} + \Delta_t \\ \frac{1}{1 - L_t} &= \lambda_t W_t \end{aligned}$$

Aggregate production function

The log-linearized production function is given by:

$$\check{Y}_t = \gamma(\alpha \check{K_{t-1}} + (1-\alpha)\check{L}_t) + \gamma \check{Z}_t$$

Having IRS has a direct effect on output since Z_t is multiplied by $\gamma > 1$. When $\gamma > 1$, labor demand L^d is flatter, and MPL is more sensitive to Z_t. The social MPL is flatter than individual MPLs, as showed on Figure 4.1.

Thus:

$$\begin{aligned} G \nearrow &\Rightarrow Y \nearrow \\ Y \nearrow &\Rightarrow E \nearrow \Rightarrow L^d \nearrow \end{aligned}$$

If γ is really large, the social labor demand could even be upward-sloping., i.e. $\gamma(1-\alpha) > 1$. Then demand-side shocks could increase both wages and output. This is very important because before we discounted demand shocks as they qualitatively generated countercyclical movements in wages. Let's make an even bolder assumption and suppose that the social L^d is steeper than L^s.

Suppose that there is a shock to preferences due to "animal spirits": $\Delta_t \searrow \Rightarrow \lambda_t \searrow$. Now shocks in expectations are driving business cycles. Consumers feel wealthy because they may think that output is higher. They consume more and indeed output becomes higher. Shocks to expectations become self-fulfilling.

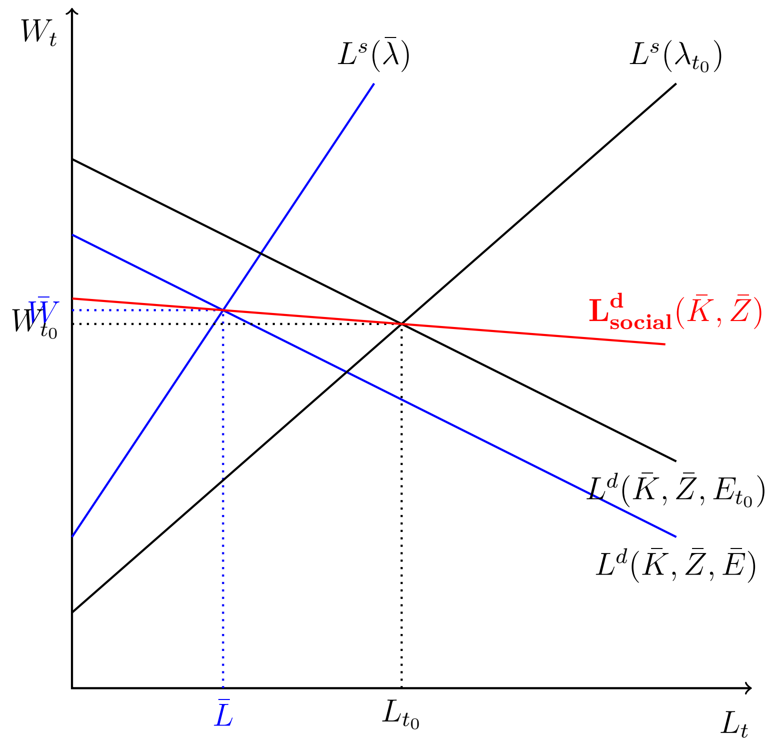


Figure 4.1: Social labor demand.

4.2.2 Recap

In our analysis of the basic RBC model, we found that the basic RBC model lacks strong internal amplification and propagation mechanisms. We considered two extensions that improve the performance of the RBC model:

- capital utilization
- increasing return-to-scale (IRS) in production

These two mechanisms enhance the properties of the model as they generate more endogenous amplification and propagation. These modifications in the model were done in competitive equilibrium. Now we extend our framework to models where firms (and maybe workers) have market power. This extension will have immediate ramifications for the properties of the basic RBC model and it will also help us later when we will analyze models where prices are not entirely flexible. Table 4.1 puts our approach into perspective.

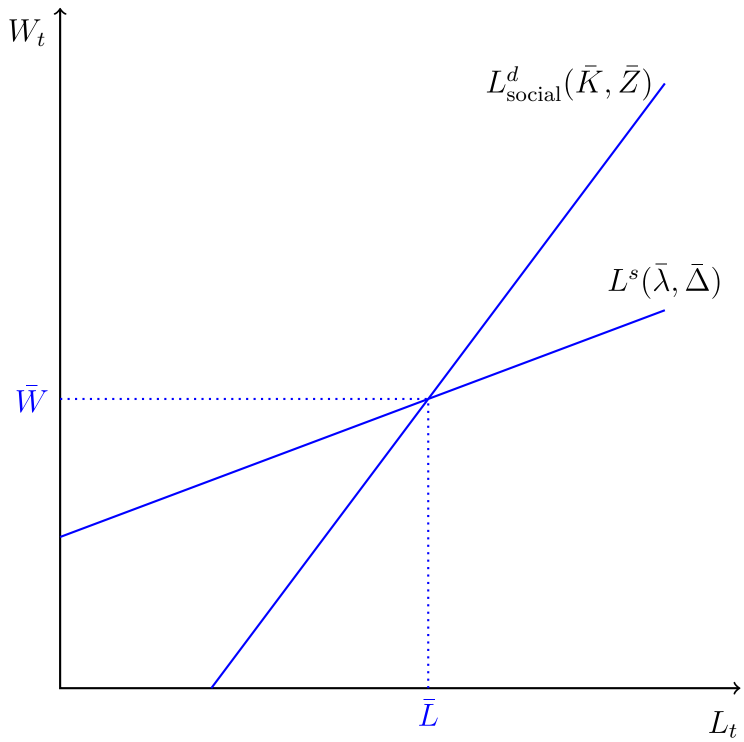


Figure 4.2: Multiple equilibria with social labor demand.

4.3 Imperfect Competition

4.3.1 Producer's behavior

We will assume that firms maximize profits, hire labor and rent capital from households.

$$\begin{aligned} \max \quad & P(Y)Y - RK_i - WL_i \\ \text{s.t.} \quad & Y_i = F(K_i, L_i) \quad (\eta) \end{aligned}$$

where $P(Y)$ is the function describing the demand curve and Lagrange multiplier η is interpreted as marginal cost of producing one extra unit of Y. The FOCs are:

$$\begin{aligned} P(Y) + P'(Y)Y &= \eta \\ R &= \eta F'_K \end{aligned}$$

		Money neutral?	
		Yes	No
Competitive markets?	Yes	RBC ⓐ	
	No	Externalities ⓑ Imperfect competition	Sticky price models ⓒ

Table 4.1: Our approach: ⓐ→ⓑ→ⓒ

$$W = \eta F'_L$$

This yields:

$$\eta = P(1 + P'(Y)Y/P) = P(1 + 1/\epsilon^D) = \frac{P}{\mu}$$

$\mu = P/\eta = P/MC$ is the markup over marginal cost. Importantly, the markup summarizes a great deal of information about the elasticity of demand and it can serve as a sufficient statistic for demand conditions and the intensity of competition. [3]

Note that profit is:

$$\begin{aligned}
\Pi &= P(Y)Y - RK - WL \\
&= P(Y)Y - \eta F'_K K - \eta F'_L L = P(Y)Y - \eta Y\Big(\frac{F'_K K}{Y} + \frac{F'_L L}{Y}\Big) = P(Y)Y - \eta Y\gamma \\
&= P(Y)Y\Big(1 - \frac{\gamma MC}{P}\Big) \\
&= P(Y)Y\Big(1 - \frac{\gamma}{\mu}\Big)
\end{aligned}$$

The firm with increasing returns to scale in production $\gamma > 1$ can make a positive profit only if $\mu > 1$. We can rewrite this condition and find that the share of ***economic*** profits in total

[3]We can get the same result if we relax the assumption of profit maximization and impose a weaker condition that firms minimize costs.

revenue is intimately related to returns to scale in production and markup:

$$s_\pi = \Pi/[P(Y)Y] = \left(1 - \frac{\gamma}{\mu}\right)$$

Hence, we can use the profit share as a useful check for estimates of returns to scale and markup.

4.3.2 Demand side

We will assume that consumers have a love for variety. They want to consume a bit of every good. The idea here is that the marginal utility of consuming any given good may be quickly declining in the consumption of this good, but you can attenuate this problem by consuming more goods. Although there are many formulations of the love for variety, we will consider only the standard one which is due to Dixit and Stiglitz (1977). We assume that households maximize the following utility function

$$\begin{aligned}
\sum \beta^t U(C_t, L_t) &= \sum \beta^t (\ln(C_t) + \ln(1 - L_t)) \\
C_t &= \left[\int_0^1 C_{it}^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}}
\end{aligned}$$

Here households consume C_t which is a bundle or composite of various goods. To remove the strategic effects of consuming any particular good on the bundle C_t, we assume a continuum of goods. Parameter σ gives the elasticity of substitution across goods and also the slope of the demand curve. In other words $\sigma = |\epsilon^D|$. In this setup, the optimal markup is given by:

$$\begin{aligned}
\eta &= P(1 + 1/\epsilon^D) \\
\mu &= P/\eta = \frac{1}{1 + 1/\epsilon^D} \\
\mu &= \frac{\sigma}{\sigma - 1}
\end{aligned}$$

Note that in this formulation markup does not depend on any aggregate variable. How-

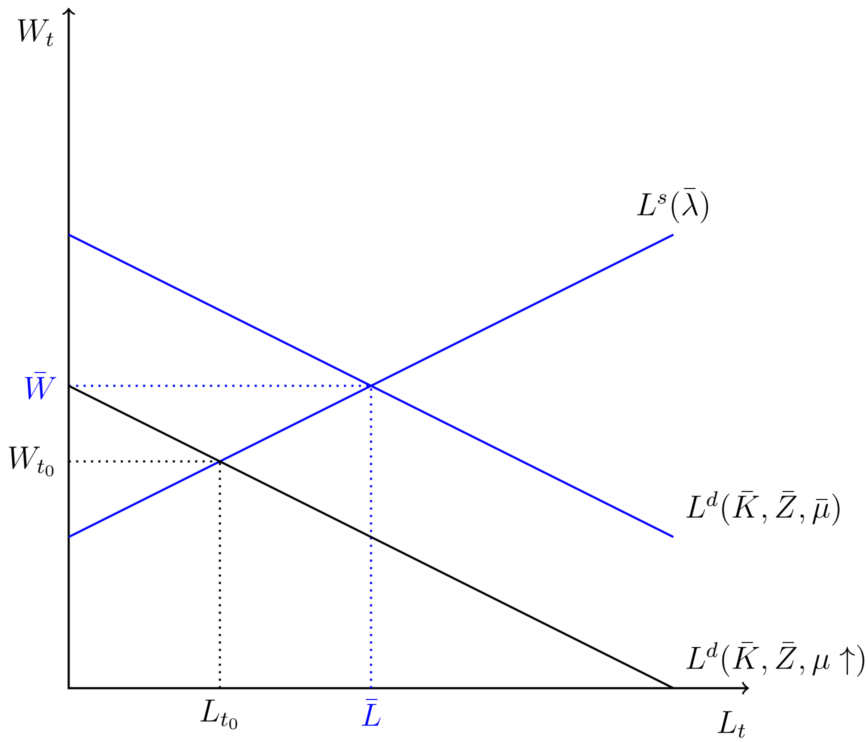


Figure 4.3: Response of labor demand to increase in markup μ.

ever, suppose markup can vary. What would be the effects of changes in markup on wages and employment? To answer this question consider the FOC for labor demand:

$$\begin{aligned} W &= \eta F'_L \\ &= P(1 + 1/\epsilon^D)F'_L \\ W/P &= (1 + 1/\epsilon^D)F'_L \\ &= \frac{F'_L}{\mu} \\ \ln(W/P) &= -\ln(\mu) + \alpha \ln(K/L) + \ln(Z) \end{aligned}$$

Hence a variable markup would work as another shift variable which is illustrated in Figure 4.3.

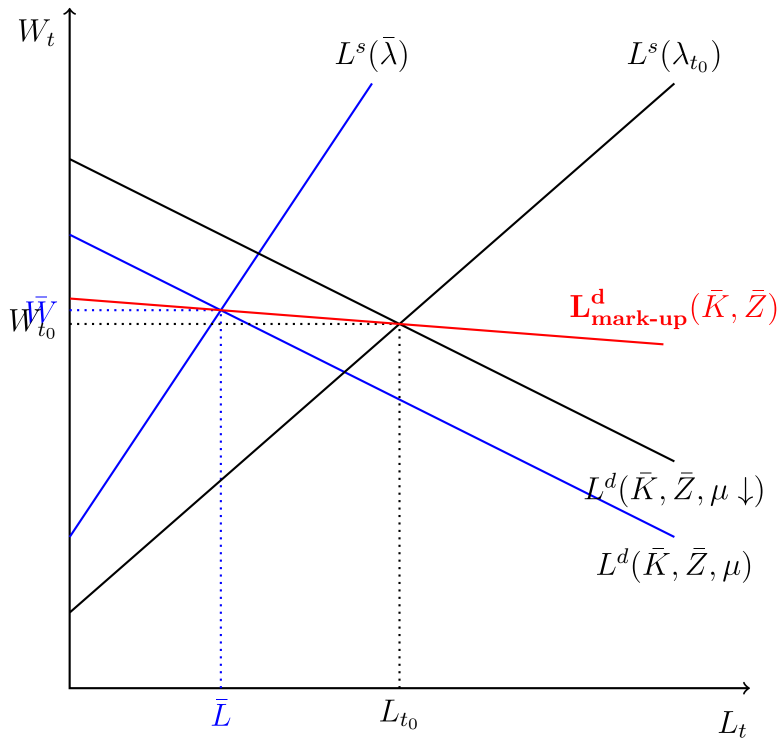


Figure 4.4: Labor demand with countercyclical mark-ups.

4.3.3 Countercyclical markups

Now suppose that the markup is countercyclical, i.e. $\mu'(Y) < 0$. With countercyclical markups, we can generate flatter labor demand curve, as shown in Figure 4.4.

1. $G \uparrow \Rightarrow \lambda \downarrow \Rightarrow L \uparrow$

2. $L \uparrow \Rightarrow Y \uparrow \Rightarrow \mu \downarrow \Rightarrow L^d \uparrow$

Hence with countercyclical markups, labor demand curve get flatter and we can achieve results similar to the results we have in models with externalities and variable capital utilization. We will study models of variable markups in the next section.

4.4 Models of Markup

Recall that markup is given $\mu = (1 + 1/\epsilon^D)^{-1}$ where ϵ^D is the elasticity of demand. If ϵ^D is countercyclical then μ is countercyclical. However, we have to be careful here. If output is non-stationary and grows over time, we cannot simply have $\mu'(Y) < 0$ because we would converge to $\mu = 1$ as output keeps growing. We need to formulate the relationship as $\mu = \mu(\check{Y}_t)$, where $\check{Y}_t$ is deviation from steady-state or balanced growth path.

4.4.1 Cyclical Composition of Demand

Let $s_t^I = \frac{I_t}{Y_t}$ be the share of investment in output. Suppose consumers use $C_i \in [0,1]$ for $C_t = \left[\int_0^1 C_{it}^{\frac{\sigma_C - 1}{\sigma_C}} di\right]^{\frac{\sigma_C}{\sigma_C - 1}}$. Firms use the same variety of goods to construct composite investment:

$$I_t = \left[\int_0^1 I_{it}^{\frac{\sigma_I - 1}{\sigma_I}} di\right]^{\frac{\sigma_I}{\sigma_I - 1}}$$

Now suppose that $\sigma_I > \sigma_C$, i.e. the elasticity of substitution in the investment sector is larger than the elasticity of substitution in the consumption sector. Then the aggregate markup is the average of markups for firms and consumers:

$$\mu_t \simeq s_t^I \frac{\sigma_I}{\sigma_I - 1} + s_t^C \frac{\sigma_C}{\sigma_C - 1}$$

Since s_t^I is procyclical, μ_t is countercyclical if $\sigma_I > \sigma_C$. See Gali (1994) for more details.

4.4.2 Sticky prices

If prices are inflexible (or sticky), one can also generate countercyclical markups. Recall that $\mu = \frac{\bar{P}}{MC(Y)}$ and $MC'(Y) > 0$. Thus $\mu'(Y) < 0$. Most prices do not change frequency and so this explanation is particularly appealing on empirical grounds. We will cover models with inflexible prices in more details later in the class.

