Economics 210C - Macroeconomics Spring 2018-19

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Take-Home Final Exam - Solutions

1. Technology Growth Regimes

a. Clearly, the planner chooses $L_t = 1$ for all t, and hence the Lagrangian for the planner's problem may be written as

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \Lambda_t \left(Z_t K_{t-1}^{\alpha} X_t{}^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t \right) \right).$$

The first order necessary conditions for C_t, K_t and Λ_t are, after canceling terms and rearranging:

$$C_t^{-\sigma} = \Lambda_t,$$

$$\beta \mathbb{E}_t \Lambda_{t+1} \left(\alpha Z_{t+1} K_t^{\alpha-1} X_{t+1}^{1-\alpha} + 1 - \delta \right) = \Lambda_t,$$

$$Z_t K_{t-1}^{\alpha} X_t^{1-\alpha} + (1-\delta) K_{t-1} = C_t + K_t.$$

b. Substituting $C_t = c_t X$, $K_t = k_t X_t$ and $\Lambda_t = \lambda_t X_t^{-\sigma}$ into the necessary conditions, canceling and rearranging gives

$$c_t^{-\sigma} = \lambda_t,$$

$$\beta \mathbb{E}_t \lambda_{t+1} \left(\frac{X_{t+1}}{X_t} \right)^{-\sigma} \left(\alpha Z_{t+1} k_t^{\alpha-1} \left(\frac{X_{t+1}}{X_t} \right)^{1-\alpha} + 1 - \delta \right) = \lambda_t, \tag{1}$$

$$Z_t k_{t-1}^{\alpha} \left(\frac{X_t}{X_{t-1}} \right)^{-\alpha} + (1-\delta) k_{t-1} \left(\frac{X_t}{X_{t-1}} \right)^{-1} = c_t + k_t, \tag{2}$$

or

$$c_t^{-\sigma} = \lambda_t, \tag{3}$$

$$\beta \mathbb{E}_t \lambda_{t+1} \mu_{t+1}^{-\sigma} \left(\alpha Z_{t+1} k_t^{\alpha-1} \mu_{t+1}^{1-\alpha} + 1 - \delta \right) = \lambda_t, \tag{4}$$

$$Z_t k_{t-1}^{\alpha} \mu_t^{-\alpha} + (1-\delta) k_{t-1} \mu_t^{-1} = c_t + k_t. \tag{5}$$

The nonstochastic steady state is thus determined by

$$c^{-\sigma} = \lambda,$$

$$\beta\mu^{-\sigma}\left(\alpha k^{\alpha-1}\mu^{1-\alpha} + 1 - \delta\right) = 1,$$

$$k^{\alpha}\mu^{-\alpha} + (1-\delta)k\mu^{-1} = c + k.$$

c. Under perfect foresight, (3)-(5) may be used to obtain

$$\Delta\lambda_t = \lambda_t\left(1 - \beta\mu_t^{-\sigma}\left(\alpha Z_t k_{t-1}^{\alpha-1}\mu_t^{1-\alpha} + 1 - \delta\right)\right),$$

$$\Delta k_t = -\lambda_t^{-\frac{1}{\sigma}} + \left(Z_t k_{t-1}^{\alpha-1}\mu_t^{-\alpha} + (1-\delta)\mu_t^{-1} - 1\right)k_{t-1}.$$

For period length Δ, the net growth rate of X_t between periods $t-\Delta$ and t may be expressed as[1]

$$\frac{X_t}{X_{t-\Delta}} - 1 = (\mu_t - 1)\,\Delta \cong e^{(\mu_t-1)\Delta} - 1,$$

or

$$\frac{X_t}{X_{t-\Delta}} = e^{(\mu_t-1)\Delta}.$$

Thus, equations (1) and (2) may be restated as

$$e^{-r\Delta}\lambda_{t+\Delta}e^{-\sigma\left(\mu_{t+\Delta}-1\right)\Delta}\left(\alpha Z_{t+\Delta}k_t^{\alpha-1}e^{-(1-\alpha)\left(\mu_{t+\Delta}-1\right)\Delta}\Delta + 1 - \delta\Delta\right) = \lambda_t,$$

