Solution of RBC Model with Government Spending

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1 Equilibrium conditions

$$\chi C_t L_t^{1/\eta} = (1-\alpha)\frac{Y_t}{L_t} = W_t,$$

$$\frac{1}{C_t} = \mathbb{E}_t \beta (1+R_{t+1}) \frac{1}{C_{t+1}},$$

$$\mathbb{E}_t \frac{1}{1+R_{t+1}} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) = 1,$$

$$Y_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} = C_t + I_t + G_t,$$

$$K_t = (1-\delta) K_{t-1} + I_t,$$

$$Z_t = Z^{1-\rho} Z_{t-1}^{\rho} e^{\varepsilon_t},$$

$$G_t = G^{1-\gamma} G_{t-1}^{\gamma} e^{\nu_t}.$$

The parameters satisfy

$$\chi, \eta, Z, G > 0, \quad 0 < \alpha, \beta, \delta, \rho, \gamma < 1.$$

$\{\varepsilon_t\}$ and $\{\nu_t\}$ are uncorrelated white noise processes.

2 Nonstochastic steady state

$$\frac{K}{L} = \kappa = \left(\frac{1/\beta - (1-\delta)}{\alpha Z} \right)^{\frac{1}{\alpha - 1}},$$

$$Z \kappa^{\alpha} L = \frac{(1-\alpha) Z \kappa^{\alpha}}{\chi} L^{-1/\eta} + \delta \kappa L + G,$$

$$C = \frac{(1-\alpha)\, Z\kappa^{\alpha}}{\chi} L^{-1/\eta},$$

$$I = \delta\kappa L,$$

$$Y = Z\kappa^{\alpha} L = C + I + G,$$

$$R = \alpha Z\kappa^{\alpha-1} - \delta.$$

Also:

$$\beta = \left(\alpha Z\kappa^{\alpha-1} + 1 - \delta\right)^{-1}.$$

3 Log-linearized equilibrium conditions

$$\hat{y}_t = \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{\imath}_t + \frac{G}{Y}\hat{g}_t,$$

$$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta\hat{\imath}_t,$$

$$\hat{c}_t + \frac{1}{\eta}\hat{l}_t = \hat{y}_t - \hat{l}_t = \hat{w}_t,$$

$$\mathbb{E}_t\hat{c}_{t+1} - \hat{c}_t = \beta\alpha\frac{Y}{K}(\mathbb{E}_t\hat{y}_{t+1} - \hat{k}_t) = (1-\beta)\mathbb{E}_t\hat{r}_{t+1},$$

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1} + (1-\alpha)\hat{l}_t,$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \varepsilon_{t+1},$$

$$\hat{g}_{t+1} = \gamma\hat{g}_t + \nu_{t+1}.$$

Resource constraint:

$$\begin{aligned}
&\hat{z}_t + \alpha\hat{k}_{t-1} + (1-\alpha)\hat{l}_t \\
&= \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\frac{1}{\delta}\left(\hat{k}_t - (1-\delta)\hat{k}_{t-1}\right) + \frac{G}{Y}\hat{g}_t \\
&= \frac{C}{Z\kappa^{\alpha}L}\hat{c}_t + \kappa^{1-\alpha}\hat{k}_t - (1-\delta)\kappa^{1-\alpha}\hat{k}_{t-1} + \frac{G}{Z\kappa^{\alpha}L}\hat{g}_t,
\end{aligned}$$

$$\hat{z}_t + \left(\alpha + (1-\delta)\kappa^{1-\alpha}\right)\hat{k}_{t-1} + (1-\alpha)\hat{l}_t = \frac{C}{Z\kappa^{\alpha}L}\hat{c}_t + \kappa^{1-\alpha}\hat{k}_t + \frac{G}{Z\kappa^{\alpha}L}\hat{g}_t.$$

Five-variable system:

$$\hat{c}_t + \left(\alpha + \frac{1}{\eta}\right)\hat{l}_t = \hat{z}_t + \alpha\hat{k}_{t-1},$$

$$\mathbb{E}_t\hat{c}_{t+1} - \hat{c}_t = \beta\alpha\kappa^{\alpha-1}\left(\mathbb{E}_t\hat{z}_{t+1} + \alpha\hat{k}_t + (1-\alpha)\mathbb{E}_t\hat{l}_{t+1} - \hat{k}_t\right),$$

$$\hat{z}_t + \left(\alpha + (1-\delta)\kappa^{1-\alpha}\right)\hat{k}_{t-1} + (1-\alpha)\hat{l}_t = \frac{C}{Z\kappa^\alpha L}\hat{c}_t + \kappa^{1-\alpha}\hat{k}_t + \frac{G}{Z\kappa^\alpha L}\hat{g}_t,$$

plus the $\hat{z}_{t+1}$ and $\hat{g}_{t+1}$ equations. This system determines $\hat{c}_t$, $\hat{l}_t$, $\hat{k}_t$, $\hat{z}_t$ and $\hat{g}_t$ given ε_t and ν_t.