4.5 Evidence on Returns to Scale

Since qualitative behavior of business cycle models can depend on aggregate returns to scale in production, it is useful to examine empirical evidence on this matter. However, before we look at the empirical estimates of the returns to scale, having some theory on what is meant by returns to scale is helpful. Consider the following production function:

$$Y = F(K_t, L_t) - \Phi = K^\alpha L^\beta - \Phi$$

where Φ is the fixed cost of running a business and does not depend on produced quantity $F(K_t, L_t)$ as long as it is positive. It turns out that returns to scale depend on α, β and Φ. Consider these combinations:

(a) CRS: $\alpha + \beta = 1$, $\Phi = 0$

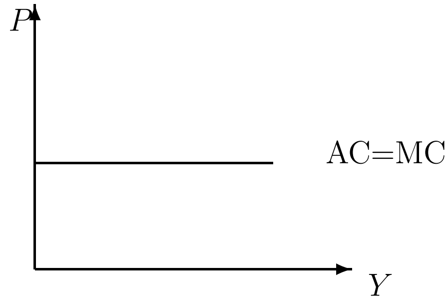
(b) $\alpha + \beta = 1$, $\Phi > 0$. $\gamma = \frac{AC}{MC} > 1$. Hence, Φ causes $s_\pi \Rightarrow \mu \Rightarrow \gamma$. So γ can be an endogenous parameter here because the MC is "normal", L^d is going to have a normal slope.

(c) $\alpha + \beta > 1$, $\Phi = 0$. L^d is going to be flatter.

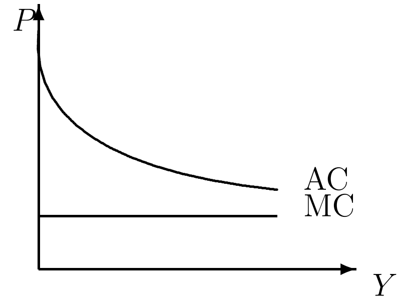
(d) $\alpha + \beta < 1$, $\Phi > 0$. IRS is possible but L^d is going to be steeper because $MC \uparrow$. If $MC \downarrow$, production, inventory is procyclical (evidence from Bils and Klenow (AER 2000)).

We can link returns to scale γ to markup μ and the share of profits in sales s_π:

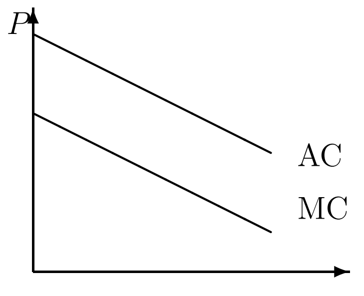
$$\begin{aligned}
\gamma &= \frac{F'_K K}{Y} + \frac{F'_L L}{Y} = \frac{(R/\eta)K + (W/\eta)L}{Y} \\
&= \frac{(RK + WL)/Y}{\eta} \\
\gamma &= \frac{AC}{MC} \\
\mu &= \frac{P}{MC} = \frac{P}{AC}\frac{AC}{MC} = \frac{P}{AC}\gamma = \frac{PY}{PY - \Pi}\gamma
\end{aligned}$$



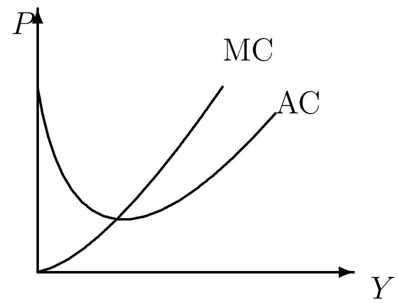
(a) $\alpha + \beta = 1$, $\Phi = 0$



(b) $\alpha + \beta = 1$, $\Phi > 0$



(c) $\alpha + \beta > 1$, $\Phi = 0$



(d) $\alpha + \beta < 1$, $\Phi > 0$

$$= \gamma \frac{1}{1 - s_\pi}$$

If $s_\pi \approx 0$, then $\gamma \approx \mu$. There is evidence that $s_\pi \approx 0$:

- $Q \approx 1$ in the stock market
- Differences between producer and consumer prices are small.

One can use firm level data to estimate returns to scale at the micro level. Much of the previous literature argued that at the firm level production function is best characterized as CRS. In short, the evidence amount to finding CRS when one regresses appropriately deflated sales on measures of inputs such as employment (or hours), capital, materials, electricity, etc. Gorodnichenko (2007) argues that this interpretation of CRS applies for the revenue function, not production functions. Since CRS in revenue weakly implies IRS in production, the previous evidence suggest locally increasing returns to scale.

While the evidence from firm level data is certainly useful, we know that micro-level elasticities could provide a distorted picture of macro-level elasticities (recall our discussion of labor supply elasticity) and so ideally we would like to have estimates based on aggregate data. In a series of papers, Robert Hall (Hall (1988), Hall (1990)) uses demand shifters as instruments to estimate returns to scale: world oil prices, government defense spending, changes in monetary policy. These variables are unlikely to be correlated with technology (the error term in the regression) and hence are valid instruments. Hall finds that $\mu \in [2, 4]$ which is a very high estimate of markups. Since $s_\pi \approx 0$, it must be that γ is very high.

Hall does not make a distinction about whether returns to scale are internal or external. However, the source of increasing returns to scale is important. To establish the source, Caballero and Lyons (1992) estimate the following regression

$$y_i = \mu_i x + \beta_i \tilde{y} + \tilde{z}$$

where x are inputs and $\tilde{y}$ is total industry output. β is a measure of external returns to scale (RTS), which can be directly related to externality in the Baxter-King model. They estimate $\hat{\beta} \approx 0.3 - 0.4$. Hence, the implied returns to scale at the aggregate level is about $1.3 - 1.4$.

Much of the evidence on increasing returns to scale was undermined by works of Susanto Basu and John Fernald (e.g. Basu and Fernald (1997)). They argue that previous estimates of increasing returns to scale based on aggregaget data are likely to suffer from measurement errors and misspecification and one has to correct these issues to obtain correct estimates. For exaple, Basu and Fernald correct for inputs (RHS variable in the regression) for variable utilization rates by using directly observed utilization (Shapiro (1993), Shapiro (1986)) or using proxies for utilization such as consumption of electricity (Burnside and Eichenbaum (1996)) or hours of work per employees (Basu and Kimball (1997)). They find that $\hat{\mu} = 1.2 \Rightarrow \hat{\beta} \approx 0$. Thus, $\hat{\mu} = 1.2$ is consistent with $P > MC$ but it does not imply implausibly large markups. In summary, while increasing returns to scale is a potentially power amplification and propagation mechanism in theoretical models, it appears that the evidence to support it is weak.

4.6 Labor Markets

One of the problems in the basic RBC model was that wages were two procyclical while employment was not sufficiently procyclical. In the end we typically need L^s being flat and L^d being flat. We solved this problem partially by making labor demand very elastic when we considered countercyclical markups, increasing returns to scale and variable capital utilization. We can also consider several modifications that make aggregate labor supply more elastic even when labor supply for each worker is highly inelastic.[4]

- **Shirking model**: Shapiro and Stiglitz (1984) model of efficiency wages.
- **Gift exchange model**: Akerlof and Yellen (1990).
- **Effort model**: Solow (1979).

To get a flavor of how these models operate, consider the Solow model. Let $L^* = eL$ where L^* is effective labor and e is effort, which depends on wage w: $e = e(w)$, $e'(w) > 0$. The firm maximizes:

$$\begin{aligned} \max_{w} \quad & PF(e(w)L) - wL \\ \text{FOC:} \quad & PF'(e(w)L)e(w) = w \quad (\text{w.r.t. } L) \\ & PF'(e(w)L)e'(w)L = L \quad (\text{w.r.t. } w) \end{aligned}$$

Taking the ratio of the two FOCs yields:

$$\frac{e(w)}{e'(w)} = w \Rightarrow \epsilon_w := \frac{e'(w)w}{e(w)} = 1$$

[4]See Romer's textbook for more details.

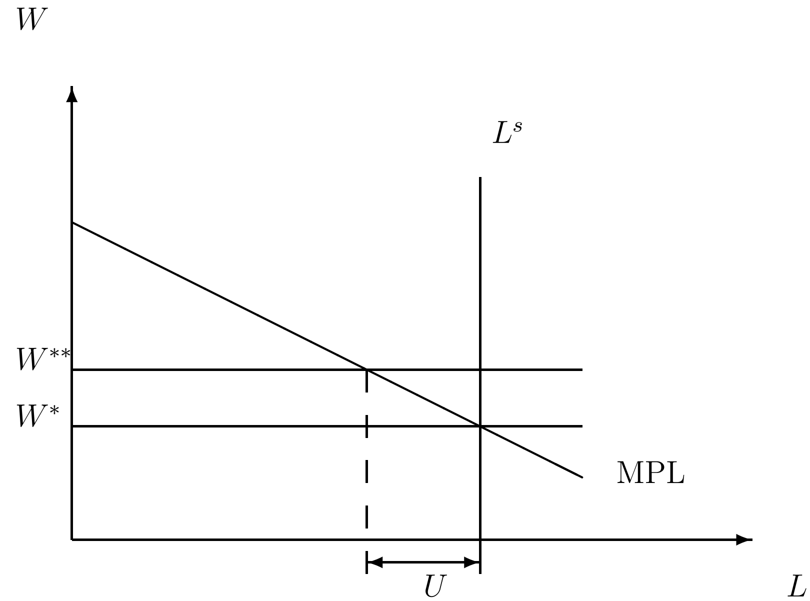


Figure 4.5: Unemployment in efficiency wage model.

So the optimality condition for the wage is:

$$\frac{e'(w^*)w^*}{e(w^*)} = 1$$

Note that the wage is not tied to the reservation wage of workers. In fact, the wage in this model does not necessarily fluctuate with the aggregate conditions as long as effort function is insensitive to these fluctuations. This makes wages very stable.

In general, models of efficiency wages (Shapiro-Stiglitz, Akerlof-Yellen, Solow) lead to some from of unemployment because either marginal product is decoupled from wages or because effective labor supply faced by firms is decoupled from the labor supply of workers. In these models, it is typical that fluctuations in demand can lead to fluctuations in unemployment (see Figure 5).

One can also make labor supply more elastic by introducing a home production sector. The idea here is that if one has an alternative of working at home then his or her supply of labor in formal markets is going to be more elastic. This is similar in spirit to why competition makes elasticities of demand larger.

Chapter 5

Price Rigidity

5.1 Stylized Facts about Pricing

As we discussed in the previous lecture, inflexible prices can lead to ex-post counter-cyclical markups and thus can serve as a useful propagation and amplification mechanism. In this lecture, we will try to formalize the idea that inflexible prices could be an optimal response to shocks. Specifically, we will model price setting behavior more explicitly and will study the effects of inflexible prices on macroeconomic outcomes.

On the empirical side, casual observations of prices in stores suggest that prices are not changing every instant. If anything, prices are rather stable over time. Stylized facts about pricing are presented in Table 5.1. Keep in mind that there is a lot of heterogeneity across goods in terms of rigidity:

- prices in the service sector are sticky (e.g. medical care);
- prices of raw goods are not sticky (e.g. gas);
- prices of manufactured goods fall in between.

	Bils and Klenow (2004)	Nakamura and Steinsson (2008)
Average duration of price spell	≈ 4 months	≈ 11 months
Average price change $\lvert\Delta P\rvert/P$		≈ 10 %
Hazard rate of price adjustment		decreasing
Mean frequency of price adjustment	20%/month	11%/month

Table 5.1: STYLIZED FACTS ABOUT PRICING

5.2 A Static Menu-Cost Model

We consider a simple model of price setting. From this model, we will get a basic intuition for why prices could be sticky and possibly insensitive to nominal shocks such as changes in money supply. Suppose that identical "yeoman farmers" maximize utility:

$$\begin{aligned} \max \quad & U_i = C_i - \frac{L_i^{1+1/\eta}}{1+1/\eta} \\ \text{s.t.} \quad & PC_i = (P_iY_i - WL_i) + WL_i \end{aligned}$$

where P is the price of the consumption bundle, Y_i is the output produced by the farmers, $(P_iY_i - WL_i)$ is the (nominal) profit from production, η is the elasticity of labor supply.

The aggregate consumption good C_i consists of a continuum of individual varieties,

$$C_i = \Big(\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj \Big)^{\frac{\sigma}{\sigma-1}}$$

where C_{ij} is the variety $j \in [0,1]$ from the continuum of goods, σ is the elasticity of substitution across goods in the Dixit-Stiglitz aggregator, which can also be interpreted as the elasticity of demand. In allocating consumption across varieties, the consumer will minimize expenditure given the optimal choice of aggregate consumption,

$$\min \int_{j=0}^1 P_j C_{ij}$$

$$\text{s.t. } \left(\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj\right)^{\frac{\sigma}{\sigma-1}} = C_i \quad (\mu)$$

The first order conditions are,

$$C_{ij}^{-\frac{1}{\sigma}} C_i^{\frac{1}{\sigma}} = P_j \mu^{-1}.$$

Taking the power $1-\sigma$ we find,

$$P \equiv \mu = \left(\int_0^1 P_j^{1-\sigma} dj\right)^{\frac{1}{1-\sigma}}.$$

Recall that the Lagrange multiplier has the interpretation that relaxing the constraint by one unit reduces the objective function by μ. So reducing C_i by one unit reduces expenditure by μ. But this is just the price of the consumption bundle, so we set $P \equiv \mu$. It also has the appropriate interpretation that total expenditure is $\int_{j=0}^1 P_j C_{ij} = PC_i$. Therefore relative demand by consumer i for variety j is,

$$C_{ij} = C_i (P_j/P)^{-\sigma}.$$

Returning to the original optimization problem, we assume that the individual has the linear production function: $Y_i = L_i$. Thus:

$$U_i = \frac{(P_i - W)Y_i + WL_i}{P} - \frac{L_i^{1+1/\eta}}{1+1/\eta}$$

We will also assume for now that the demand for good i is given by the integral of consumption demand by all consumers:

$$Y_i = (\int_k C_k)(P_i/P)^{-\sigma} = Y(P_i/P)^{-\sigma}$$

P_i/P is the relative price for good i. $Y = \int_0^1 C_k dk$ is the output in the economy. Now combine this demand condition with the utility:

$$U_i = \frac{(P_i - W)Y(P_i/P)^{-\sigma} + WL_i}{P} - \frac{L_i^{1+1/\eta}}{1+1/\eta}$$

Here the farmer has to control:

- price of the good P_i (we need to have firms with some market power if we want to study price setting);

- labor supply L_i.

The FOC for the price P_i yields:

$$Y(P_i/P)^{-\sigma} - (P_i - W)\sigma(P_i/P)^{-\sigma-1}(1/P) = 0$$

Multiplying by $(P_i/P)^{\sigma+1}P$ and dividing by Y yields:

$$\frac{P_i}{P} = \frac{\sigma}{\sigma-1}\frac{W}{P}$$

(P_i/P) is the relative price. $\frac{\sigma}{\sigma-1}$ is the markup and W/P is the marginal cost. The FOC for the labor supply decision:

$$W/P = L_i^{1/\eta} \Rightarrow L_i \;\; = \;\; (W/P)^{\eta}$$

W/P is the marginal benefit from supplying more labor and $L^{1/\eta}$ is the marginal disutility of labor.

In a symmetric equilibrium $Y_i = Y_j = Y$:

$$\begin{aligned} \frac{W}{P} &= L_i^{1/\eta} = Y_i^{1/\eta} = Y^{1/\eta} \\ \frac{P_i}{P} &= \frac{\sigma}{\sigma-1}\frac{W}{P} = \frac{\sigma}{\sigma-1}Y^{1/\eta} \end{aligned}$$

Also because the equilibrium is symmetric and thus $P_i = P$, we get:

$$Y = \Big(\frac{\sigma - 1}{\sigma}\Big)^{\eta}$$

Note that as $\sigma \to +\infty$, $Y \to 1$ (perfect competition). Also note that in this model no firm has an effect on aggregate demand since there is a continuum of firms. There is also a form of market demand externality: when I lower my own price, I ignore the positive effects on other producers; but if everybody does it, we all can be better off.

To quickly summarize, we have the following results at this point:

- In equilibrium output is lower than in perfect competition:
  - asymmetric effects of booms and recessions;
  - unemployment.
- market demand externality:
  - we could improve welfare if everybody cuts his price (raises real wages and output);
  - everybody has an incentive to deviate from the actions of the other firms;
  - when a firm deviates, it does not take into account the actions of the other firms.