$$Z_t k_{t-\Delta}^{\alpha}e^{-\alpha(\mu_t-1)\Delta}\Delta + (1-\delta\Delta)k_{t-\Delta}e^{-(\mu_t-1)\Delta} = c_t\Delta + k_t.$$

where r is determined by $e^{-r} = \beta$. Rearranging gives

$$\frac{\lambda_{t+\Delta} - \lambda_t}{\Delta} = \frac{1 - e^{-\left(r+\sigma\left(\mu_{t+\Delta}-1\right)\right)\Delta}}{\Delta}\lambda_{t+\Delta}$$

$$-e^{-\left(r+\sigma\left(\mu_{t+\Delta}-1\right)\right)\Delta}\lambda_{t+\Delta}\left(\alpha Z_{t+\Delta}k_t^{\alpha-1}e^{-(1-\alpha)\left(\mu_{t+\Delta}-1\right)\Delta} - \delta\right),$$

[1] Alternatively, note that we may write $\ln X_t - \ln X_{t-1} = \ln \mu_t \cong u_t - 1$, and hence $X_t/X_{t-1} = e^{\mu_t - 1}$ holds as an approximation.

$$\frac{k_t - k_{t-\Delta}}{\Delta} = -c_t + \left(Z_t k_{t-\Delta}^{\alpha} e^{-\alpha(\mu_t - 1)\Delta} - \delta k_{t-\Delta} e^{-(\mu_t - 1)\Delta} \right) - \frac{1 - e^{-(\mu_t - 1)\Delta}}{\Delta} k_{t-\Delta}.$$

If $\mu_{t+\Delta} = \mu_t$ for small Δ (which will be true for the paths considered in parts g and h), then taking the limit as $\Delta \to 0$, and substituting from (3), gives

$$\dot{\lambda}_t = \lambda_t \left(r + \sigma \left(\mu_t - 1 \right) - \left(\alpha Z_t k_t^{\alpha - 1} - \delta \right) \right), \tag{6}$$

$$\dot{k}_t = -\lambda_t^{-\frac{1}{\sigma}} + Z_t k_t^{\alpha} - \left(\delta + \mu_t - 1 \right) k_t. \tag{7}$$

d. Implicitly differentiating (6) for $\dot{\lambda}_t = 0$ gives

$$0 = \lambda_t \sigma - (\alpha - 1) \alpha Z_t k_t^{\alpha - 2} \frac{\partial \bar{k}_t}{\partial \mu_t},$$

or

$$\frac{\partial \bar{k}_t}{\partial \mu_t} = -\frac{\lambda_t \sigma}{(1 - \alpha) \alpha Z_t k_t^{\alpha - 2}} < 0.$$

e. Implicitly differentiating (7) for $\dot{k}_t = 0$ gives

$$0 = \frac{1}{\sigma} \bar{\lambda}_t^{-\frac{1}{\sigma} - 1} d\lambda_t + \left(\alpha Z_t k_t^{\alpha - 1} - (\delta + \mu_t - 1) \right) dk_t - k_t d\mu_t.$$

Thus,

$$\frac{\partial \bar{\lambda}_t}{\partial k_t} = -\bar{\lambda}_t^{\frac{1+\sigma}{\sigma}} \sigma \left(\alpha Z_t k_t^{\alpha - 1} - (\delta + \mu_t - 1) \right),$$

$$\frac{\partial \bar{\lambda}_t}{\partial \mu_t} = \bar{\lambda}_t^{\frac{1+\sigma}{\sigma}} \sigma k_t > 0.$$

