Economics 210C - Macroeconomics Spring 2018-19

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Stochastic Components Exercise - Solution

a. The Lagrangian for the planner's problem is

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t + \Lambda_t \left(Z_t K_{t-1} + (1-\delta)K_{t-1} - C_t - K_t\right)\right).$$

The first order necessary conditions for C_t, K_t and Λ_t are, after canceling terms and rearranging:

$$\frac{1}{C_t} = \Lambda_t,$$

$$\beta \mathbb{E}_t \Lambda_{t+1}(Z_{t+1} + 1 - \delta) = \Lambda_t,$$

$$Z_t K_{t-1} + (1-\delta)K_{t-1} = C_t + K_t.$$

b. Restated necessary conditions:

$$\frac{1}{c_t} = \lambda_t, \tag{1}$$

$$\beta \mathbb{E}_t \lambda_{t+1}(Z_{t+1} + 1 - \delta) = \mu_t \lambda_t, \tag{2}$$

$$Z_t + 1 - \delta = c_t + \mu_t. \tag{3}$$

c. Log linearized necessary conditions:

$$-\hat{c}_t = \hat{\lambda}_t, \tag{4}$$

$$\mathbb{E}_t \hat{\lambda}_{t+1} + \frac{\beta}{\mu} \mathbb{E}_t \hat{z}_{t+1} = \hat{\mu}_t + \hat{\lambda}_t, \tag{5}$$

$$\hat{z}_t = c\hat{c}_t + \mu\hat{\mu}_t. \tag{6}$$

d. We can write

$$\ln K_{t-1} = \ln K_{t-1} - \ln K_{t-2} + \ln K_{t-2} - \ln K_{t-3} +$$

$$\cdots + \ln K_1 - \ln K_0 + \ln K_0 \;=\; \sum\nolimits_{j=1}^{t-1} \Delta \ln K_j + \ln K_0.$$

Moreover,

$$\ln \mu_t \;=\; \ln K_t - \ln K_{t-1} \;=\; \Delta \ln K_t,$$

and hence

$$\Delta \ln K_t \;=\; \ln \mu_t - \ln \mu + \ln \mu \;=\; \hat{\mu}_t + \ln \mu.$$

Combining these expressions with $\hat{\mu}_t = \phi \hat{z}_t$ gives

$$\ln Y_t = \ln Z_t + \ln K_{t-1}$$

$$= \hat{z}_t + \sum\nolimits_{j=1}^{t-1} \Delta \ln K_j + \ln K_0$$

$$= \hat{z}_t + \sum\nolimits_{j=1}^{t-1} \left(\hat{\mu}_j + \ln \mu\right) + \ln K_0$$

$$= \hat{z}_t + \phi \sum\nolimits_{j=1}^{t-1} \hat{z}_j + (t-1) \ln \mu + \ln K_0.$$

e. Deterministic trend and stochastic components:

$$y_t^{tr} \;=\; \mathbb{E} \ln Y_t \;=\; (t-1) \ln \mu + \ln K_0,$$

$$y_t^{s} \;=\; \ln Y_t - \mathbb{E} \ln Y_t \;=\; \hat{z}_t + \phi \sum\nolimits_{j=1}^{t-1} \hat{z}_j.$$

f. The specification of $\{Z_t\}$ implies

$$\hat{z}_t = \rho \hat{z}_{t-1} + \varepsilon_t. \tag{7}$$

Thus, we have $\hat{z}_t = \rho^{t-1} \varepsilon_1$ for $t \geq 1$ under the given path of ε_t. The implied stochastic component is

$$y_t^{s} \;=\; \rho^{t-1} \varepsilon_1 + \phi \sum\nolimits_{j=1}^{t-1} \rho^{j-1} \varepsilon_1 = \rho^{t-1} \varepsilon_1 + \phi \sum\nolimits_{j=0}^{t-2} \rho^{j} \varepsilon_1$$

$$= \rho^{t-1} \varepsilon_1 + \phi \left(\sum\nolimits_{j=0}^{\infty} \rho^{j} \varepsilon_1 - \rho^{t-1} \sum\nolimits_{k=0}^{\infty} \rho^{k} \varepsilon_1 \right)$$

$$= \rho^{t-1}\varepsilon_1 + \phi\frac{1-\rho^{t-1}}{1-\rho}.$$

The permanent component is

$$y_t^p \;=\; \lim_{k\to\infty} \mathbb{E}_t \ln y_{t+k}^s$$

$$= \rho^{t+k-1}\varepsilon_1 + \phi\frac{1-\rho^{t+k-1}}{1-\rho} \;=\; \frac{\phi}{1-\rho}\varepsilon_1,$$