To close the model on the nominal side, we use the quantity theory of money $MV = PY$ where V is the velocity of money such that with $M = P$ we have $V = Y = (\frac{\sigma-1}{\sigma})^{\eta}$. In this exercise, we will assume that velocity is fixed. From the quantity theory of money, output is equal to real money balances $Y = MV/P$ and:

$$\frac{P_i}{P} = \frac{\sigma}{\sigma - 1}\Big(\frac{MV}{P}\Big)^{1/\eta} = \Big(\frac{M}{P}\Big)^{1/\eta}$$

Given the optimal price P_i and the state of demand, we compute the (real) profit for each firm

$$\Pi_i \;=\; Y_i(P_i - W)/P = Y\Big(\frac{P_i}{P}\Big)^{-\sigma}\Big(\frac{P_i}{P} - Y^{1/\eta}\Big)$$

$$= \frac{MV}{P}\Big(\frac{P_i}{P}\Big)^{1-\sigma} - \Big(\frac{P_i}{P}\Big)^{-\sigma}\Big(\frac{MV}{P}\Big)^{1+1/\eta}$$

In equilibrium, $P_i = P_j = P$ so:

$$\Pi = \frac{MV}{P} - \Big(\frac{MV}{P}\Big)^{1+1/\eta} = \Big(\frac{MV}{P}\Big)\Big[1 - \Big(\frac{MV}{P}\Big)^{1/\eta}\Big]$$

Now suppose we start from a state where the prices were set to equilibrium levels. Denote the level of prices in this equilibrium with P_0. Suppose that money supply increases from M_0 to M_1 and, as a result, the price level increases from P_0 to P_1. Firms that do not adjust prices get:

$$\Pi_{\text{Fixed}} = \frac{M_1V}{P_1}\Big(\frac{P_0}{P_1}\Big)^{1-\sigma} - \Big(\frac{P_0}{P_1}\Big)^{-\sigma}\Big(\frac{M_1V}{P_1}\Big)^{1+1/\eta}$$

If a firm adjusts its price then:

$$\Pi_{\text{Adjust}} = \frac{M_1V}{P_1}\Big(\frac{P^*}{P_1}\Big)^{1-\sigma} - \Big(\frac{P^*}{P_1}\Big)^{-\sigma}\Big(\frac{M_1V}{P_1}\Big)^{1+1/\eta}$$

where P^* is the optimal price to be chosen by firms for the given level of aggregate demand and price level P_1. Note that P^*, which is often called the optimal reset price, satisfies

$$\frac{P^*}{P_1} = \frac{\sigma}{\sigma-1}\frac{W_1}{P_1} = \frac{\sigma}{\sigma-1}\Big(\frac{M_1V}{P_1}\Big)^{1/\eta}$$

and hence

$$\begin{aligned}\Pi_{\text{Adjust}} &= \frac{M_1V}{P_1}\Big(\frac{\sigma}{\sigma-1}\Big(\frac{M_1V}{P_1}\Big)^{1/\eta}\Big)^{1-\sigma} - \Big(\frac{M_1V}{P_1}\Big)^{1+1/\eta}\Big(\frac{\sigma}{\sigma-1}\Big(\frac{M_1V}{P_1}\Big)^{1/\eta}\Big)^{-\sigma} \\ &= \frac{1}{\sigma-1}\Big(\frac{\sigma}{\sigma-1}\Big)^{-\sigma}\Big(\frac{M_1V}{P_1}\Big)^{1+(1-\sigma)/\eta}\end{aligned}$$

Obviously, $\Pi_{\text{Adjust}} > \Pi_{\text{Fixed}}$ and hence if prices can be adjusted frictionlessly every firm should adjust their prices. Note that in this case, $\frac{P_i^*}{P_1} = \frac{\sigma}{\sigma-1}(\frac{M_1V}{P_1})^{1/\eta}$ and given the symmetry of the equilibrium $P_i^* = P_1$, we have $P_1 = M_1$ and hence nominal shocks have no effect on output.

Now suppose that there is a fee ϕ to changing the price level. Then the decision rule of firm is:

- $\phi > \Pi_{\text{Adjust}} - \Pi_{\text{Fixed}}$: firms do not adjust the price
- $\phi < \Pi_{\text{Adjust}} - \Pi_{\text{Fixed}}$: firms adjust their price

To give our analysis a more dynamic flavor, we can assume that

1. Firms set their prices based on expected money supply $M_0 = M^e = M_1$.
2. Money shock is realized with possibly $M_1 \neq M^e$.
3. Firms decide whether to change the price or not.

Suppose that P_0 is the price which was set in expectation of $M_0 = M^e = M_1$. The question is how much a firm would gain in terms of profits if it adjusts its price from P_0 to P^*. Alternatively, how much does a firm lose when it is not adjusting its price P_0 to the optimal level P^*? We could evaluate the difference $\Pi_{\text{Adjust}} - \Pi_{\text{Fixed}}$ numerically, but it is more convenient and useful to work with the 2nd order Taylor expansion. According to the 2nd order approximation:

$$\Pi(P_0) \approx \Pi(P^*) + \Pi'(P^*)(P_0 - P^*) + \frac{1}{2}\Pi''(P^*)(P_0 - P^*)^2$$

Since firms are optimizing, $\Pi'(P^*) = 0$. Thus:

$$\Pi(P_0) - \Pi(P^*) \approx \frac{1}{2}\Pi''(P^*)(P_0 - P^*)^2$$

Since the profit function is concave, $\Pi''(P^*) < 0$ and hence $\Pi(P^*) - \Pi(P_0) > 0$.

Recall that the log-linearized demand for firms is given by $\check{P}_i^* = (1/\eta)(\check{M} - \check{P}) + \check{P}$ which we will write as $\check{P}_i^* = \psi(\check{M} - \check{P}) + \check{P}$ since eventually we want to have a more general model (i.e., here $\psi = 1/\eta$ but ψ could be a much more complex function of η and other fundamental parameters). Suppose that only ***one*** firm is considering a price change and every other firm keeps its price fixed at $P_0 = M_0 = M_e$. For this one adjusting firm we have:

$$\check{P}^* - \check{P}_0 = \psi(\check{M}_1 - \check{P}_0) = \psi(\check{M}_1 - \check{M}_0) = \psi((\check{M}_1 - \check{P}_0) - (\check{M}_0 - \check{P}_0)) = \psi(\check{Y}_1 - \check{Y}_0)$$

Given $(P_0/M_0) = 1$, we have

$$P^* - P_0 = (\check{P}^* - \check{P}_0)P_0 = \psi(\check{M}_1 - \check{M}_0)M_0 = \psi(M_1 - M_0)$$

and hence

$$\begin{aligned} \Pi(P^*) - \Pi(P_0) &\approx -\frac{1}{2}\Pi''(P^*)\psi^2(M_0 - M_1)^2 \\ &= -\frac{1}{2}\Pi''(P^*)\psi^2(Y_0 - Y_1)^2\left(\frac{P_0}{Y_0}\right)^2 \end{aligned}$$

ψ is a measure of real rigidity. $\Pi''(P^*)$ is the sensitivity of the profit function. Keeping a fixed price is a Nash equilibrium if:

$$-\frac{1}{2}\Pi''(P^*)\psi^2(Y_0 - Y_1)^2\left(\frac{P_0}{Y_0}\right)^2 < \phi$$

This finding is due to seminal papers by Mankiw (1985) and Akerlof and Yellen (1985).

To make non-adjustment an equilibrium outcome for low menu costs, we need small ψ and $|\Pi''(P^*)|$. In this model, the curvature is coming from two places:

- Elasticity of substitution (σ)
- Labor supply elasticity (η)

By adjusting these two parameters we can control the size of the coefficient that multiplies $(Y_0 - Y_1)^2$. For example, if elasticity of labor supply is high, the cost of deviating from the optimal price is low. Why? When the price is fixed and demand changes, then demand induces big changes in labor supply. If I dislike labor, then having artificially low price can be very costly in terms of disutility.

The problem here is that for this approach to work, it must be that the profit function is relatively flat. This is not realistic. We need extra mechanisms that make non-adjustment of prices an equilibrium outcome. These mechanisms that make ψ small are called "real rigidity". In general, the relationship between non-neutrality of nominal/monetary shocks (i.e., whether nominal shocks have real effects), real rigidity (i.e., how sensitive the optimal price is to changes in demand conditions; captured by ψ), and nominal rigidity (i.e., the cost of changing a nominal price) can be summarized as: **Non-neutrality = real rigidity × nominal rigidity**. If nominal rigidity = 0, then real rigidity does not make money important. If real rigidity is zero, then even costly nominal price changes do not make nominal shocks to have real effects. ***Both*** rigidities (nominal and real) are needed for non-neutrality.

5.3 Sources of Real Rigidity

What can make profit function flat:

1. flat marginal cost (MC)

2. steep marginal revenue (MR)

More precisely:

1. Diminishing marginal cost: $Y_i = L^\gamma$, $\gamma > 1$ makes the MC flatter or even downward sloping. As $\gamma \uparrow$, $\psi \downarrow$.

2. Intermediate inputs (Basu (1995)): $Y_i = L_i^\alpha Q_i^{1-\alpha}$. $MC \propto W^\alpha P^{-\alpha}$. $P_i^* - P = const + (\alpha/\eta)Y$. $\psi = \alpha/\eta$. As long as $\alpha < 1$, then ψ is smaller. Intuitively, to produce 1% of output, one only needs (α/η)% increase in labor and other inputs have the same price because other firms' prices are unchanged.

3. Kinked demand curves (Warner and Barsky (1995), Kimball (1995)): this makes MR steep (oligopoly).

5.4 Social Welfare

Recall that in our analysis we continuously have market demand externality. This externality has important social welfare implications. Suppose there is a negative shock to money:

- need to cut prices
- firms fail to adjust their prices because their ***private*** benefit is small
- but the ***social*** cost can be large

Intuition:

- $\Pi'(P^*) = 0$ at the private level so that only second-order terms are important.
- Social welfare (SW) does not have $SW'(P^*) = 0$. Hence, the cost is first order.

This can also explain why booms are "good" and recessions are "bad". Deviation from equilibrium output have symmetric effects for firms, because output is optimal privately for firms (increase and decrease in output are both equally costly for firms). However, from a social welfare point of view, the higher the output the better because output is too low in equilibrium from a social point of view. Again this finding is due to Mankiw, Akerlof and Yellen.

5.5 Issues and open questions

1. Why is it costly to change prices and not quantities? I.e. why does the menu cost apply to prices and not quantities? The answer must be that producers find it more

convenient to post prices, change them in discrete intervals, and let demand determine quantities. Why it happens is outside the model...

2. So far we looked only at a static model. However if the game is repeated, we need to look at present values and then we need even larger menu costs.

3. Why would anyone fix the "level" of prices when inflation is positive?

4. How does the response to nominal shocks change when the economy is outside the steady state? What if there is some special cross-sectional distribution of prices at the time of the shock?

Chapter 6

Models with sticky prices

In the previous lecture, we considered why firms may find it optimal to keep their prices fixed in response to shocks. Our analysis was static in nature while much of macroeconomics emphasizes dynamics. In this lecture, we will generalize our model to a dynamic setting. Although there are several classical formulations of price setting, we will consider in detail only one due Calvo (1983), which is the workhorse of modern Keynesian economics.

6.1 Household's Problem

The household solves the following optimization problem:

$$\max \qquad \mathbb{E}\sum \beta^t U(C_t, L_t) = \mathbb{E}\sum \beta^t \Big(\frac{C_t^{1-\psi}}{1-\psi} - \frac{L_t^{1+1/\eta}}{1+1/\eta}\Big)$$

where C_t is consumption index (based on the Dixit-Stiglitz aggregator):

$$C_t = \Big(\int_0^1 C_t(i)^{\frac{\sigma-1}{\sigma}} di\Big)^{\frac{\sigma}{\sigma-1}}$$

where $C_t(i)$ is the variety $i \in [0,1]$ from the continuum of goods. Parameter ψ is the inverse of the **intertemporal elasticity of substitution** (IES) and parameter η is the **Frisch**

labor supply elasticity. The budget constraint is:

$$\int_0^1 P_t(i)C_t(i)di + Q_tB_t = B_{t-1} + W_tL_t + \Pi_t$$

Q_t is the price of zero coupon bond B_t. One can show that the optimal consumption of good i is:

$$C_t(i) = C_t\Big(\frac{P_t(i)}{P_t}\Big)^{-\sigma}.$$

The price of the optimal consumption (also the price index) is:

$$P_t \equiv \Big(\int_0^1 P_t(i)^{1-\sigma}di\Big)^{\frac{1}{1-\sigma}}$$

Then the price of the optimal bundle is:

$$\Big(\int_0^1 P_t(i)C_t(i)di\Big) = P_tC_t$$

This allows us to simplify the budget constraint. The FOCs are now familiar:

$$-\frac{U'_L}{U'_C} = \frac{W_t}{P_t} \tag{6.1}$$
$$Q_t = \beta\mathbb{E}_t\Big[\frac{U'_{C,t+1}}{U'_{C,t}}\frac{P_t}{P_{t+1}}\Big] \tag{6.2}$$

Equation 6.2 is the Euler equation governing the ***inter***temporal allocation of resources. Equation 6.1 is the ***intra***temporal condition, which can be rewritten as:

$$\frac{L_t^{1/\eta}}{C_t^{-\psi}} = \frac{W_t}{P_t} \Rightarrow \check{W}_t - \check{P}_t = \psi\check{C}_t + 1/\eta\check{L}_t$$

Let $Q_t \equiv 1/\exp(i_t) \Rightarrow \check{Q}_t = -i_t$ and define inflation rate as $\pi_{t+1} = \frac{P_{t+1}-P_t}{P_t} \approx log(P_{t+1}/P_t) \approx$

$\check{P}_{t+1} - \check{P}_t$. We will assume in all derivations that the inflation rate is **zero** in the steady state.[1] After log-linearizing the Euler equation, we have:

$$\check{C}_t = \mathbb{E}_t[\check{C}_{t+1}] - \frac{1}{\psi}(i_t - \mathbb{E}_t[\pi_{t+1}]) \tag{6.3}$$

If ψ is low, consumption is sensitive to the real interest rate $(i_t - \mathbb{E}_t[\pi_{t+1}])$. Before, we assumed log utility in consumption which gave us $\psi = 1$.

6.2 Pricing Assumption

The key ingredient of Calvo's model is the pricing assumption:

1. Each firm can reset its price only with probability $1 - \theta$.

2. This probability is ***independent*** from the time elapsed since last price adjustment.

The assumption of independent price adjustments is critical. It allows simple aggregation of prices and, hence, it leads to a straightforward dynamics of the aggregate price level. This assumption will also be the critical problem of the model since in reality it appears that adjustment of prices depends on how far a given price is from the optimal price. In any case, given this assumption, the probability of price non-adjustment is θ and the average duration of the price is $\frac{1}{1-\theta}$. We can interpret θ as an index of price stickiness.

Let's determine the aggregate price index dynamics in a symmetric equilibrium:

$$\begin{aligned} P_t &= \left(\int_0^1 P_t(i)^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} \\ &= \left(\int_0^{1-\theta} P_t^*(i)^{1-\sigma} di + \int_{1-\theta}^1 P_{t-1}(i)^{1-\sigma} di\right)^{\frac{1}{1-\sigma}} \\ &= \left((1-\theta)(P_t^*)^{1-\sigma} + \theta(P_{t-1})^{1-\sigma}\right)^{\frac{1}{1-\sigma}} \end{aligned}$$

[1]The assumption of zero steady state inflation is not innocuous. See Cogley and Sbordone (2008) and Coibion and Gorodnichenko (2011). While clearly unrealistic, this assumption simplifies derivations tremendously.

where we used the facts that i) all firms choose the same reset price P_t^* because there is no firm-specific state variable, and ii) price adjustment is random and therefore the composition of prices in $\int_{1-\theta}^1 P_{t-1}(i)^{1-\sigma}di$ is identical to the composition of prices in $\int_0^1 P_{t-1}(i)^{1-\sigma}di$. Now divide both sides of this equation by P_{t-1} to get:

$$\left(\frac{P_t}{P_{t-1}}\right)^{1-\sigma} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\sigma}$$

Log-linearize the equation around steady-state with zero inflation to get:

$$\pi_t = \check{P}_t - \check{P}_{t-1} = (1-\theta)(\check{P}_t^* - \check{P}_{t-1})$$

Note that the inflation rate is the combination of two margins:

- extensive margin $= (1-\theta)$
- intensive margin $= (\check{P}_t^* - \check{P}_{t-1})$

The extensive margin is fixed in this model while in reality this margin can respond to shocks.

6.3 Firm's Problem

We now focus on the firms side to figure out how the optimal price is chosen. We'll make the following assumptions:

- Continuum of firms, $i \in [0, 1]$.
- Each firm i is producing a fixed variety.
- Production function is $Y_t(i) = Z_t L_t(i)^{1-\alpha}$ where Z_t is the level of technology ($\alpha > 0$ yields an upward-sloping marginal cost).
- No capital.

- Labor is homogenous across firms (hence, firms face the same wage and there is perfect competition in the labor market).

- All firms face the same demand curve: $Y_t(i) = Y_t\left(\frac{P_t(i)}{P_t}\right)^{-\sigma}$.