The continuous time steady state satisfies $\dot{\lambda}_t = 0$, which implies, using (6),

$$r + \sigma (\mu - 1) - \left(\alpha k^{\alpha - 1} - \delta \right) = 0.$$

In a neighborhood of the steady state, we have

$$\alpha Z_t k_t^{\alpha - 1} - (\delta + \mu_t - 1) \cong \alpha k^{\alpha - 1} - \delta - (\mu - 1)$$

$$= r + (\sigma - 1)(\mu - 1).$$

Moreover, we can write

$$e^{-r} = \beta < \mu^{\sigma-1} \cong e^{(\sigma-1)(\mu-1)},$$

and hence

$$-r < (\sigma - 1)(\mu - 1).$$

It follows that $\partial\bar{\lambda}_t/\partial k_t < 0$ in a neighborhood of the steady state.

g. When μ_t rises from μ to μ', the $\dot{\lambda}_t = 0$ line shifts in and the $\dot{k}_t = 0$ curve shifts up. Thus the new steady state has $k' < k$ and $\lambda' > \lambda$. Moreover, (3) implies $c' < c$.

Intuitively, equations (6) and (7) show that μ_t and δ have similar effects on the detrended variables. However, depreciation affects the variables' absolute levels, while technology growth affects their values relative to trend. When μ_t is higher, it becomes more costly for consumption and capital stock to keep up with the trend, so the values of these variables relative to trend are reduced.

h. λ_t may rise or fall on impact, depending on the position of the new saddlepoint path relative to the original one. After impact, k_t falls and λ_t rises, reflecting the greater cost of keeping up with faster technology growth. At t_1, the path reverses, and the economy follows the original saddlepoint path back to the original steady state, with k_t rising and λ_t falling.

2. Optimal policy in the New Keynesian model

a. The policymaker's problem is

$$\max_{a_0,a_1} -\mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t \left(\varphi\left(\chi_t - \chi^*\right)^2 + \hat{\pi}_t^2\right),$$

$$\text{s.t.} \quad \hat{\pi}_t = \theta\chi_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + \varepsilon_t,$$

$$\hat{\pi}_t = a_0 + a_1\varepsilon_t.$$

The problem may be equivalently expressed as

$$\max_{a_0,a_1} \mathcal{U} = -\mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t \left(\frac{\varphi}{\theta^2}\left(\left(1-\beta\right)a_0 - \theta\chi^* + a_1\varepsilon_t - \varepsilon_t\right)^2 + \left(a_0 + a_1\varepsilon_t\right)^2\right).$$

First order necessary conditions for a maximum are

$$\frac{\partial \mathcal{U}}{\partial a_0} = -\mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t \left(\frac{2\varphi}{\theta^2} \left(\left(1-\beta\right) a_0 + a_1 \varepsilon_t - \varepsilon_t - \theta \chi^* \right) \left(1-\beta\right) \right.$$
$$\left. +2 \left(a_0 + a_1 \varepsilon_t \right) \right) = 0,$$

$$\frac{\partial \mathcal{U}}{\partial a_1} = -\mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t \left(\frac{2\varphi}{\theta^2} \left(\left(1-\beta\right) a_0 - \theta \chi^* + a_1 \varepsilon_t - \varepsilon_t \right) \varepsilon_t \right.$$
$$\left. +2 \left(a_0 + a_1 \varepsilon_t \right) \varepsilon_t \right) = 0.$$

Manipulating these equations gives

$$\frac{\varphi \left(1-\beta\right)}{\theta^2} \left(\left(1-\beta\right) a_0 - \varphi \theta \chi^* \right) + a_0 = 0,$$
$$\frac{\varphi}{\theta^2} \left(a_1 - 1\right) \sigma_\varepsilon^2 + a_1 \sigma_\varepsilon^2 = 0,$$

or

$$a_0 = \frac{\left(1-\beta\right) \varphi \theta}{\left(1-\beta\right)^2 \varphi + \theta^2} \chi^*, \quad a_1 = \frac{\varphi}{\varphi + \theta^2}.$$

(i) Effect of increase in θ:

$$\frac{\partial a_0}{\partial \theta} = \frac{\left(1-\beta\right) \varphi \left(\left(1-\beta\right)^2 \varphi - \theta^2 \right)}{\left(\left(1-\beta\right)^2 \varphi + \theta^2 \right)^2} \chi^* \begin{cases} > 0, & \text{low } \theta, \\ < 0, & \text{high } \theta. \end{cases}$$

Low θ means that prices are rigid, and thus average inflation has a large effect on the output gap. As θ rises, this effect weakens, and the average inflation coefficient a_0 increases to maintain an optimal tradeoff. Eventually θ becomes so high that further increases in θ lead the policymaker to shift toward inflation reduction.