and the transitory component is

$$y_t^c \;=\; y_t^s - \lim_{k\to\infty} \mathbb{E}_t \ln y_{t+k}^s$$

$$\rho^{t-1}\varepsilon_1 + \phi\frac{1-\rho^{t-1}}{1-\rho} - \frac{\phi}{1-\rho}\varepsilon_1$$

$$= \frac{\rho^{t-1}}{1-\rho}\left(1-\rho-\phi\right)\varepsilon_1.$$

Both the stochastic and permanent components increase at $t = 1$. The permanent component remains at its period 1 value, since there are no future shocks. Given $\phi < 1-\rho$, the transitory component also increases at $t = 1$, and then decreases to zero as $t \to \infty$.

Part of the transitory technology shock is consumed in short run, and part is saved, in line with the usual consumption smoothing motive. In this model, however, increased saving has a permanent effect, due to the AK technology.

The predicted IRFs fail to match the empirical IRFs estimated in Cogley and Nason (AER 1995) when ε_1 is identified as either permanent or transitory shock.

Rational Expectations Equilibrium

The REE of the model is derived as follows. First, to calculate the nonstochastic steady state equilibrium, solve the necessary conditions (1)-(3) for the steady state variables c, λ and μ:

$$\frac{1}{c} = \lambda,$$

$$\beta(2-\delta) = \mu,$$

$$2-\delta = c+\mu.$$

The soutions for c and μ are

$$\mu = \beta(2-\delta),$$

$$c = (1-\beta)(2-\delta).$$

Next, in the log-linearized conditions (4)-(6) there are two endogenous variables determined at time t, $\hat{c}_t$ and $\hat{\mu}_t$, and no endogenous variables determined at time $t-1$. Eliminating $\hat{c}_t$ and $\hat{\mu}_t$, and adding equation (7), gives the following two variable reduced system:

$$\mathbb{E}_t\hat{\lambda}_{t+1} + \frac{\beta}{\mu}\mathbb{E}_t\hat{z}_{t+1} = \left(\frac{c}{\mu}+1\right)\hat{\lambda}_t + \frac{1}{\mu}\hat{z}_t, \tag{8}$$

$$\hat{z}_t = \rho\hat{z}_{t-1} + \varepsilon_t.$$

The model is now easily solved using the method of undetermined coefficients. Substituting for $\mathbb{E}_t\hat{z}_{t+1}$ in (8) gives

$$\mathbb{E}_t\hat{\lambda}_{t+1} = \left(\frac{c}{\mu}+1\right)\hat{\lambda}_t + \frac{1}{\mu}\left(1-\beta\rho\right)\hat{z}_t. \tag{9}$$

Guess $\hat{\lambda}_t = \chi\hat{z}_t$, and substitute into (9):

$$\chi\rho\hat{z}_t = \left(\frac{c}{\mu}+1\right)\chi\hat{z}_t + \frac{1}{\mu}\left(1-\beta\rho\right)\hat{z}_t.$$

Canceling $\hat{z}_t$ and solving for χ gives

$$\chi = -\frac{1-\beta\rho}{c+\left(1-\rho\right)\mu}.$$

We may now solve for $\hat{\mu}_t$. Substitute $\hat{\lambda}_t = \chi\hat{z}_t$ into (5):

$$\chi\rho\hat{z}_t + \frac{\beta}{\mu}\rho\hat{z}_t = \hat{\mu}_t + \chi\hat{z}_t.$$

Solving for $\hat{\mu}_t$ gives

$$\hat{\mu}_t = \phi\hat{z}_t,$$

where

$$\phi = \frac{\beta\rho c + (1-\rho)\,\mu}{(c + (1-\rho)\,\mu)\,\mu}.$$

A sufficient condition for $\phi < 1 - \rho$ is obtained by substituting for c and μ:

$$\begin{aligned}\phi &= \frac{(\beta\rho(1-\beta) + (1-\rho)\,\beta)\,(2-\delta)}{(1-\beta+(1-\rho)\,\beta)\,(2-\delta)\cdot\beta(2-\delta)}\\ &= \frac{1}{2-\delta} < 1-\rho,\end{aligned}$$

or $(1-\rho)(2-\delta) > 1$.