How do firms set prices with the Calvo pricing assumption? The problem is dynamic because the same price will be charged for some time and hence firms must take into account that for a while they won't be able to change their prices. The optimal reset price maximizes the present value of profits:

$$\max_{P_t^*} \mathbb{E}_t \sum_{k=0}^{+\infty} \theta^k \left[Q_{t,t+k}(P_t^* Y_{t+k|t} - \Sigma_{t+k}(Y_{t+k|t}))\right]$$

Σ_{t+k} is the nominal cost function in period $t+k$ and $Y_{t+k|t}$ is the output in period $t+k$ of the firm that resets its price at time t. $Q_{t,t+k}$ is the stochastic discount factor. It is obtained from the consumer's Euler equation:

$$Q_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\psi} \frac{P_t}{P_{t+k}}$$

The firm is subject to the demand constraint:

$$Y_{t+k|t} = Y_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\sigma}.$$

The first order condition for P_t^* is:

$$\mathbb{E}_t \sum_{k=0}^{+\infty} \theta^k \left[Q_{t,t+k} Y_{t+k|t}(P_t^* - \frac{\sigma}{\sigma - 1}\Sigma'_{t+k}))\right] = 0 \tag{6.4}$$

where $\mu = \frac{\sigma}{\sigma-1}$ is the desired markup over cost and Σ'_{t+k} is the marginal cost at time $t+k$. This yields:

$$P_t^* = \frac{\mathbb{E}_t \sum \theta^k \left(Q_{t,t+k} Y_{t+k|t}\right) \frac{\sigma}{\sigma-1} \Sigma'_{t+k}}{\mathbb{E}_t \sum \theta^k Q_{t,t+k} Y_{t+k|t}}$$

Note that for $\theta = 0$ we find the familiar static condition:

$$P_t^* = \frac{\sigma}{\sigma - 1} \times MC$$

Divide equation (6.4) by P_{t-1} to get:

$$\mathbb{E}_t \sum_{k=0}^{+\infty} \theta^k \Big[Q_{t,t+k} Y_{t+k|t} \Big(\frac{P_t^*}{P_{t-1}} - \frac{\sigma}{\sigma - 1} \frac{\Sigma'_{t+k}}{P_{t+k}} \frac{P_{t+k}}{P_{t-1}} \Big) \Big] = 0$$

Define the ***real*** marginal cost:

$$MC_{t+k|t} = \frac{\Sigma'_{t+k}}{P_{t+k}}$$

Log-linearize the FOC for optimal reset price around the steady state with ***zero*** inflation and get:

$$\check{P}_t^* - \check{P}_{t-1} = (1 - \beta\theta) \sum_{k=0}^{+\infty} (\beta\theta)^k \mathbb{E}_t \big(\check{MC}_{t+k|t} + (\check{P}_{t+k} - \check{P}_{t-1}) \big)$$

To have a more convenient expression, rewrite:

$$\check{P}_t^{\,*} = (1 - \beta\theta) \sum_{k=0}^{+\infty} (\beta\theta)^k \mathbb{E}_t \big(\check{MC}_{t+k|t} + \check{P}_{t+k} \big)$$

Firms that reset will choose a price that corresponds to the weighted average of desired future markups. Here weights depend on θ, which governs the effective horizon. With low θ, the future does not matter as firms can effectively set prices period by period. With high θ, the future is very important.

Since we assume no capital in the model, equilibrium in the goods market implies:

$$\begin{aligned} Y_t(i) &= C_t(i) \\ \Big(\int_0^1 Y_t(i)^{\frac{\sigma-1}{\sigma}} di \Big)^{\frac{\sigma}{\sigma-1}} &\equiv Y_t = C_t \\ \check{Y}_t &= \check{C}_t \end{aligned}$$

Equilibrium in the labor market:

$$
\begin{aligned}
L_t &= \left(\int_0^1 L_t(i)di\right) = \left(\int_0^1 \left(\frac{Y_t(i)}{Z_t}\right)^{\frac{1}{1-\alpha}} di\right) \\
&= \left(\frac{Y_t}{Z_t}\right)^{\frac{1}{1-\alpha}} \left(\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{-\sigma}{1-\alpha}} di\right) \Rightarrow \\
(1-\alpha)\check{L}_t &= \check{Y}_t - \check{Z}_t + \check{d}_t
\end{aligned}
$$

Here term $\check{d}_t$ arises from $\left(\int_0^1 (\frac{P_t(i)}{P_t})^{\frac{-\sigma}{1-\alpha}} di\right)$ which captures the effect of price dispersion. We will ignore this term (by assuming that $\check{d}_t$ is a negligible, higher-order term) since inflation rate is zero in the steady state and hence variation in d is small (see Gali's textbook for more rigorous treatment of this term).

Bonds are in zero net supply. You can check that the equilibrium in other markets implies that this market also clears, $B_t = 0$ (Walras Law).

Now the aggregate marginal cost is:

$$
\begin{aligned}
\check{MC}_t &= (\check{W}_t - \check{P}_t) - \check{MPL}_t \\
&= (\check{W}_t - \check{P}_t) - (\check{Z}_t - \alpha \check{L}_t) \\
&= (\check{W}_t - \check{P}_t) - \frac{1}{1-\alpha}(\check{Z}_t - \alpha \check{Y}_t)
\end{aligned}
$$

where $\check{MPL}_t$ is the average marginal product in the economy. Therefore,

$$
\begin{aligned}
\check{MC}_{t+k|t} &= (\check{W}_{t+k} - \check{P}_{t+k}) - \check{MPL}_{t+k|t} \\
&= (\check{W}_{t+k} - \check{P}_{t+k}) - \frac{1}{1-\alpha}(\check{Z}_{t+k} - \alpha \check{Y}_{t+k|t}) \\
&= \check{MC}_{t+k} + \frac{\alpha}{1-\alpha}(\check{Y}_{t+k|t} - \check{Y}_{t+k}) \\
&= \check{MC}_{t+k} - \frac{\alpha\sigma}{1-\alpha}(\check{P}_t^* - \check{P}_{t+k})
\end{aligned}
$$

where we used both the expression for firm's demand and the expression for the marginal cost derived above. Note that if you have CRS ($\alpha = 0$), MC does not depend on the level of production.

Using this formula for the markups, we can compute the optimal reset price as:

$$\check{P}_t^* - \check{P}_{t-1} = (1-\beta\theta)\sum_{k=0}^{+\infty}(\beta\theta)^k \mathbb{E}_t\big(\Theta \check{MC}_{t+k} + (\check{P}_{t+k} - \check{P}_{t-1})\big)$$

where $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\sigma}$. Then:

$$\begin{aligned}
\check{P}_t^* - \check{P}_{t-1} &= (1-\beta\theta)\Theta\check{MC}_t + (1-\beta\theta)\mathbb{E}_t(\check{P}_t - \check{P}_{t-1}) \\
&\quad +(1-\beta\theta)\sum_{k=1}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big(\Theta\check{MC}_{t+k} + (\check{P}_{t+k} - \check{P}_{t-1})\big) \\
\check{P}_t^* - \check{P}_{t-1} &= (1-\beta\theta)\Theta\check{MC}_t + (1-\beta\theta)\mathbb{E}_t(\check{P}_t - \check{P}_{t-1}) \\
&\quad +(1-\beta\theta)\beta\theta\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big(\Theta\check{MC}_{t+k+1} + (\check{P}_{t+k+1} - \check{P}_{t-1})\big) \\
\check{P}_t^* - \check{P}_{t-1} &= (1-\beta\theta)\Theta\check{MC}_t + (1-\beta\theta)\pi_t + (1-\beta\theta)\beta\theta\sum_{k=0}^{+\infty}(\beta\theta)^k(\check{P}_t - \check{P}_{t-1}) \\
&\quad +(1-\beta\theta)\beta\theta\sum_{k=0}^{+\infty}(\beta\theta)^k\mathbb{E}_t\big(\Theta\check{MC}_{t+k+1} + (\check{P}_{t+k+1} - \check{P}_t)\big) \\
\check{P}_t^* - \check{P}_{t-1} &= (1-\beta\theta)\Theta\check{MC}_t + \pi_t + \beta\theta\mathbb{E}_t[\check{P}_{t+1}^* - \check{P}_t]
\end{aligned}$$

Now use the law of motion for aggregate price level $\pi_t = (1-\theta)(\check{P}_t^* - \check{P}_{t-1})$ to get:

$$\check{\pi}_t = \beta\mathbb{E}_t\check{\pi}_{t+1} + \kappa\check{MC}_t$$

where $\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$. By iterating this equation forward, we find that inflation is the present value of ***future*** marginal costs:

$$\check{\pi}_t = \kappa\sum\beta^s\mathbb{E}_t\check{MC}_s$$

Observe that inflation is a ***forward looking variable***, which may mean that inflation should have low inertia in the model. Importantly, inflation in our model results from the aggregate consequences of purposeful ***optimizing*** price-setting decisions of firms which adjust prices in light of current and future cost conditions.

Now let's relate the marginal cost to measures of aggregate economic activity:

$$\begin{aligned} \check{MC}_t &= (\check{W}_t - \check{P}_t) - \check{MPL}_t \\ &= (\psi \check{Y}_t + 1/\eta \check{L}_t) - (\check{Y}_t - \check{L}_t) \\ &= \Big(\psi + \frac{1/\eta + \alpha}{1-\alpha}\Big)\check{Y}_t - \frac{1 + 1/\eta}{1-\alpha}\check{Z}_t, \end{aligned}$$

where we have used the "aggregate" production function analogue to compute the marginal product of labor $\check{MPL}_t$ and first-order condtion for labor supply to replace $(\check{W}_t - \check{P}_t)$.

Now let's denote the level of output under flexible prices as Y^N. In a frictionless world: $P_i = P,\ Y_i = Y,\ \theta = 0$. Thus from the FOC for the optimal reset price we have $1 - \frac{\sigma}{\sigma-1}MC^N = 0$ and consequently $MC^N = \frac{\sigma-1}{\sigma} = 1/\mu$. Hence, the marginal cost in the frictionless world MC^N is fixed. At the same time, if we replace output with the natural rate of output in the equation for the percent deviation of the marginal cost, we get

$$\check{MC}_t^N = (\psi + \frac{1/\eta + \alpha}{1-\alpha})\check{Y}_t^N - \frac{1 + 1/\eta}{1-\alpha}\check{Z}_t$$

Since MC^N does not vary over time, $\check{MC}^N = 0$ and hence:

$$\begin{aligned} \check{Y}_t^N &= \frac{1 + 1/\eta}{(1-\alpha)\psi + 1/\eta + \alpha}\check{Z}_t \Rightarrow \\ \check{MC}_t &= \Big(\psi + \frac{1/\eta + \alpha}{1-\alpha}\Big)(\check{Y}_t - \check{Y}_t^N) \end{aligned}$$

where $\check{Y}_t - \check{Y}_t^N$ is the ***output gap***, i.e. the discrepancy between the actual output and output that should have been observed if prices were fully flexible. Let $\kappa^* = \kappa(\psi + \frac{1/\eta+\alpha}{1-\alpha})$ and derive the ***New Keynesian Phillips Curve*** (NKPC):

$$\pi_t = \beta \mathbb{E}_t \pi_t + \kappa^*(\check{Y}_t - \check{Y}_t^N)$$

This is the key equation in the Keynesian economics as it links nominal variables (here, inflation) and real variables (here, output gap or marginal cost). Coefficient κ^* (or κ) is intimately related to the amount of real rigidity in the model. Lower values of κ^* correspond to higher levels of real rigidity, i.e., prices (and hence inflation) are less sensitive to movements in real variables with high real rigidity.

Likewise we can modify the Euler equation (6.3) to express it in terms of output gap:

$$(\check{Y}_t - \check{Y}_t^N) = -\frac{1}{\psi}(i_t - \mathbb{E}_t\pi_{t+1} - R_t^N) + \mathbb{E}_t(\check{Y}_{t+1} - \check{Y}_{t+1}^N)$$

where $R_t^N \equiv \psi\mathbb{E}_t\Delta\check{Y}_{t+1}^N = \frac{\psi(1+1/\eta)}{\psi(1-\alpha)+1/\eta+\alpha}\mathbb{E}_t\Delta Z_{t+1}$ and hence $\check{Y}_t^N = \mathbb{E}_t(\check{Y}_{t+1}^N) - \frac{1}{\psi}R_t^N$. This gives us the ***IS curve***:

$$(\check{Y}_t - \check{Y}_t^N) = -\mathbb{E}_t\frac{1}{\psi}(\sum_{k=0}^{+\infty}(R_{t+k} - R_{t+k}^N)) = -\frac{1}{\psi}(R_t - R_t^N) + \mathbb{E}_t(\check{Y}_{t+1} - \check{Y}_{t+1}^N)$$

where $R_t = i_t - \mathbb{E}_t\pi_{t+1}$ is the real interest rate. Here the current output gap $(\check{Y}_t - \check{Y}_t^N)$ depends on the ***path*** of future short-term interest rates (not just on the current rate!). Why is this so important for macroeconomics? Central bankers control only the short-term rate but if they can promise the ***path*** of interest rates they can anchor expectations in such a way that they will effectively stabilize output.

To close the model, we have several options:

- ***Taylor rule***: $i_t = \phi_\pi\pi_t + \phi_y(\check{Y}_t - \check{Y}_t^N) + v_t$ where v_t is a monetary policy shock.
- Money demand: $(\check{Y}_t - \check{Y}_t^N) - \zeta i_t = (\check{M}_t - \check{P}_t) - \check{Y}_t^N$ where $\check{M}_t - \check{P}_t$ is real money balances (or liquidity) and ΔM_t is a money shock: $\Delta M_t = \rho_M\Delta M_{t-1} + \epsilon_t^M$.

The Taylor rule has more empirical appeal since present-day central bankers' behavior is reasonably well approximated with this rule.

In summary, the basic New Keynesian model has three equation: IS curve, NKPC curve, and policy reaction function such as the Taylor rule.

6.4 Interpretation and solution

In the previous lecture we derived the three equation system of equations which constitute the simplest New Keynesian macroeconomic model. In the end we have the following equations:

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa^*\check{X}_t \tag{6.5}$$

$$\check{X}_t = -\frac{1}{\psi}(i_t - \mathbb{E}_t\pi_{t+1} - R_t^N) + \mathbb{E}_t\check{X}_{t+1} \tag{6.6}$$

$$i_t = \phi_\pi\pi_t + \phi_y\check{X}_t + v_t \tag{6.7}$$

where $\check{X}_t = (\check{Y}_t - \check{Y}_t^N)$ is output gap, v_t is a monetary policy (or liquidity) shock, i_t is the deviation of the nominal interest rate from its steady state value. Equation (6.5) is the New Keynesian Phillips Curve (supply side), equation (6.6) is the IS curve (demand side). Equation (6.7) describes the reaction function (Taylor rule) for monetary policy. This modern formulation of old ideas from Keynes has several advantages:

- **Results of optimization problems:** all these curves are results of optimization problems. This is good since old-school Keynesian analysis "started with curves": the macroeconomic relationships started with assumed equations rather than equations derived from micro foundations.

- **Structural parameters:** we can do policy experiments. We derive equations with coefficients based on structural ("deep", "fundamental") parameters of the production function, utility, and process of price adjustment (not quite derived in the Calvo model).

- **Role of expectations:** In contrast to old-style Keynesian, forward looking behavior plays the fundamental role.

Now we can eliminate the interest rate equation and reduce the system of equations to:

$$\begin{bmatrix} \check{X}_t \\ \pi_t \end{bmatrix} = A\mathbb{E}_t \begin{bmatrix} \check{X}_{t+1} \\ \pi_{t+1} \end{bmatrix} + B(\check{R}_t^N - v_t) \tag{6.8}$$

$$
\begin{aligned}
A &= \Omega \begin{bmatrix} \psi & 1-\beta\phi_\pi \\ \psi\kappa^* & \kappa^* + \beta(\psi+\phi_y) \end{bmatrix} &(6.9)\\
\Omega &= \frac{1}{\psi+\phi_y+\kappa^*\phi_\pi} &(6.10)\\
B &= \Omega \begin{bmatrix} 1 \\ \kappa^* \end{bmatrix} &(6.11)
\end{aligned}
$$

We know from Blanchard and Kahn (1980) that to have a unique stable rational expectations equilibrium, we must have that the number of stable eigenvalues must be equal to the number of predetermined variables when we work with a system like:

$$
\mathbb{E}_t \begin{bmatrix} \check{X}_{t+1} \\ \pi_{t+1} \end{bmatrix} = \Pi_0 \begin{bmatrix} \check{X}_t \\ \pi_t \end{bmatrix} \tag{6.12}
$$

In our system all variables are jump variables (non-predetermined). Hence all eigenvalues must be unstable (i.e., greater than one; recall that $\Pi_0 = A^{-1}$ and hence [unstable] eigenvalues in Π_0 correspond to [stable] eigenvalues in A.). One can show that a necessary and sufficient condition for this to hold is:

$$
(1-\beta)\phi_y + \kappa^*(\phi_\pi - 1) > 0 \tag{6.13}
$$

Intuitively, monetary authority should respond to deviations of inflation and output gap from their natural levels by adjusting interest rate sufficiently strongly. How does it work? Suppose that inflation increases permanently by $\Delta\pi$. Then:

$$
\begin{aligned}
\Delta i &= \phi_\pi \Delta\pi + \phi_y \Delta X \\
&= \phi_\pi \Delta\pi + \frac{\phi_y(1-\beta)}{\kappa^*}\Delta\pi \\
&= \Big(\phi_\pi + \frac{\phi_y(1-\beta)}{\kappa^*}\Big)\Delta\pi
\end{aligned}
$$

where the second equality follows from the IS curve. Note that equation (6.13) is equivalent to:

$$\phi_\pi + \frac{\phi_y(1-\beta)}{\kappa^*} > 1 \tag{6.14}$$

which is often called the (modified) Taylor principle.[2] This condition implies that for any increase in inflation $\pi > 0$, the ***real*** interest increases too. That is, the nominal rate increases so much that the real rate increases with rising inflation. This increased real interest rate depresses economic activity, output gap falls, inflation pressure cools down and, hence, the policy maker keeps inflation in check. In summary, if we want to have a unique determinate rational expectations equilibrium, the policy reaction function has to be sufficiently aggressive to inflation.