The stabilization coefficient a_1 decreases in θ for all θ. As θ rises, stabilization efforts have a lower cost in terms of the output gap, so shocks are more heavily dampened.

(ii) Effect of an increase in φ: Both a_0 and a_1 are increasing in φ, reflecting greater importance of the output gap term in the objective function.

(iii) Effect of an increase in χ^*: An increase in the target value of the output gap causes the policymaker to choose higher mean inflation, while there is no effect on the stabilization tradeoff.

b. The policymaker's problem becomes, for each t,

$$\max_{\hat{\pi}_t} -\left(\varphi\left(\chi_t - \chi^*\right)^2 + \hat{\pi}_t^2\right),$$

$$\text{s.t.} \quad \hat{\pi}_t = \theta\chi_t + \beta\mathbb{E}_t\hat{\pi}_{t+1} + \varepsilon_t,$$

which is equivalent to

$$\max_{\hat{\pi}_t} \mathcal{V}_t = -\left(\frac{\varphi}{\theta^2}\left(\hat{\pi}_t - \beta\mathbb{E}_t\hat{\pi}_{t+1} - \varepsilon_t - \theta\chi^*\right)^2 + \hat{\pi}_t^2\right).$$

The first order necessary condition for a maximum is

$$\frac{\partial\mathcal{V}_t}{\partial\hat{\pi}_t} = -\left(\frac{2\varphi}{\theta^2}\left(\hat{\pi}_t - \beta\mathbb{E}_t\hat{\pi}_{t+1} - \varepsilon_t - \theta\chi^*\right) + 2\hat{\pi}_t\right) = 0,$$

which implies

$$\hat{\pi}_t = \frac{\varphi}{\varphi + \theta^2}\left(\beta\mathbb{E}_t\hat{\pi}_{t+1} + \theta\chi^* + \varepsilon_t\right). \tag{8}$$

Substituting $\hat{\pi}_t = b_0 + b_1\varepsilon_t$ into (8) gives

$$b_0 + b_1\varepsilon_t = \frac{\varphi}{\varphi + \theta^2}\left(\beta b_0 + \theta\chi^* + \varepsilon_t\right),$$

and hence

$$b_0 = \frac{\varphi\theta}{\left(1-\beta\right)\varphi + \theta^2}\chi^* = \frac{\left(1-\beta\right)\varphi\theta}{\left(1-\beta\right)^2\varphi + \left(1-\beta\right)\theta^2} > a_0,$$

$$b_1 = \frac{\varphi}{\varphi + \theta^2} = a_1.$$

The policymaker chooses higher mean inflation in the absence of policy commitment. Since policy no longer directly affects inflation expectations, the policymaker does not take expectations into account when determining policy in period t. The stabilization tradeoff is unaffected, however, because inflation expectations do not interact with ε_t.

c. Substituting the policy rule from part b into the NKPC and solving for χ_t gives

$$\chi_t = \frac{1}{\theta}(1-\beta) b_0 - \frac{1}{\theta}(1-b_1)\varepsilon_t.$$

Substituting the latter equation into the bond pricing equation and solving for $\hat{r}_t^n$ gives

$$\hat{r}_t^n = b_0 + \frac{\xi}{\theta}(1-b_1)\varepsilon_t + \xi u_t.$$

If $\varepsilon_t > 0$, then the policy rule raises $\hat{\pi}_t$ by a smaller amount, since $0 < b_1 < 1$. Thus $\hat{r}_t^n$ is raised in order to lower χ_t and limit the increase in $\hat{\pi}_t$ implied by the NKPC.

If $u_t > 0$, then $\hat{r}_t^n$ is increased to offset u_t directly and prevent χ_t from increasing, which would raise $\hat{\pi}_t$ via the NKPC.