Rational Expectations Solutions of Linear Dynamic Stochastic Models

Garey Ramey
University of California, San Diego

April 2019 - Version 2

1 Linear dynamic stochastic model

Consider the following model:

$$0 = A_v v_t + A_\lambda \lambda_t + A_k k_t + A_z z_t, \tag{1}$$

$$B_v E_t v_{t+1} + B_{\lambda 1} E_t \lambda_{t+1} + B_{k1} k_{t+1} + B_z E_t z_{t+1} = B_{\lambda 0} \lambda_t, \tag{2}$$

$$C_{k1} k_{t+1} = C_v v_t + C_{k0} k_t + C_z z_t, \tag{3}$$

where v_t is an $l \times 1$ vector of endogenous variables determined at t, λ_t is an $m \times 1$ vector of Lagrangian multipliers, k_t is an $s \times 1$ vector of endogenous variables determined at $t-1$, and z_t is a $d \times 1$ vector of exogenous variables determined at t. In terms of macroeconomic models, (1) represents l linearized intratemporal Euler equations corresponding to the elements of v_t, (2) gives s linearized intertemporal Euler equations associated with the elements of k_{t+1}, and (3) gives m linearized constraints. A_v is assumed to be invertible, and $\{z_t\}$ is a stationary vector process.

The system may be expressed as

$$v_t = -A_v^{-1} \begin{bmatrix} A_\lambda & A_k \end{bmatrix} \begin{bmatrix} \lambda_t \\ k_t \end{bmatrix} - A_v^{-1} A_z z_t, \tag{4}$$

$$\begin{bmatrix} B_v \\ 0 \end{bmatrix} E_t v_{t+1} + \begin{bmatrix} B_{\lambda 1} & B_k \\ 0 & C_{k1} \end{bmatrix} \begin{bmatrix} E_t \lambda_{t+1} \\ k_{t+1} \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} 0 \\ C_v \end{bmatrix} v_t + \begin{bmatrix} B_{\lambda 0} & 0 \\ 0 & C_{k0} \end{bmatrix} \begin{bmatrix} \lambda_t \\ k_t \end{bmatrix}$$

$$+ \begin{bmatrix} -B_z \\ 0 \end{bmatrix} E_t z_{t+1} + \begin{bmatrix} 0 \\ C_z \end{bmatrix} z_t.$$

Using (4) to substitute for $E_t v_{t+1}$ and v_t and rearranging gives

$$D_1 \begin{bmatrix} E_t \lambda_{t+1} \\ k_{t+1} \end{bmatrix} = D_0 \begin{bmatrix} \lambda_t \\ k_t \end{bmatrix} \tag{6}$$

$$+ D_{z1} E_t z_{t+1} + D_{z0} z_t,$$

where

$$D_1 = \begin{bmatrix} B_{\lambda 1} & B_k \\ 0_{m \times m} & C_{k1} \end{bmatrix} - \begin{bmatrix} B_v \\ 0_{m \times l} \end{bmatrix} A_v^{-1} \begin{bmatrix} A_\lambda & A_k \end{bmatrix},$$

$$D_0 = \begin{bmatrix} B_{\lambda 0} & 0_{s \times s} \\ 0_{m \times m} & C_{k0} \end{bmatrix} - \begin{bmatrix} 0_{s \times l} \\ C_v \end{bmatrix} A_v^{-1} \begin{bmatrix} A_\lambda & A_k \end{bmatrix},$$

$$D_{z1} = \begin{bmatrix} -B_z \\ 0_{m \times d} \end{bmatrix} + \begin{bmatrix} B_v \\ 0_{m \times l} \end{bmatrix} A_v^{-1} A_z,$$

$$D_{z0} = \begin{bmatrix} 0_{s \times d} \\ C_z \end{bmatrix} - \begin{bmatrix} 0_{s \times l} \\ C_v \end{bmatrix} A_v^{-1} A_z.$$

D_1 and D_0 are $(m+s) \times (m+s)$ matrices, and D_{z1} and D_{z0} are $(m+s) \times d$ matrices. Note that (6) reduces the system to m costate variables and s endogenous state variables, combined with the exogenous process $\{z_t\}$.

2 Rational expectations solution

Suppose $z_{t+1} = \Pi z_t + \varepsilon_{t+1}$, where $\{\varepsilon_t\}$ is vector white noise. Then (6) and the z_t process may be expressed jointly as

$$A_1 \begin{bmatrix} E_t \lambda_{t+1} \\ k_{t+1} \\ z_{t+1} \end{bmatrix} = A_0 \begin{bmatrix} \lambda_t \\ k_t \\ z_t \end{bmatrix} + B_0 \varepsilon_{t+1}, \tag{7}$$

where

$$A_1 = \begin{bmatrix} D_1 & 0_{(m+s)\times d} \\ 0_{d\times(m+s)} & I_{d\times d} \end{bmatrix},$$

$$A_0 = \begin{bmatrix} D_0 & D_{z0} + D_{z1}\Pi \\ 0_{d\times(m+s)} & \Pi \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0_{(m+s)\times d} \\ I_{d\times d} \end{bmatrix}$$

A_1 and A_0 are $(m+s+d)\times(m+s+d)$ matrices.