Interestingly, if the Taylor principle is satisfied, then the central bank can set policy such that both output gap and inflation will be zero (e.g., implement an inflation target). That is, the output gap and inflation are simultaneously stabilized at their steady state level. This is sometimes called the "***divine coincidence***". When there are also no shocks to the natural level of output (no supply-side shocks), then is equivalent to stabilizing inflation and output.

We first study the simplest version of the new Keynesian model, where all shocks v_t and $\check{Z}_t$ are iid with mean zero. In this case all expectations of variables dated $t+1$ are zero. Why? Then model is entirely forward looking (there are no $t-1$ variables), so anything that happens at t has no bearing on what happens at $t+1$. So then we expect that at $t+1$ we are back at the steady-state. Second, we will for now ignore the feedback effects of inflation and output to the nominal interest rate. Our model then becomes:

$$\begin{aligned} \pi_t &= \kappa^*(\check{Y}_t - \check{Y}_t^N) && (6.15) \\ \check{Y}_t &= -\frac{1}{\psi} i_t && (6.16) \\ i_t &= v_t && (6.17) \end{aligned}$$

[2]Coibion and Gorodnichenko (2011) argue that with positive trend inflation the policymaker has to be much more aggressive on inflation than predicted by the Taylor principle.

We can know qualitatively examine what happens in this model given a one-time interest rate shock v_t or a one-time technology shock $\check{Z}_t$. First, an increase in v_t raises the nominal interest rate which lowers output. This is because the real interest rate rises ($R_t = i_t - E_t\pi_{t+1} = i_t$), so consumers reduce consumption today relative to consumption tomorrow. The decline in output in turn lowers inflation. By contrast, a positive technology shock raises the natural level of output, *but not the actual level of output.* Here output is tied down by the nominal (and real) interest rate. Thus, the behavior of this sticky price model is very different to that of the RBC model. In fact the comparative statics are very similar to an undergraduate ISLM model, but these are derived based on micro-foundations and market clearing.

To study the dynamics of this model with policy rule feedback, suppose that:

$$v_t = \rho_v v_{t-1} + \epsilon_t^v \tag{6.18}$$

which is the shock to liquidity or monetary policy shock. Innovation $\epsilon_t^v > 0$ is interpreted as a monetary contraction. Here, the only predetermined variable is the exogenous liquidity shock v_t. Given our previous experience with solving rational expectations models, we guess a solution:

$$\begin{aligned}
\check{X}_t &= c_{xv} v_t \\
\pi_t &= c_{\pi v} v_t
\end{aligned}$$

and using the method of undetermined coefficients, we find that:

$$\begin{aligned}
\check{X}_t &= -(1-\beta\rho_v)\Gamma_v v_t \\
\pi_t &= -\kappa^* \Gamma_v v_t \\
\Gamma_v &= \left\{(1-\beta\rho_v)\left[\psi(1-\rho_v)+\phi_y\right] + \kappa^*\left[\phi_\pi - \rho_v\right]\right\}^{-1}
\end{aligned}$$

If the stability condition (here the Taylor principle) is satisfied, $\Gamma_v > 0$. The expression for the nominal and real interest rate is:

$$\begin{aligned} R_t &= \psi(1-\beta\rho_v)(1-\rho_v)\Gamma_v v_t \\ i_t &= R_t + \mathbb{E}_t\pi_{t+1} = \left[\psi(1-\beta\rho_v)(1-\rho_v) - \rho_v\kappa^*\right]\Gamma_v v_t \end{aligned}$$

Note that if ρ_v is sufficiently large, the nominal rate can decrease when v rises. Intuitively, the direct effect of the increase due to v_t is offset by declines in inflation and output gap.

6.5 Analysis of the model

The model does not have a simple algebraic solution and taking derivatives is cumbersome. We will solve the model numerically. We calibrate the model as follows:

- $\psi = 1$ (log utility)
- $\eta = 1$ (unit elasticity of labor)
- $\sigma = 6$
- $\alpha = 1/3$
- $\phi_y = 0.125,\ \phi_\pi = 1.5$ (roughly the Taylor rule estimated in the data)
- $\theta = 2/3$ (roughly the duration of the price spell equal to 3 quarters)

Why should nominal shocks have real effects in this model? When $\Delta M > 0$ (or $\Delta v_t < 0$), a fraction of firms can change their prices. These firms could have adjusted their prices by ΔM but they don't want to do it. Why? Because by setting a high nominal price, you also set a high relative price P_{it}/P_t, and lose demand. Hence the gain in profit from setting the price to new steady state level isn't the maximum a firm can achieve. The price-adjusting firm has to take into account that other firms do not reset their prices. In other words, firms are looking backward (at non-adjusting firms) and forward (into future periods when the now price-adjusting firms may be unable to adjust their prices again). This is where **strategic complementarity** is important. The actions of price-adjusting firms depend on actions of other firms that cannot adjust prices at the time of the shock. Therefore even when a firm can adjust its price this firm doesn't do it by ΔM. Next period, another subset of firms is going to reset their prices and again they will refrain from moving their prices by ΔM and they will look at the price level P_t.

Figure 6.1: Response to contractionary monetary shock.

This ***staggered price setting*** (i.e., the timing of price changes is not synchronized across firms) creates incentives to dampen price changes of firms that adjust prices. If firms were able to adjust prices simultaneously, then they could have reached higher profits (this is where strategic complementarity plays the key role) but to do this firms would need to coordinate. Note that coordination is prevented by the assumption that only a fraction of firms is allowed to adjust prices (this is where ***staggered*** price setting is important).

Figure 1 presents the impulse responses of inflation, output, interest rate, and the liquidity shock to a shock in liquidity ("tight" monetary policy).

We can perform a similar analysis for the technology shock: $\check{Z}_t = \rho_Z \check{Z}_{t-1} + \epsilon_t^z$. Recall that:

$$\begin{aligned} R_t^N &= -\psi \mathbb{E}_t \Delta \check{Y}_{t+1}^N = -\psi \frac{(1+1/\eta)}{\psi(1-\alpha)+1/\eta+\alpha} \mathbb{E}_t \Delta \check{Z}_{t+1} \\ &= \psi \frac{(1+1/\eta)}{\psi(1-\alpha)+1/\eta+\alpha} (1-\rho_z) \check{Z}_t \end{aligned}$$

Without loss of generality, abstract from monetary shocks v_t. Let's guess a solution:

$$\begin{aligned} \check{X}_t &= c_{xz} \check{Z}_t \\ \pi_t &= c_{\pi z} \check{Z}_t \end{aligned}$$

Using the method of undetermined coefficients we can find that:

$$\begin{aligned}
\check{X}_t &= (1-\beta\rho_z)\Gamma_z \mathbb{R}_t^N \\
&= -\psi\frac{(1+1/\eta)}{\psi(1-\alpha)+1/\eta+\alpha}(1-\rho_z)(1-\beta\rho_z)\Gamma_z \check{Z}_t \\
\pi_t &= \kappa^*\Gamma_z R_t^N \\
&= -\psi\frac{(1+1/\eta)}{\psi(1-\alpha)+1/\eta+\alpha}(1-\rho_z)\kappa^*\Gamma_z \check{Z}_t \\
\Gamma_z &= \left\{(1-\beta\rho_z)\left[\psi(1-\rho_z)+\phi_y\right]+\kappa^*\left[\phi_\pi-\rho_z\right]\right\}^{-1}
\end{aligned}$$

We can also derive the evolution for the level of output.

$$\begin{aligned}
\check{Y}_t &= \check{Y}_t^N + \check{X}_t \\
&= \psi\frac{(1+1/\eta)}{\psi(1-\alpha)+1/\eta+\alpha}\left[1-\psi(1-\rho_z)(1-\beta\rho_z)\Gamma_z\right]\check{Z}_t
\end{aligned}$$

And employment:

$$(1-\alpha)\check{L}_t = \check{Y}_t - \check{Z}_t$$

Figure 6.2: Response to 1% technology shock.

Why does employment fall at the time of a technology shock? Generally, a firm with a positive technology shock would like to produce and sell more. However, in the sticky price model when the price is fixed, the amount of demand is effectively fixed. Given that capital is also fixed in the short run, we have from the production function:

$$\bar{Y} = F(\bar{K}, L)Z$$

So when $Z\uparrow$, $L\downarrow$. While some firms in the Calvo model can adjust prices, they will not lower them as much as when prices are flexible because they may not be able to raise prices again once technology dissipates. Consequently prices fall less and the (real) interest rates declines by less than in a flexible-price (RBC) world. This limits demand and implies lower employment. In contrast in RBC models, $Z \uparrow$ is almost always associated with $L \uparrow$.

6.6 Model evaluation

We are back to the question: would sticky prices alter the basic RBC model to produce long-lasting effects in response to shocks (which are weakly correlated)? Recall that Cogley and Nason, Rotemberg and Woodford, and others suggested that the basic RBC model to a 1st order approximation is equal to the exogenous shock, i.e. internal propagation mechanism is weak. We argued before that sticky prices lead to countercyclical markups which make labor demand curve flatter and thus help to amplify and propagate shocks in the model.[3] Hence, we know that qualitatively this mechanism (sticky prices) helps to match the properties of the data. The question is then whether quantitatively sticky prices make a difference.

Chari, Kehoe, and McGrattan (1998) suggest that sticky price models are unable to match the volatility and persistence observed in the data. The low volatility (low to some extent persistence) in their model stems from low levels of real rigidity (they call it "contract multiplier"). They consider various mechanisms (in isolation) and conclude that sticky price models don't have a strong "contract multiplier" (i.e., high real rigidity). In other words, the model needs very low values of κ (which corresponds to high real rigidity) to match the data. It is true that inflation is much more persistent in the data than in the model. In the model, serial correlation of the inflation rate is typically somewhere near 0.5 (recall that in our model inflation is a forward looking variable and this can explain intuitively the low serial correlation). In the data, the inflation rate has enormous serial correlation:

[3]In this respect, sticky prices are similar to IRS in production or variable capital utilization as these two alternative mechanisms also make labor demand curve flatter. In other words, models with sticky prices, IRS and variable capital utilization will have similar properties for key macroeconomic variables such as output, labor supply, etc.

$\rho(\pi_t, \pi_{t-1}) \approx 0.9 - 0.95$. In fact, the persistence of inflation in the data used to be almost a unit root but fell recently.

It's possible that none of the individual mechanisms considered in Chari et al. is capable of generating low κ^* plausibly, but combinations can lead to plausibly very low κ^*. Recall that sources of high real rigidity could be:

Flat marginal cost:

- variable capital utilization
- sticky intermediate goods prices (includes sticky wages)
- elastic labor supply

Steep marginal revenue:

- countercyclical target markups
- kinked/concave demand curves

What is the size of the real rigidity parameter κ^* in the data? We can estimate κ^* directly from the Phillips curve:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa^* \check{X}_t$$

Small κ^* is associated with large real rigidity and we need small $\kappa^* > 0$ to have interesting and important implications from the sticky price models. Chari et al. (1998) argue we need $\kappa^* = 0.05$ to generate persistent response to shocks.[4] In the data, κ^* is close to zero and, in fact, it's so close to zero that it is barely statistically significant (sometimes has the wrong sign). So the issue is not that the real rigidity is small in the data. Instead the issue seems

[4]They use Taylor pricing frictions rather than Calvo pricing frictions, so this is only an approximate mapping. In fact, in our simple Calvo model the persistence of output is exogenously given by the persistence of the shock and independent of κ^*. Nevertheless, κ^* is still an important determinant of the amplification of shocks in the Calvo model.

to be that the real rigidity is very high; so high that we start to question its plausibility given our educated guess about how structural parameters ***in our model*** should look like. We can fix this problem by considering more sophisticated models that lead to greater real rigidities (flatter marginal cost or steeper marginal revenue) and then we can generate low κ^* with plausible structural parameters.

A more important issue is that inflation is quite persistent in the data while in the model it is not. This result is not surprising given that inflation is a forward looking variable. To make the model more persistent, we can allow "backward looking" (not fully optimizing) firms. To capture inflation persistence, it is typical to add:

- "rule-of-thumb" firms
- "indexation": firms adjust their prices by $\bar{\pi}$ when they cannot fully reoptimize

The implied Phillips curve is then:

$$\pi_t = (1-\phi)\beta\mathbb{E}_t\pi_{t+1} + \phi\pi_{t-1} + \kappa\check{X}_t$$

The unappealing feature of the backward looking component of this modified Phillips curve is that it is ad hoc (why are some firms not optimizing?), but it makes inflation more persistent. In the data, $\phi \approx 0.25$ but this estimate is sensitive to variations in estimation technique, sample, etc. This is a difficult problem to address in the framework of the Calvo model. The inability of the Calvo model to generate persistent inflation induced the search for pricing models with persistent inflation rates and we discuss one of the models which can resolve this problem below.

6.7 A Model With Informational Friction

One of the issues we discovered in the New Keynesian model with sticky prices was that inflation does not have enough persistence. This is not surprising because inflation in that model is a forward looking variable and persistence in the inflation rate can arise only through the persistence of forcing variables and endogenous state variables like capital stock (which we did not have in the baseline Calvo model). In the previous lecture, we noted that adding backward-looking firms may help to generate higher persistent of inflation in the model. However, this modification in the model was rather ad hoc while our objective is to derive firm's actions as optimizing responses. A promising line of research which can lead to greater persistence of inflation is incorporation of informational frictions into pricing decisions. Intuitively, with informational frictions, firms which adjust their prices may use incomplete or outdated information sets which do not contain information about aggregate shocks such as shock to monetary policy. One popular model with informational frictions is due to Mankiw and Reis (2002). The key advantage of this model is that (apart from generating persistent inflation) it leads to tractable dynamics. Although we are going to make a number of simplifying assumptions in this model, many of these assumptions were later derived as optimizing responses. Hence, we should not worry too much about the shortcuts we make. To simplify notation, we will assume that all variables are already log-linearized around steady state.

6.7.1 Firms

Firm's pricing and information update decisions are summarized as follows:

- fully flexible prices, there is not menu cost;
- information is not perfect;
- update information with Calvo probability $1-\theta$ (note the difference with the Calvo model in which information is perfect but price is reset with probability $1-\theta$);
- optimal desired price is $p_t^* = p_t + \phi y_t$ with no frictions, where ϕ captures real rigidity;
- a firm that updated its information set j periods ago chooses $x_t^j = \mathbb{E}_{t-j} p_t^*$ as its price.

6.7.2 Sticky Information Phillips Curve (SIPC)

The price level in this economy is:

$$p_t = \sum_{j=0}^{+\infty} w_j x_t^j = \sum_{j=0}^{+\infty}(1-\theta)\theta^j x_t^j = \sum_{j=0}^{+\infty}(1-\theta)\theta^j \mathbb{E}_{t-j}(p_t + \phi y_t)$$

where x_t^j is the optimal reset price for period t given information set updated at period $t-j$. After some algebra (please consult Mankiw and Reis (2002) for more details), we can derive the sticky-information Phillips curve (SIPC):

$$\pi_t = \frac{\phi(1-\theta)}{\theta} y_t + (1-\theta)\sum_{j=0}^{+\infty}\theta^j \mathbb{E}_{t-1-j}(\pi_t + \phi \Delta y_t)$$

There are several key differences between sticky-information (SIPC) and sticky-price (NKPC) Phillips curves:

- NKPC: π_t depends on ***current*** expectations about the ***future*** economic conditions
- SIPC: π_t depends on **past** expectations about the **current** economic conditions

As a result, the dependance on the past is much stronger in the SIPC than in the NKPC.