Auxiliary variables:

$$\hat{l}_t = \frac{1}{\psi_1}\hat{z}_t + \frac{\alpha}{\psi_1}\hat{k}_{t-1} - \frac{1}{\psi_1}\hat{c}_t,$$

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1} + (1-\alpha)\hat{l}_t,$$

$$\hat{\imath}_t = \frac{1}{\delta}\left(\hat{k}_t - (1-\delta)\hat{k}_{t-1}\right),$$

$$\hat{w}_t = \hat{c}_t + \frac{1}{\eta}\hat{l}_t = \hat{y}_t - \hat{l}_t = \hat{p}_t.$$

4 Reduced-form system

Define

$$\psi_1 = \alpha + \frac{1}{\eta}, \quad \psi_2 = \beta\alpha\kappa^{\alpha-1},$$

$$\psi_3 = \alpha + (1-\delta)\kappa^{1-\alpha}, \quad \psi_4 = \frac{C}{Z\kappa^\alpha L}, \quad \psi_5 = \frac{G}{Z\kappa^\alpha L}.$$

The system maybe written as

$$\hat{c}_t + \psi_1\hat{l}_t = \hat{z}_t + \alpha\hat{k}_{t-1},$$

$$\mathbb{E}_t\hat{c}_{t+1} - \hat{c}_t = \psi_2\left(\mathbb{E}_t\hat{z}_{t+1} + \alpha\hat{k}_t + (1-\alpha)\mathbb{E}_t\hat{l}_{t+1} - \hat{k}_t\right),$$

$$\hat{z}_t + \psi_3\hat{k}_{t-1} + (1-\alpha)\hat{l}_t = \psi_4\hat{c}_t + \kappa^{1-\alpha}\hat{k}_t + \psi_5\hat{g}_t,$$

plus the $\hat{z}_{t+1}$ and $\hat{g}_{t+1}$ equations. Eliminate $\hat{l}_t$:

$$\hat{l}_t = \frac{1}{\psi_1}\hat{z}_t + \frac{\alpha}{\psi_1}\hat{k}_{t-1} - \frac{1}{\psi_1}\hat{c}_t,$$

$$\mathbb{E}_t\hat{c}_{t+1}-\hat{c}_t=\psi_2\left(\mathbb{E}_t\hat{z}_{t+1}-(1-\alpha)\left(1-\frac{\alpha}{\psi_1}\right)\hat{k}_t+\frac{1-\alpha}{\psi_1}\mathbb{E}_t\hat{z}_{t+1}-\frac{1-\alpha}{\psi_1}\mathbb{E}_t\hat{c}_{t+1}\right),$$

$$\left(1+\frac{1-\alpha}{\psi_1}\right)\hat{z}_t+\left(\psi_3+\frac{(1-\alpha)\alpha}{\psi_1}\right)\hat{k}_{t-1}-\frac{1-\alpha}{\psi_1}\hat{c}_t=\psi_4\hat{c}_t+\kappa^{1-\alpha}\hat{k}_t+\psi_5\hat{g}_t.$$

Substitute $\mathbb{E}_t\hat{z}_{t+1}=\rho\hat{z}_t$, and express the system as a first-order expectational difference system:

$$\left(1+\frac{(1-\alpha)\,\psi_2}{\psi_1}\right)\mathbb{E}_t\hat{c}_{t+1}+(1-\alpha)\psi_2\left(1-\frac{\alpha}{\psi_1}\right)\hat{k}_t=\hat{c}_t+\rho\psi_2\left(1+\frac{1-\alpha}{\psi_1}\right)\hat{z}_t,$$

$$\kappa^{1-\alpha}\hat{k}_t=-\left(\psi_4+\frac{1-\alpha}{\psi_1}\right)\hat{c}_t+\left(\psi_3+\frac{(1-\alpha)\alpha}{\psi_1}\right)\hat{k}_{t-1}+\left(1+\frac{1-\alpha}{\psi_1}\right)\hat{z}_t-\psi_5\hat{g}_t,$$

$$\hat{z}_{t+1}=\rho\hat{z}_t+\varepsilon_{t+1},$$

$$\hat{g}_{t+1}=\gamma\hat{g}_t+\nu_{t+1}.$$

In terms of matrices and vectors:

$$\begin{bmatrix} 1+\frac{(1-\alpha)\psi_2}{\psi_1} & (1-\alpha)\psi_2\left(1-\frac{\alpha}{\psi_1}\right) & 0 & 0 \\ 0 & \kappa^{1-\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbb{E}_t\hat{c}_{t+1} \\ \hat{k}_t \\ \hat{z}_{t+1} \\ \hat{g}_{t+1} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & \rho\psi_2\left(1+\frac{1-\alpha}{\psi_1}\right) & 0 \\ -\left(\psi_4+\frac{1-\alpha}{\psi_1}\right) & \psi_3+\frac{(1-\alpha)\alpha}{\psi_1} & 1+\frac{1-\alpha}{\psi_1} & -\psi_5 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} \hat{c}_t \\ \hat{k}_{t-1} \\ \hat{z}_t \\ \hat{g}_t \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \varepsilon_{t+1} \\ \nu_{t+1} \end{bmatrix}.$$

5 REE Solution

The solution takes the form

$$\hat{c}_t = \Phi_U \begin{bmatrix} \hat{k}_{t-1} \\ \hat{z}_t \\ \hat{g}_t \end{bmatrix},$$

$$\begin{bmatrix} \hat{k}_t \\ \hat{z}_{t+1} \\ \hat{g}_{t+1} \end{bmatrix} = \Phi_S \begin{bmatrix} \hat{k}_{t-1} \\ \hat{z}_t \\ \hat{g}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1} \\ \nu_{t+1} \end{bmatrix}.$$

Let the numerical parameter values be

$$\begin{matrix} \alpha & \beta & \delta & \eta & \rho & \sigma_\varepsilon \\ .34 & 1.04^{-1/4} & .02 & .25 & .95 & .007 \end{matrix}.$$

Also:

$$\begin{matrix} \chi & Z & G & \gamma & \sigma_\nu \\ 1 & 1 & 0 & 0 & 0 \end{matrix}.$$

Then the solution is given by

$$\Phi_U = \begin{bmatrix} .592 & .311 & 0 \end{bmatrix}, \quad \Phi_S = \begin{bmatrix} .966 & .076 & 0 \\ 0 & .95 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$