A rational expectations solution of (7) takes the form (see "Notes on Linear Rational Expectations Equilibria"):

$$\lambda_t = \Phi_U \begin{bmatrix} k_t \\ z_t \end{bmatrix},$$

$$\begin{bmatrix} k_{t+1} \\ z_{t+1} \end{bmatrix} = \Phi_S \begin{bmatrix} k_t \\ z_t \end{bmatrix} + \begin{bmatrix} 0_{s\times d} \\ I_{d\times d} \end{bmatrix} \varepsilon_{t+1}.$$

The associated solution for v_t is determined by (4).

3 RBC model with government spending

Consider the following standard specification of the Real Business Cycle model with government spending: For $t = 0, 1, 2, ...$, the social planner solves

$$\max_{C_t, L_t, K_{t+1}} E_t \sum\nolimits_{s=0}^{\infty} \beta^s \left(\ln C_{t+s} - \frac{\chi L_{t+s}^{1+1/\eta}}{1+1/\eta} \right)$$

subject to

$$Z_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t = C_t + G_t + K_{t+1},$$

$$Z_t = Z_{t-1}^{\rho} e^{\varepsilon_t},$$

$$G_t = G^{1-\gamma} G_{t-1}^{\gamma} e^{\nu_t},$$

where $\chi, G > 0$, $0 < \beta, \alpha, \delta, \rho, \gamma < 1$, and $\{[\varepsilon_t \ \nu_t]'\}$ is exogenous vector white noise. First-order necessary conditions for a solution are

$$\frac{1}{C_t} = \lambda_t,$$

$$\chi L_t^{1/\eta} = (1-\alpha) \lambda_t Z_t K_t^{\alpha} L_t^{-\alpha},$$

$$\lambda_t = \beta E_t \lambda_{t+1} \left[\alpha Z_{t+1} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} + 1 - \delta \right],$$

$$Z_t K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t = C_t + G_t + K_{t+1},$$

together with the Z_t and G_t processes, where λ_t is the Lagrange multiplier on the resource constraint in period t.

Log-linearizing the necessary conditions gives

$$-\hat{c}_t = \hat{\lambda}_t,$$

$$\frac{1}{\eta} \hat{l}_t = \hat{\lambda}_t + \hat{z}_t + \alpha \hat{k}_t - \alpha \hat{l}_t,$$

$$E_t \hat{\lambda}_{t+1} + \beta \alpha \kappa^{\alpha-1} \left(E_t \hat{z}_{t+1} - (1-\alpha) \hat{k}_{t+1} + (1-\alpha) E_t \hat{l}_{t+1} \right) = \hat{\lambda}_t,$$

$$\kappa^{\alpha} L \left(\hat{z}_t + \alpha \hat{k}_t + (1-\alpha) \hat{l}_t \right) + (1-\delta) \kappa L \hat{k}_t = C \hat{c}_t + G \hat{g}_t + \kappa L \hat{k}_{t+1},$$

$$\hat{z}_t = \rho \hat{z}_{t-1} + \varepsilon_t,$$

$$\hat{g}_t = \gamma \hat{g}_{t-1} + \nu_t,$$

where κ, L and C are nonstochastic steady state values:

$$\frac{K}{L} = \kappa = \left(\frac{1/\beta - (1-\delta)}{\alpha} \right)^{1/(\alpha-1)},$$

$$\left(\kappa^\alpha - \delta\kappa\right) L = \frac{1-\alpha}{\chi} \kappa^\alpha L^{-1/\eta} + G,$$

$$\frac{1}{\lambda} = C = \frac{1-\alpha}{\chi} \kappa^\alpha L^{-1/\eta}.$$

For this example, the vectors v_t, λ_t, k_t and z_t are given by

$$v_t = \begin{bmatrix} \hat{c}_t \\ \hat{l}_t \end{bmatrix}, \quad \lambda_t = \hat{\lambda}_t, \quad k_t = \hat{k}_t, \quad z_t = \begin{bmatrix} \hat{z}_t \\ \hat{g}_t \end{bmatrix}.$$

Expressing the linearized necessary conditions in the form (1)-(3) gives:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(\alpha + \frac{1}{\eta}) \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ \hat{l}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \hat{\lambda}_t$$
$$+ \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \hat{k}_t + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{z}_t \\ \hat{g}_t \end{bmatrix},$$

$$\begin{bmatrix} 0 & \beta\alpha(1-\alpha)\kappa^{\alpha-1} \end{bmatrix} \begin{bmatrix} E_t\hat{c}_{t+1} \\ E_t\hat{l}_{t+1} \end{bmatrix} + E_t\hat{\lambda}_{t+1}$$
$$-\beta\alpha(1-\alpha)\kappa^{\alpha-1}\hat{k}_{t+1} + \begin{bmatrix} \beta\alpha\kappa^{\alpha-1} & 0 \end{bmatrix} \begin{bmatrix} E_t\hat{z}_{t+1} \\ E_t\hat{g}_{t+1} \end{bmatrix} = \hat{\lambda}_t,$$

$$\kappa L \hat{k}_{t+1} = \begin{bmatrix} -C & (1-\alpha)\kappa^\alpha L \end{bmatrix} \begin{bmatrix} \hat{c}_t \\ \hat{l}_t \end{bmatrix}$$
$$+ \left(\alpha\kappa^{\alpha-1} + 1 - \delta\right) \kappa L \hat{k}_t + \begin{bmatrix} \kappa^\alpha L & -G \end{bmatrix} \begin{bmatrix} \hat{z}_t \\ \hat{g}_t \end{bmatrix}.$$