To close the model, Mankiw and Reis assume: $m = p + y$ and get:

$$p_t^* = \phi m_t + (1-\phi)p_t$$

6.7.3 Response to shocks

Consider a positive shock to the stock of money Δm_t. Eventually, all prices will increase by Δm_t but this will take time.

- Period 0 (impact response):

- At the time of the shock, only a fraction of firms updates information and hence responds to the shock.
- Response is not full strength (i.e., Δm_t) because of the relative demand constraints ($\phi \neq 1$). With a small ϕ, adjusting firms will only have a small adjustment. This is the place where strategic complementarity (i.e., real rigidity) is important.
- Other firms are ignorant and do not adjust their prices.
- In summary, price level moves only by a small amount.

◦ Period 1:

- More firms learn about the shock to Δm_t.
- They adjust prices. So do firms that have updated information in period 0 because they know about the shock and can compute and charge the optimal path of their prices in all future periods (recall that prices are flexible).
- However, adjustment is not complete because there is still a fraction of firms that do not know about the shock and hence continue to charge low (relative to new steady state) prices. Since there is strategic complementarity, full adjustment of prices by Δm_t is not optimal.

◦ Period 2:

- more firms learn about price adjustment
- learning continues
- more price adjustment

Over time, as more and more firms learn about the price change, the rate at which price level changes (i.e., the inflation rate) can accelerate. After some point in time, sufficiently many firms have learned about the shock and the rate at which price level changes (i.e., the inflation rate) slows down and gradually converges to zero. In this model, it is possible to generate a hump-shaped response of inflation (which we see in the data) to a nominal shock which is very hard to generate in the forward-looking NKPC model. Note that the gradual adjustment in the SIPC stems from two things:

- real rigidities (which is determined by strategic complementarity in the demand functions for firms).
- some firms use outdated information sets.

6.7.4 Realism

Is it realistic and optimal to assume that firms update their information infrequently and in some random fashion? It seems to be a realistic assumption given that firms face significant costs of collecting and processing information. However, firms may keep track of some key variables and hence can respond to news (esp. big news much faster than the sticky information model predicts). It's less realistic that information updates are random. Relaxing this assumption is an area of active research.[5] In any case, the central contribution of the sticky-information model is that imperfect information can help to generate more persistent deviations of variables (esp. inflation) from their steady states.

6.8 Taylor Model

We considered a model where price adjustment was random (Calvo model). A popular alternative is the Taylor (1979) model. Instead of stochastic duration of the price spell (contract between sellers and buyers), the duration of the price spell (contract) is fixed at a known number of periods (say, 1 or 3 years). After a shock, only a fraction of firms reset their prices because other firms are going to be bound by their contracts. Price adjustment

[5] Gorodnichenko (2008) shows how endogenizing the timing of information updates can affect the response of inflation to nominal shocks. He also show that learning about nominal shocks from price adjustment of other firms can greatly increase the rigidity of price adjustment, can lead to hump-shaped response of inflation to nominal shocks and these results can be achieved even when there is no strategic complementarity in demand functions.

for firms that can adjust is only partial because these firms do not want to set a very high **relative** price. In other words, inability of some firms to change their prices will slow down price adjustment of firms that can adjust their prices. This is the essence of **staggered price adjustment**.[6] The intuition is very similar to the Calvo model. Many of the results analyzed in the Calvo framework can be reproduced in the Taylor framework, which is effectively a model with **overlapping generations** (OLG) of price setters.

However, this model cannot solve the persistence problem. So the staggered wage or price setting as in the Taylor model does not help to improve much upon the Calvo model. The disadvantage of working with the overlapping contracts models is that the state space can be quite large (number of vintages of firms), and it maybe hard to see the intuition. At the same time, the model maybe more realistic relative to the random probability of adjusting prices. There are different flavors of OLG-type models:

- prices are fixed for the duration of the contract (Taylor model).
- prices are predetermined (i.e. can grow at some rate) during the contract (Fischer model).

Despite these differences, models with overlapping contracts lead to very similar quantitative results.

6.9 State-dependent Models

So far we have assumed that the timing of price adjustment was a function of time (duration of the contract), or some fixed probabilities that do not vary with prices or macroeconomic conditions. These models are often called models with **time-dependent** pricing. On the other hand, it may be more realistic to model firms as changing prices in response to current

[6] In the Mankiw-Reis model, it is staggered information updating which leads to nominal non-neutrality. Recall that in their model prices are entirely flexible.

conditions, i.e. **state-dependent pricing**. One of these models is due to Caplin and Spulber (1987). This model is not highly realistic but it makes a striking point!

Assume that nominal GDP grows at a constant rate and hence M and P grow at constant rates in steady state. It is optimal for firms to follow an Ss policy: as long as $p^* - p \in [\underline{S}, \overline{S}]$, the firm does not adjust its price. In other words, a firm does adjust its price when the price moves outside of $[\underline{S}, \overline{S}]$ and this firm does not adjust its price if its price is within $[\underline{S}, \overline{S}]$, which is called the **band of inaction**.

We need to make a few more assumptions to make the analysis tractable. We will assume "no bunching" (i.e., prices are smoothly distributed over $[\underline{S}, \overline{S}]$). Specifically, we will assume that there is a uniform distribution of prices on $[\underline{S}, \overline{S}]$. Similar to our previous analyses, assume $p^* = (1-\phi)p + \phi m$ and $y - p = m$.

Now if $\Delta m \uparrow$ and $\Delta m < (\overline{S} - \underline{S})$: $\Delta p^* = (1-\phi)\Delta p + \phi \Delta m$. Firms with the initial price p_i: $p_i^* - p_i > \overline{S} - [(1-\phi)\Delta p + \phi \Delta m]$ change their price. How many of these firms do we have? Prices are distributed uniformly on $[\underline{S}, \overline{S}]$, so $[(1-\phi)\Delta p + \phi \Delta m]/[\underline{S}, \overline{S}]$ firms will be pushed out of the band of inaction. When $p_i^* - p_i$ hit $\overline{S}$, price is increased by $\overline{S} - \underline{S}$.

Therefore the change in price level is:

$$\begin{aligned} \pi &= \Delta p = \text{ \# firms changing prices } \times \text{ average price change} \\ &= \frac{\Delta p(1-\phi) + \phi \Delta m}{\overline{S} - \underline{S}}(\overline{S} - \underline{S}) = \Delta p(1-\phi) + \phi \Delta m = \Delta m \end{aligned}$$

It follows that money is neutral:

$$\Delta y = \Delta m - \Delta p = 0$$

This is a very striking result! There are (a) ***staggered*** price changes (i.e., firms change prices in different times), (b) a ***menu cost*** (nominal rigidity), (c) ***real rigidity*** (i.e., $\phi \in (0,1)$ in $p^* = (1-\phi)p + \phi m$), and yet there is complete neutrality of nominal shocks.

Why does this happen? The distribution of prices in the band $[\underline{S}, \overline{S}]$ does not change.

Firms that adjust the price adjust it so much that it compensates for the non-adjustment of other firms. Which firms adjust? These are firms with prices ***furthest*** away from the optimal price. These firms will have big price adjustments. So big that these price adjustments are sufficient to compensate for not adjusting firms. In other words, the intensive margin has such a big response that it dominates a relatively weak response of the extensive margin. This is called the "selection effect" because only a special group of firms will adjust prices.

This is an amazing result, especially if put in the historical context. Economists expected that state-dependent pricing was going to produce more sluggish price adjustment than time-dependent pricing. The Caplin-Spulber model defeated this notion. Furthermore, if firms face idiosyncratic shocks (not just global money shocks), then nominal non-neutrality can be even smaller (see e.g. Golosov and Lucas (2007)). However, this result is somewhat fragile once information is not perfect.[7]

[7] One example is Gorodnichenko (2008).

Chapter 7

Monetary Policy

In this lecture, we will briefly look into design of optimal monetary policy when prices are sticky. We will use the basic Calvo pricing model in our analysis. Although some details depend on specifics of the Calvo model, many results will hold qualitatively for other models with inflexible prices.

7.1 A Place for Monetary policy

There are two types of distortions in the Calvo model:

1. monopolistic competition and hence underproduction relative to perfect competition;

2. sticky prices lead to a) dispersion of relative prices (recall that price setting is staggered) which can lead to inefficient allocation of resources and b) variable levels of markups so that output can be different from output we would observe when prices are flexible.

We cannot fix (1) with monetary policy, but we can fix (2) in some cases. The inefficiency (2) justifies why government should step in and improve welfare. To understand the intuition, suppose we have no price stickiness. Then optimality conditions from consumer and firm problems would imply that:

$$P_t = \mu \frac{W_t}{MPL_t}$$

where $\mu > 1$ is the markup over costs, W/MPL is the nominal marginal cost. Note that the *real* marginal cost is constant! From the household side we have

$$-\frac{U'_{L,t}}{U'_{C,t}} = \frac{W_t}{P_t} = \frac{MPL_t}{\mu} < MPL_t$$

This result is just a manifestation of inefficiency (1). Since we cannot fix (1),[1] we will take this condition and the corresponding level of output which we denote with Y_t^N as the best case we can achieve with optimal monetary policy. Given our functional forms Y_t^N is given by:[2]

$$-\frac{U'_{L,t}}{U'_{C,t}} = \frac{MPL_t}{\mu} \tag{7.1}$$

$$\iff C_t^{\psi} L_t^{\eta^{-1}} = \mu^{-1}(1-\alpha) Z_t L_t^{-\alpha} \tag{7.2}$$

$$\iff (Y_t^N)^{\psi} \left(\frac{Y_t^N}{Z_t}\right)^{\frac{1}{\eta(1-\alpha)}} = \mu^{-1}(1-\alpha) Z_t \left(\frac{Y_t^N}{Z_t}\right)^{\frac{-\alpha}{1-\alpha}} \tag{7.3}$$

$$\iff Y_t^N = \mu^{-1}(1-\alpha) Z_t^{\frac{1+\eta^{-1}}{\psi(1-\alpha)+\eta^{-1}+\alpha}} \tag{7.4}$$

where the third step uses $C_t = Y_t$ as well as the production function to substitute for L_t. Note that high mark-ups cause underproduction relative to the first-best.

[1] We can fix (1) using an employment subsidy τW_t for labor. This yields

$$P_t = \mu(1-\tau)\frac{W_t}{MPL_t}$$

and $\tau = (1-\mu)^{-1}$ will then eliminate the distortion. Intuitively, through the employment subsidy we encourage higher employment and output, thereby eliminating the underproduction.

[2] With the optimal employment subsidy, this simply becomes $Y_t^N = (1-\alpha) Z_t^{\frac{1+\eta^{-1}}{\psi(1-\alpha)+\eta^{-1}+\alpha}}$.

Now if there are sticky prices, then the average markup in the economy becomes:

$$\mu_t = \frac{P_t}{W_t/MPL_t}$$

where μ is the steady-state (desired) markup. We know from previous lectures that the level of output is affected by the variability in actual markup. We need to do something to make $\mu_t = \mu$ at all times to reach the flexible-price levels of output.

There is a related source of inefficiency arising from the dispersion of relative prices. Because firms are going to adjust their prices in different times: $P_t(i) \neq P_t(j) \Rightarrow C_t(i) \neq C_t(j) \Rightarrow L_t(i) \neq L_t(j)$. This violates the efficiency condition that $C_t(i) = C_t$ and $L_t(i) = L_t$ at all times. Recall that since there are diminishing marginal utility to each variety $C_t(i)$, it is most efficient to consume the same of each variety, $C_t(i) = C_t$. To achieve efficiency, we need to equalize the price and marginal cost across firms. Hence, we need to introduce a policy that puts induces not only the stability of markups and marginal costs across firms, but also equality of these things across firms.

Suppose that we attain an equilibrium in which everybody charges the same price: $P_t(i) = P_t$. If this policy is in place forever, no firm will adjust its price and hence we will have no change in markups and no change in the price level:

$$P_t^* = P_{t-1} \Rightarrow P_t = P_{t-1}$$

Price stability follows as a consequence of no firm willing to adjust prices. Thus, if we stabilize the price level, there will be ("divine coincidence"):

- no relative price distortions;
- actual markup is equal to the desired markup, $\mu_t = \mu$;
- $Y_t = Y_t^N$, i.e., output will be at its natural level which can be achieved when all prices are flexible;

- $L_t = L_t^N$, i.e., output will be at its natural level which can be achieved when all prices are flexible;
- output gap is zero, $\check{X}_t = \check{Y}_t - \check{Y}_t^N = 0$;
- inflation rate is zero, $\pi_t = 0$, $i_t = R_t^N$.

Two points here:

- There is no need to stabilize output fully (i.e., $Y = \bar{Y}$); Y_t can fluctuate when Y_t^N is moving. Why? Because the natural rate of output Y_t^N reflects agents *optimal* response to shocks. For instance, when Z is high it makes sense to produce more because you can do so with less labor, thereby enjoying both more leisure and higher consumption.
- Policy makers have generally no particular reason to care about the stability of the price level but price level stability ***becomes*** an objective if they want to track Y_t^N.

Note that monetary policy implements the flexible price allocation (i.e., $Y_t = Y_t^N$) even while keeping prices fixed. When Z is high today relative to tomorrow, then with flexible prices $P_t = \mu\Sigma_t$ will be low relative to tomorrow $P_{t+1} = \mu\Sigma_{t+1}$.[3] Thus $\mathbb{E}_t\pi_{t+1} > 0$, which lowers the natural rate of interest R_t^N and stimulates the demand for the extra output ($\check{Y}_t^N = \check{Y}_{t+1}^N - \psi^{-1}R_t^N$).

When prices are sticky, our optimal policy sets $i_t = R_t^N$. This raises demand by the same amount as the decline in prices did, i.e. $Y_t = Y_t^N$. Then, $\check{X}_t = 0$ and real marginal costs are constant. Therefore there is no reason for firms to charge different prices at different times and $\pi_t = 0$. The fact that $Y_t = Y_t^N$ implies $\pi_t = 0$ and thus an absence of relative price distortions is known as the "Divine Coincidence." It implies that a single instrument (i_t) can simultaneously eliminate both the mark-up distortion ($\mu_t \neq \mu$) and the relative price-distortion ($P_t(i) \neq P_t$). This is a property of our current model but it is not universal. For instance, adding time-variation to σ (the demand elasticity) will break the divine coincidence.

[3] Implicitly we hold monetary policy fixed here, but this is inconsequential since monetary policy is neutral when prices are flexible (see chapter 6).

The central bank would then face a trade-off between achieving $Y_t = Y_t^N$ and minimizing price dispersion.

7.2 Monetary Policy Rules

We need to find rules delivering that Y_t tracks Y_t^N well. Although it is possible to derive optimal rules based on the utility of households and the structure of the model, we will constrain our focus to "simple" interest rate rules, i.e., central bankers commit to a policy which uses the interest rate as the policy instrument. These rules are simple in the sense that the rule is a reaction function of a few (rather than many) macroeconomic variables.

One important question is why we care about rules if we can optimize period by period instead of committing ourselves to a rigid rule. The answer is that commitment to a rule can resolve dynamic inconsistencies in policy. Specifically, one can show that commitment (unlike discretion) does not lead to the inflation bias in the steady state and it does lead to ***less*** volatility of macroeconomic variables.

Recall that, in addition to a policy reaction function, the Calvo model had two key equations (IS and NKPC curves):

$$\begin{aligned} \check{X}_t &= \mathbb{E}_t \check{X}_{t+1} - \frac{1}{\psi}(i_t - \mathbb{E}_t \pi_{t+1} - R_t^N) \\ \pi_t &= \beta E \pi_{t+1} + \kappa^* \check{X}_t \end{aligned}$$

We close this model with the interest rule which must satisfy two minimum requirements:

1. Unique rational expectation equilibrium (REE): no indeterminacy/sunspots.

2. Stable REE: no explosive solutions.

Now we are ready to consider several alternative rules.

7.2.1 Exogenous interest rate

Suppose that the central bank tries to set the interest rate equal to the natural interest rate at all times, i.e.,

$$i_t = R_t^N$$

Let:

$$A = \begin{bmatrix} 1 & 1/\psi \\ \kappa^* & \beta + \kappa^*/\psi \end{bmatrix}$$

Plug $i_t = R_t^N$ in the system of IS and NKPC equations:

$$\begin{bmatrix} \check{X}_t \\ \pi_t \end{bmatrix} = A \times \mathbb{E}_t \begin{bmatrix} \check{X}_{t+1} \\ \pi_{t+1} \end{bmatrix} + \text{ forcing terms}$$

There are two non-predetermined variables. To obtain a unique stable REE we need two stable eigenvalues in the matrix A. But A has one eigenvalue greater than 1; hence there exist multiple equilibria. We can have volatility not associated with changes in fundamentals ("sunspots"). This is a bad policy since it induces unnecessary volatility in the economy.

7.2.2 Current response to economic conditions

Suppose that the central bank allows some feedback from economic conditions into its reaction function. Specifically, consider

$$i_t = R_t^N + \phi_\pi \pi_t + \phi_y \check{X}_t$$

where ϕ_π is the strength of the reaction to inflation and ϕ_y is the strength of the reaction to output gap. In the previous lectures, we had:

$$\begin{aligned}\begin{bmatrix}\check{X}_t \\ \pi_t\end{bmatrix} &= \Omega \begin{bmatrix}\psi & 1-\beta\phi_\pi \\ \psi\kappa^* & \kappa^* + \beta(\psi+\phi_y)\end{bmatrix} \mathbb{E}_t \begin{bmatrix}\check{X}_{t+1} \\ \pi_{t+1}\end{bmatrix} \\ \Omega &= \frac{1}{\psi + \phi_y + \kappa^*\phi_\pi}\end{aligned}$$

The Taylor principle told us:

$$(1-\beta)\phi_y + \kappa^*(\phi_\pi - 1) > 0$$

With full stabilization (i.e., $\pi_t = 0, \hat{X}_t = 0$), we have optimal $i_t = R_t^N$.

7.2.3 Forward-looking interest rate rule

In principle, the policymakers could be forward looking and they could kill inflation before it starts. One way to model this is to assume that the current policy instrument responds to expectations of macroeconomic variables in the future. For example,

$$i_t = R_t^N + \phi_\pi \mathbb{E}_t \pi_{t+1} + \phi_y \mathbb{E}_t \check{X}_{t+1}$$

We use this rule to eliminate the nominal interest rate from IS and NKPC curves to get

$$\begin{bmatrix}\check{X}_t \\ \pi_t\end{bmatrix} = \begin{bmatrix}1-\phi_y/\psi & (1-\phi_\pi)/\psi \\ \kappa^*(1-\phi_y/\psi) & \beta + \kappa^*(1-\phi_\pi)/\psi\end{bmatrix} \mathbb{E}_t \begin{bmatrix}\check{X}_{t+1} \\ \pi_{t+1}\end{bmatrix}$$

Conditions to ensure two stable eigenvalues (i.e. unique stable REE) are:

$$(1-\beta)\phi_y + \kappa^*(\phi_\pi - 1) > 0 \tag{7.5}$$

$$(1+\beta)\phi_y + \kappa^*(\phi_\pi - 1) < 2(1+\beta)\psi \tag{7.6}$$

Equation (7.5) is similar to Taylor's principle. Equation (7.6) requires that the central bank does not respond to shocks too strongly. Otherwise, expectations can become self-fulfilling and we are back to undesirable sunspot volatility.

We should also keep in mind that the strength of the response necessary to guarantee determinacy depends on the level of inflation in the steady state. In general, holding everything else constant, the response to inflation in the Taylor rule is sharply increasing in the level of the steady state level of inflation. Figure 1 (based on Coibion and Gorodnichenko (2011)) shows that with steady state inflation as low as 3 percent per year the response to inflation has to be 2.5 rather than 1 as prescribed by the Taylor principle.

Figure 7.1: Minimum response ϕ_π in $i_t = \phi_\pi \pi_t$ required for determinacy.

7.3 Are Optimal Rules Implementable?

The question now is whether rules which we considered above are actually feasible to use. There are several issues:

- need to observe/know output gap
- need to observe/know real natural interest rate
- need to know variables in real time

These are difficult issues to address and these difficulties motivate simpler rules based only on observables. For example, the Taylor rule can be modified in a variety of ways:

- based on lagged inflation and output: $i_t = \bar{R} + \phi_\pi \pi_{t-1} + \phi_y \check{Y}_{t-1}$

- based on output growth rate rather than output gap: $i_t = \bar{R} + \phi_\pi \pi_t + \phi_y \Delta Y_t$, and the hope is that $\Delta Y_t^N \approx 0$.

We sidestep the issues related to implementation and turn to the question of which rule maximizes welfare.

7.4 Social Welfare

To evaluate the performance of alternative rules, we need a welfare metric. In old Keynesian literature, the social welfare was approximated with the following social loss function:

$$\mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t (\check{X}_t^2 + \omega \pi_t^2)$$

where ω gives the importance of inflation volatility relation to output gap volatility. This turns out to be the right functional form when we do the second order approximation of the consumer's utility. One can show that in our basic New Keynesian model the 2nd order approximation to consumer utility is the following loss function:

$$W = \frac{1}{2}\mathbb{E}_0 \sum_{t=0}^{+\infty} \beta^t \Big(\Big(\psi + \frac{1/\eta + \alpha}{1-\alpha} \Big) \check{X}_t^2 + \frac{\sigma}{\kappa} \pi_t^2 \Big)$$

where σ is the elasticity of substitution across varieties, ψ is the intertemporal elasticity of substitution, η is the Frisch labor supply elasticity, α is the elasticity of output with respect to labor, and κ measures the degree of real rigidity. If we take the expected value, the average loss is:

$$\frac{1}{2}\Big(\Big(\psi + \frac{1/\eta + \alpha}{1-\alpha} \Big) \mathrm{Var}(\check{X}_t) + \frac{\sigma}{\kappa} \mathrm{Var}(\pi_t) \Big)$$

- $\left(\psi + \frac{1/\eta+\alpha}{1-\alpha}\right)\mathrm{Var}(\check{X}_t)$ comes from the markups departing from the desired level. Note that the coefficient on $\mathrm{Var}(\check{X}_t)$ depends on parameters which for a given deviation of output from its natural level amplify the gap between the marginal rate of substitution and the marginal rate of transformation (which is here the marginal product of labor).

- $\frac{\sigma}{\kappa}\text{Var}(\pi_t)$ stems from the dispersion of relative prices. Note that the coefficient on $\text{Var}(\pi_t)$ is a function of the elasticity of substitution across varieties σ (which exacerbates welfare losses when price dispersion increases) and the sensitivity of inflation to marginal cost in the the Phillips curve κ (recall that $\pi_t = \beta E_t \pi_{t+1} + \kappa \check{MC}_t$ where $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta}\Theta$). Note that when $\theta = 0$ (i.e., all prices are flexible), we have $\frac{\sigma}{\kappa} = 0$.

Consider a rule in the spirit of Taylor:

$$i_t = \bar{R} + \phi_\pi \check{\pi}_t + \phi_y \check{Y}_t$$

We use $\check{Y}_t$ instead of $\check{X}_t$ because $\check{X}_t$ is not observed.

$$\begin{bmatrix} \check{X}_t \\ \pi_t \end{bmatrix} = A_t \times \mathbb{E}_t \begin{bmatrix} \check{X}_{t+1} \\ \pi_{t+1} \end{bmatrix} + B_t(\check{R}_t^N - \xi_t)$$

where $\xi_t = \phi_y \check{Y}_t^N$. Note that as ϕ_y increases, the volatility of ξ_t also increases so that the volatility of the error term depends on the policy reaction function. We assume there is only one source of shocks: technology. To assess the properties of this rule, we

- calibrate;
- find reduced-form solution;
- simulate the model;
- compute moments (variance);
- evaluate the social loss function for different choices of ϕ_π and ϕ_y.

Given our standard calibration, simulation results can be summarized as follows:

ϕ_π	1.5	1.5	5	1.5
ϕ_y	0.125	0	0	1
$\mathrm{Var}(\check{X}_t)$	0.55×10^{-2}	0.28×10^{-2}	0.04×10^{-2}	1.4×10^{-2}
$\mathrm{Var}(\pi_t)$	2.6×10^{-2}	1.33×10^{-2}	0.21×10^{-2}	6.55×10^{-2}
Welfare loss	0.3	0.08	0.002	1.92

Table 7.1: Welfare Loss for Various Interest-Rate Rules

- more aggressive response to output leads to a larger welfare loss;
- we obtain the smallest welfare loss with (very) aggressive responses to inflation ($\approx$ optimal policy)

The results can change when we modify the model and introduce more shocks, esp. shocks that move inflation and output gap in different directions. One example is markup shocks (or more generally cost-push shocks):

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa^*\check{X}_t + u_t$$

u_t is a stochastic wage markup - shock to MRS between labor and consumption. For example:

$$\begin{aligned}
\frac{L_t^{1/\eta}}{C_t^{-\psi}}\exp\tilde{u}_t &= \frac{W_t}{P_t} \\
1/\eta\check{L}_t + \psi\check{C}_t + \tilde{u}_t &= \check{W}_t - \check{P}_t \Rightarrow \\
\check{MC}_t &= [(1/\eta)\check{L}_t + \psi\check{C}_t] - (\check{Y}_t - \check{N}_t) + \tilde{u}_t \Rightarrow \\
\pi_t &= \beta\mathbb{E}_t\pi_{t+1} + \kappa^*\check{X}_t + \kappa^*\tilde{u}_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa^*\check{X}_t + u_t
\end{aligned}$$

where $u_t = \kappa^*\tilde{u}_t$. With these additional shocks (like u_t) it is optimal to have a positive response to output. Nevertheless, in general we will find that the model (for plausible calibrations) really dislikes price dispersion and will push us towards a high ϕ_π. Remember that we needed a low κ (high real rigidities) to generate sufficient persistence and amplitude to shocks. But in our model a low κ also implies that the costs of business cycles come primarily from price dispersion rather than the variation in output; a point that New Keynesian models are criticized for.

Keep in mind that we assumed zero inflation in the steady state. The welfare implications of policies aggressive on inflation or/and output can change substantively when we allow inflation to be positive in the steady state. For example, Coibion and Gorodnichenko (2011) argue that it is optimal to have a strong response to the real side of the economy (in particular to the output growth rate) to improve welfare. At the same time, they show that aggressive responses to output gap can be highly destabilizing.

7.5 Persistence of Optimal Policy Rules

Recall that we can use repeated substitutions to express the IS curve as follows

$$\check{X}_t = -\frac{1}{\psi}\sum_{s=0}^{+\infty}(R_{t+s} - R^N_{t+s})$$

which means that output gap is a function of current and future ***short-term*** interest rates. Hence, if the central banker controls the short rate only in the current period, his or her policy may be relatively ineffective because it will not be able to move output gap by large amounts ceteris paribus. To make the policy more powerful, the central banker must control not only the current interest rates but also the future interest rates (which means the central banker must influence the long-term interest rates). One way a central banker can promise the path of short-term interest rates is to to introduce "interest rate smoothing":

$$i_t = (1-\rho)(\bar{R} + \phi_\pi \pi_t + \phi_y \check{X}_t) + \rho i_{t-1}$$

In this formulation, the short run responses to inflation and output gap are $(1-\rho)\phi_\pi,\ (1-\rho)\phi_y$. The long run responses are $\phi_\pi,\ \phi_y$ which could be much larger than the short run response. More importantly, parameter ρ controls not only the size of the LR and SR responses but also the ***path*** of future interest rates. Hence with interest rate smoothing the

Figure 7.2: Response of the price level to an inflationary shock.

central banker has more power to control expectations about future interest rates.

Another powerful alternative is to target the price level:

$$i_t = \phi_p \check{P}_t + \phi_\pi \pi_t + \phi_y \check{X}_t$$

The difference between price level targeting (PLT) and inflation targeting (IT) is summarized by Figure 2. An inflation shock has a permanent effect on the level of prices under IT and a transitory effect under PLT. Price level targeting is particularly useful in controlling inflationary expectations since any inflation today will be followed by a period with below-average inflation which greatly reduces incentives to adjust prices by large amounts at the time when shocks occur.[4] In other words, the policy reaction function based on the price level which accumulates past deviations of inflation from the steady state compensates above-average inflation (relatively high output) with below-average inflation (relatively low output). Price level targeting is also a very attractive alternative to inflation targeting when nominal rates are close to zero since price level targeting is much less likely to hit zero lower bound for nominal rates and much more likely to escape a deflationary spiral.[5]

A general theme that emerges from this informal discussion is that for policy rules to be effective in stabilizing the economy they must be ***history dependent***. Here history dependence basically means that policy is anchored to some extent to the past and this link

[4] Recall that there is a trade off between losing market share and anticipation that need to revise the price later.

[5] There are other benefits of using PLT. For example, Gorodnichenko and Shapiro (2007) argue that the Fed under Alan Greenspan had an element of price level targeting when it set the policy rate. This element of PLT helped the Fed to sustain a low-inflation expansion in the US economy in the late 1990s when the benefits of technology improvements were not obvious and, at the time, it looked that the policymakers were overly optimistic. The main downside of PLT is that inflationary shocks may force the central bank to abandon its price level target. This problem, however, is typical for any commitment to a policy rule.

to the past helps to set (or promise) the path of the policy instrument in the future and, hence, to control long-term interest rates.

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Combining equations (??), (??) and (??), we get the expression for $\bar{\omega}^j$:

$$(1 - F(\bar{\omega}^j))\bar{\omega}^j R^k_{t+1} Q_t K^j_{t+1} + (1-\mu)\int_0^{\bar{\omega}^j} \omega dF(\omega) R^k_{t+1} Q_t K^j_{t+1} = R_{t+1}(Q_t K^j_{t+1} - N^j_{t+1}) \quad (8.5)$$

There are two opposing effects of changing $\bar{\omega}^j$. First, a larger $\bar{\omega}^j$ increases the probability of default. Second, a larger $\bar{\omega}^j$ increases the non-default payoff. Given the assumed restrictions on the hazard function of default (given by equation (??)), the expected return reaches maximum at a unique interior value of $\bar{\omega}^j$. For simplicity, we assume that the equilibrium $\bar{\omega}^j$ is always smaller than the maximum feasible value of $\bar{\omega}^j$.

With aggregate uncertainty, $\bar{\omega}^j$ will depend on the ex post realization of R^k_{t+1}. Thus, the loan rate will be tied to macroeconomic conditions. Specifically, when R^k_{t+1} is lower than expected, then D^j_{t+1} increases (i.e., the default probability rises) and consequently $\bar{\omega}^j$ also increases which means that the loan rate will be countercyclical in general. Note that since we assume risk neutral entrepreneurs, they will bear all aggregate risk (and thus insure lending intermediaries). Since lending intermediaries are diversified, households earn risk free rate on their savings.

8.1.2 Net worth and the optimal choice of capital

Given the state-contingent debt form of the optimal contract, the expected return to the entrepreneur is

$$E\Big(\int_{\bar{\omega}^j}^{\infty} \omega R^k_{t+1} Q_t K^j_{t+1} dF(\omega) - (1 - F(\bar{\omega}^j))\bar{\omega}^j R^k_{t+1} Q_t K^j_{t+1}\Big), \quad (8.6)$$

where expectations are taken with respect to the random variable R^k_{t+1} and we assume that $\bar{\omega}^j$ can be contingent on the realization of this variable. Using this relation and equation (??), we can simplify the objective function for entrepreneurs to

$$E\Big(\Big(1 - \mu \int_0^{\bar{\omega}^j} \omega dF(\omega)\Big) U^{rk}_{t+1}\Big) E(R^k_{t+1}) Q_t K^j_{t+1} - R_{t+1}(Q_t K^j_{t+1} - N^j_{t+1}), \quad (8.7)$$

where $U^{rk}_{t+1} = R^k_{t+1}/E(R^k_{t+1})$ is the ratio of the realized return to capital to the expected return. This equation suggests that the entrepreneur internalizes the expected costs of defaults given that the intermediary must receive a competitive return.

The formal investment and contracting problem then reduces to choosing K^j_{t+1} and a schedule for $\bar{\omega}^j$ to maximize the entrepreneur's objective function given by equation (**??**) subject to the constraint in equation (**??**). Entrepreneurs take the distributions of aggregate and idiosyncratic shocks to the return on capital as given.

Let $s_t \equiv E_t(R^k_{t+1}/R_{t+1})$ be the expected discounted return to capital. For entrepreneurs to purchase capital in the competitive equilibrium it must be the case that $s_t \geq 1$. Given $s_t \geq 1$, the first order conditions yield the following relation for optimal capital purchases:

$$\begin{aligned} Q_t K^j_{t+1} &= \psi(s_t) N^j_{t+1}, \psi(1) = 1, \psi'(\cdot) > 0 \Longleftrightarrow \\ E(R^k_{t+1}) &= s\Big(\frac{N^j_{t+1}}{Q_t K^j_{t+1}}\Big) R_{t+1}, s' < 0. \end{aligned} \tag{8.8}$$

which describes the critical link between capital expenditures by the firm and financial conditions. This link is captured by the wedge between the expected return on capital and the safe rate s_t and by entrepreneurial net worth N^j_{t+1}. Given the value of K^j_{t+1} that satisfies equation (**??**), the schedule for $\bar{\omega}^j$ is fixed by the state-contingent constraint on the expected return to debt, defined by equation (**??**). Importantly, from equation (**??**), capital expenditures by each firm are proportional to the net worth of the owner (i.e., entrepreneur) with the proportionality factor that is increasing in the expected discounted return on capital s_t. The fact that the value of capital is linear in net worth permits easy aggregation of individual entrepreneurs. The entrepreneur is constrained from increasing the size of the firm by the fact expected default costs also increase with the ratio of borrowing to net worth.

Figure 1 illustrates this relationship for a calibrated model. Firm's demand for capital is on the x-axis, while the cost of funds is on the y-axis. For capital stock which can be entirely financed internally with equity ($K < 4.6$), the cost of capital is equal to the risk free rate.

Figure 8.1: Effect of an increase in net worth.

As capital stock increases, the cost of capital rises as well. An increase in the net worth N_{t+1}^j is shown as the right shift in the curve.

8.1.3 General equilibrium

The entrepreneurial sector

We will assume that the aggregate production function takes the standard Cobb-Douglas form

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha} \tag{8.9}$$

where Y_t is aggregate output of wholesale goods, K_t is the aggregate amount of capital purchased by entrepreneurs in period $t-1$, L_t is labor input, and Z_t is an exogenous technology parameter.

Let I_t denote aggregate investment expenditures. The aggregate capital stock evolves as follows:

$$K_{t+1} = \Phi(I_t/K_t) + (1-\delta)K_t \tag{8.10}$$

where Φ is the function capturing capital adjustment costs and δ is the depreciation rate. Given the adjustment costs, the price of a unit of capital in terms of the numeraire good is given by

$$Q_t = \Big(\Phi'(I_t/K_t)\Big)^{-1} \tag{8.11}$$

We normalize the adjustment costs function so that the price of capital goods is unity in the steady state.

Assume that entrepreneurs sell their output to retailers and let us denote the price of wholesale goods with $1/X_t$. Given the Cobb-Douglass production function, the capital payment to a unit of capital at time $t+1$ is

$$\frac{1}{X_{t+1}}\frac{\alpha Y_{t+1}}{K_{t+1}} \tag{8.12}$$

The expected gross return to holding a unit of capital from t to $t+1$ is

$$E(R^k_{t+1}) = E\Big(\frac{\frac{1}{X_{t+1}}\frac{\alpha Y_{t+1}}{K_{t+1}} + Q_{t+1}(1-\delta)}{Q_t}\Big) \tag{8.13}$$

The supply curve for investment finance is obtained by aggregating equation (**??**) over firms and inverting to get

$$E(R^k_{t+1}) = s\Big(\frac{N_{t+1}}{Q_t K_{t+1}}\Big)R_{t+1} \tag{8.14}$$

where the function $s(\cdot)$ is the ratio of costs of external and internal finance. We assume that s is decreasing in $\frac{N_{t+1}}{Q_t K_{t+1}}$ for $N_{t+1} < Q_t K_{t+1}$. The unusual feature of this supply curve is the dependence of the cost of funds on the aggregate financial condition of entrepreneurs, as measured by the ratio $\frac{N_{t+1}}{Q_t K_{t+1}}$. The dynamic behavior of capital demand as well as return on capital depends on net worth N_{t+1} which is the equity stake that entrepreneurs have in their firms.

We will also assume that in addition to supplying capital, entrepreneurs provide labor services to their firms. We will denote the "regular" labor input with H_t and the "entrepreneurial" labor input with H^e_t. These two inputs are aggregated into labor input L_t as follows:

$$L_t = H_t^{\Omega}(H^e_t)^{1-\Omega} \tag{8.15}$$

To simplify things, we will assume that entrepreneurs supply their labor inelastically and the mass of entrepreneurial labor is equal to one .

Let V_t denote entrepreneurial equity, W_t^e denote entrepreneurial wage, and $\bar{\omega}_t$ denote the state contingent value of ω set in period t. Then aggregate entrepreneurial net worth at the end of period t, N_{t+1} is given by

$$N_{t+1} = \gamma V_t + W_t^e \tag{8.16}$$

where

$$V_t = R_t^k Q_{t-1} K_t - \Big(R_t + \frac{\mu \int_0^{\bar{\omega}_t} \omega R_t^k Q_{t-1} K_t dF(\omega)}{Q_{t-1}K_t - N_{t-1}}\Big)(Q_{t-1}K_t - N_{t-1}) \tag{8.17}$$

where γV_t is the equity held by entrepreneurs at $t-1$ who are still in business at time t. This equation says that entrepreneurial equity is equal to gross earnings on holdings of equity from $t-1$ to t less repayment of borrowing. The ratio of default costs to quantity borrowed is

$$\frac{\mu \int_0^{\bar{\omega}_t} \omega R_t^k Q_{t-1} K_t dF(\omega)}{Q_{t-1}K_t - N_{t-1}}$$

which captures the premium for external finance.

Entrepreneurial equity not only provides the main source of variation in N_{t+1}, but it is also very sensitive to shifts in asset prices, especially when firms are leveraged. For example, let $U_t^{rk} \equiv R_t^k - E_{t-1}(R_t^k)$ be the unexpected shift in the gross return on capital and let $U_t^{dp} \equiv \int_0^{\bar{\omega}_t} \omega Q_{t-1} K_t dF(\omega) - E_{t-1}\Big(\int_0^{\bar{\omega}_t} \omega Q_{t-1} K_t dF(\omega)\Big)$ be the unexpected shift in the conditional default costs. We can express V_t as follows

$$V_t = \Big(U_t^{rk}(1 - \mu U_t^{dp})\Big) Q_{t-1} K_t + E_{t-1}(V_t). \tag{8.18}$$

Given this expression, the elasticity of entrepreneurial equity with respect to an unanticipated movement in the return on capital is

$$\frac{\frac{V_t}{E_{t-1}(V_t)}}{\frac{U_t^{rk}}{E_{t-1}(R_t^k)}} = \frac{E_{t-1}(R_t^k)Q_{t-1}K_t}{E_{t-1}(V_t)} \tag{8.19}$$

A one percent increase in the ex post return on capital leads to a change in entrepreneurial equity equal to the ratio of gross holdings of capital to equity. If this ratio is larger than one, there is a magnification effect. Thus, unexpected movements in asset prices can have large effects on firms' financial positions. In general equilibrium, this magnification can be further amplified.

The demand curves for labor are derived from the marginal product curves:

$$(1-\alpha)\Omega\frac{Y_t}{H_t}\frac{1}{X_t} = W_t \tag{8.20}$$

$$(1-\alpha)(1-\Omega)\frac{Y_t}{H_t^e}\frac{1}{X_t} = W_t^e \tag{8.21}$$

where W_t is the real wage for household labor and W_t^e is the real wage for entrepreneurial labor. Here the marginal revenue product of labor is equal to the wage.

Combining equations (**??**), (**??**), and imposing the condition that entrepreneurial labor supply is fixed at unity, we obtain the law of motion for net worth N_{t+1}:

$$N_{t+1} = \gamma\Big(R_t^k Q_{t-1}K_t - \Big(R_t + \frac{\mu\int_0^{\bar{\omega}_t}\omega R_t^k Q_{t-1}K_t dF(\omega)}{Q_{t-1}K_t - N_t}\Big)(Q_{t-1}K_t - N_{t-1})\Big) \\ +(1-\alpha)(1-\Omega)Z_t K_t^{\alpha} H_t^{(1-\alpha)\Omega} \tag{8.22}$$

This equation, which describes endogenous variation in net worth, and equation (**??**), which describes the supply of investment funds, are the key ingredients of the financial accelerator.

8.2 Discussion and analysis of impulse responses

The log-linearized model is in the appendix. In many respects, this model is very similar to the basic New Keynesian model we have considered. Many equations have the standard interpretation. There are two novel equations:

$$
\begin{aligned}
E_t(r_{t+1}^k) - r_{t+1} &= -\nu(n_{t+1} - (q_t + k_{t+1})) && (8.23) \\
n_{t+1} &= \frac{\gamma RK}{N}(r_t^k - r_t) + r_t + n_t + \phi_t^n && (8.24) \\
\nu &= \frac{\psi(R^k/R)}{\psi'(R^k/R)}
\end{aligned}
$$

Specifically, equation (**??**) shows the effect of net worth on investment. In the absence of capital frictions, this relation collapses to $E_t(r_{t+1}^k) - r_{t+1} = 0$ so that investment is done to the point where expected return $E_t(r_{t+1}^k)$ is equal to the opportunity cost of funds r_{t+1}. With capital frictions, the cost of external funds depends on the net worth relative to the gross value of capital. A rise in this ratio reduces the cost of external funds implying that investment will rise. In short, equation (**??**) embodies the financial accelerator. Another new equation is (**??**) which describes the law of motion for net worth.

Unfortunately, we cannot derive a simple closed form solution to this model. However, we can use impulse responses to understand the properties of the model. Figures 2 and 3 show the reaction of key macroeconomic variables to key shocks in a calibrated model with the financial accelerator we have developed above. The key difference between the basic New Keynesian model and the New Keynesian model with the financial accelerator is that in the latter responses are larger and more persistent. Thus we move closer to matching the "small shocks, large responses" objective. To see the intuition for this result, consider the responses to a monetary policy shock.

An unanticipated decline in the fed funds rate stimulates demand for capital which raises investment and the price of capital. If the price of capital rises, the value of the capital stock and hence net worth increase. Given an increase in the net worth, the cost of borrowing

Figure 8.2: Monetary shock. All panels: time horizon in quarters.

Figure 8.3: Output response – alternative shocks. All panels: time horizon in quarters.

falls because this cost is the sum of the risk free rate and the external finance premium and the premium is ceteris paribus decreasing in net worth. This further stimulates demand for capital and pushes the cost of capital further up and amplification continues. Thus, the initial shock has a multiplier working via relaxing the borrowing constraint. Indeed, the premium falls sharply in response to an expansionary nominal shock. Entrepreneurial net worth reverts back to a steady as firms leave the market but the effect is sufficiently slow to make the external premium persists below trend. Interestingly, to generate persistent response in the New Keynesian model with the financial accelerator, one does not need a large labor supply elasticity. Some of the responses reach trough or peak too fast relative to what we normally observe in the empirical responses, but this problem can be rectified to a certain extent by introducing various delays and adjustment costs (e.g., gestation lag for new investment).

One novel shock in this analysis is the shock to net worth. In the basic New Keynesian model, an increase in the wealth of households leads to only a small response of output. In contrast, the response of output in the New Keynesian model with the financial accelerator is much larger. The mechanics behind this much bigger response is similar to what we have in the responses to a nominal shock. More net worth reduces the external finance premium. A smaller premium generates more demand for capital which drives the price of capital up and further relaxes the borrowing constraint by increasing net worth and thus further lowering the external finance premium.

Shocks to net worth are interesting and important for several reasons. The most important one is probably the fact that net worth can respond not only to changes in the cash flow but also to changes in asset prices. Irving Fisher and other writers emphasized the deflation of asset prices as an important mechanism for generating depressions. The first formal analysis of this channel is due to **?** who develop a stylized model where land serves both as a factor of production and as a source of collateral for loans to farmers. In their model, a temporary shock to productivity decreases the value of land and thus the value of the collateral. This triggers tighter borrowing constraints and consequently less production and spending which in turn further reduces the value of land. A similar logic can be used to analyze stock market shocks which could force investors to sell assets to meet margin calls, these sales depress the value of assets and hence can start "fire sales" of assets when one sale leads to another sale and each sale deflates asset prices. Such a spiral could lead to a collapse of financial markets and frozen credit markets. This mechanism can be also used to explain sudden stops of credit flows to developing countries where a shock to beliefs of foreign investors can depress the value of domestic assets and start the spiral of "fire sales" of domestic assets and, in the worst case scenario, can devastate the financial system of a developing economy. Finally, Bernanke, Gertler and Gilchrist (REStat 1996) argue that agency costs can lead to the "flight to quality" during recessions when the share of credit going to low-agency-cost borrowers increases. One prominent example of such a flight was a massive increase in the demand for US Treasuries during the recent recession.

8.3 Empirics

One reason for why incorporation of financial frictions in the business cycle models was very sporadic until recently is the fact that the empirical support for the financial accelerator in one form or another is somewhat weak. More specifically, identification of the financial accelerator effects on lending and key macroeconomic variable is very tricky because factors triggering such effects influence the same variables through other channels. Recall, for example, that an increase in the interest rate (controlled by the central bank) not only triggers the financial accelerator but also has a direct effect on the cost of capital. Hence, one often has to rely on the changes in composition of lending to firms and other rather indirect measures of the effects of financial frictions. In addition, until recently, the U.S. economy did not have a major recession with a deep distress in financial markets and therefore there was not enough variation to find significant effects. Most of the convincing evidence comes from the analysis of the response of small vs. large firms to liquidity shocks. This analysis is built on the presumption that small firms are more vulnerable to agency problems than large firms. Interestingly, there is a bit of experimental evidence suggesting that external credit is rationed and positive net worth shocks can increase output, employment, etc. at the firm level. The general equilibrium implications of these micro level studies are however less clear cut and it is not apparent if the effects are large or small at the macroeconomic level. In summary, although at the fundamental level the financial accelerator is appealing and empirically plausible, it seems we need more (convincing) evidence at the aggregate level.

Appendix: log-linearized model

This section present log-linearized equations of the model presented in Section 2. Lower case variables denote percent deviations from the steady state. In addition to technology shocks, we introduce serially correlated government spending (G_t) with shocks e_t^g and monetary policy shocks e_t^{rn}. Consumption of entrepreneurs is C^e.

Aggregate demand

$$
\begin{aligned}
y_t &= \frac{C}{Y}c_t + \frac{I}{Y}i_t + \frac{G}{Y}g_t + \frac{C^e}{Y}c_t^e + \phi_t^y && (8.25)\\
c_t &= -r_{t+1} + E_t(c_{t+1}) && (8.26)\\
c_t^e &= n_{t+1} + \phi_t^{c^e} && (8.27)\\
E_t(r_{t+1}^k) - r_{t+1} &= -\nu(n_{t+1} - (q_t + k_{t+1})) && (8.28)\\
r_{t+1}^k &= (1-\epsilon)(y_{t+1} - k_{t+1} - x_{t+1}) + \epsilon q_{t+1} - q_t && (8.29)\\
q_t &= \phi(i_t - k_t) && (8.30)
\end{aligned}
$$

Aggregate supply

$$
\begin{aligned}
y_t &= z_t + \alpha k_t + (1-\alpha)\Omega h_t && (8.31)\\
y_t - h_t - x_t - c_t &= \eta^{-1} h_t && (8.32)\\
\pi_t &= E_{t-1}(\kappa(-x_t) + \beta\pi_{t+1}) && (8.33)
\end{aligned}
$$

State variables

$$
\begin{aligned}
k_t &= \delta i_t + (1-\delta)k_t && (8.34)\\
n_{t+1} &= \frac{\gamma RK}{N}(r_t^k - r_t) + r_t + n_t + \phi_t^n && (8.35)
\end{aligned}
$$

$$(8.36)$$

Monetary policy and shock processes

$$r_t^n = \rho r_{t-1}^n + \zeta \pi_{t-1} + e_t^{rn} \tag{8.37}$$

$$g_t = \rho_g g_{t-1} + e_t^g \tag{8.38}$$

$$z_t = \rho_z z_{t-1} + e_t^z \tag{8.39}$$

where

$$\phi_t^y = \frac{DK}{Y}\Big(\log\Big(\mu \int_0^{\bar{\omega}_t} \omega dF(\omega) R_t^k Q_{t-1} K_t / DK\Big)\Big) \tag{8.40}$$

$$D \equiv \mu \int_0^{\bar{\omega}} \omega dF(\omega) R^k \tag{8.41}$$

$$\phi_t^{c^e} = \log\Big(\frac{1 - C_t^e/N_{t+1}}{1 - C^e/N}\Big) \tag{8.42}$$

$$\phi_t^n = \frac{(R^k/R - 1)K}{N}(r_t^k + q_{t-1} + k_t) + \frac{(1-\alpha)(1-\Omega)(Y/X)}{N} y_t - x_t \tag{8.43}$$

$$\nu = \frac{\psi(R^k/R)}{\psi'(R^k/R)} \tag{8.44}$$

$$\epsilon = \frac{1-\delta}{(1-\delta) + \alpha Y/(XK)} \tag{8.45}$$

$$\phi = \frac{(\Phi(I/L)^{-1})'}{(\Phi(I/L)^{-1})''} \tag{8.46}$$

$$\kappa = \frac{1-\theta}{\theta}(1 - \theta\beta) \tag{8.47}$$

Many of these equations have the standard interpretation. For example, equation (**??**) is the resource constraint; equation (**??**) is the consumption Euler equation; equation (**??**) is the Phillips curve; equation (**??**) is a Taylor rule. Equations (**??**)-(**??**) describe investment demand.

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