Economics 210C - Macroeconomics Spring 2018-19

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Phase Diagram Exercise v.2 - Solution

a. The Lagrangian for the household's problem is:

$$\mathcal{L} = \mathbb{E}_t \sum\nolimits_{s=0}^{\infty} \beta^s \Bigg(\ln C_{t+s} - \frac{L_{t+s}^{1+1/\eta}}{1+1/\eta} + \lambda_{t+s}\big(W_{t+s}L_{t+s} + (1-\tau_{t+s})R_{t+s}K_{t+s-1}$$

$$+\Pi_{t+s} + (1-\delta)K_{t+s-1} - K_{t+s} - C_{t+s} - T_{t+s}\big)\Bigg)$$

The first order necessary conditions for C_t, L_t, K_t and λ_t are, after canceling terms and rearranging:

$$\frac{1}{C_t} = \lambda_t, \tag{1}$$

$$L_t^{1/\eta} = \lambda_t W_t, \tag{2}$$

$$\lambda_t = \beta \mathbb{E}_t \lambda_{t+1} \left((1-\tau_{t+1})R_{t+1} + 1 - \delta \right), \tag{3}$$

$$W_t L_t + (1-\tau_t)R_t K_{t-1} + \Pi_t + (1-\delta)K_{t-1} = K_t + C_t + T_t. \tag{4}$$

The first order necessary conditions for the firm's choices of L_t and K_t are

$$W_t = (1-\alpha)Z_t K_{t-1}^{\alpha} L_t^{-\alpha}, \tag{5}$$

$$R_t = \alpha Z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha}. \tag{6}$$

b. Substitute (5)-(6) into (2)-(3) to obtain

$$L_t^{1/\eta} = \lambda_t (1-\alpha) Z_t K_{t-1}^{\alpha} L_t^{-\alpha}, \tag{7}$$

$$\lambda_t = \beta \mathbb{E}_t \lambda_{t+1} \Big((1-\tau_{t+1})\alpha Z_{t+1} K_t^{\alpha-1} L_{t+1}^{1-\alpha} + 1 - \delta \Big). \tag{8}$$

Using the definition of Π_t, household income satisfies

$$W_t L_t + (1-\tau_t)R_t K_{t-1} + \Pi_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha}.$$

Substituting the latter equation and the government budget constraint into (4) gives

$$Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} + (1-\delta) K_{t-1} = K_t + C_t + G_t. \tag{9}$$

In summary, the necessary conditions are (1), (7), (8) and (9), which determine the endogenous variables C_t, L_t, λ_t and K_t given exogenous variables Z_t, τ_t and G_t.

c. Since the exogenous variables are perfectly-anticipated deterministic paths, the expectation operator can be dropped. Equation (8) implies

$$\lambda_{t-1} = \beta \lambda_t \left((1-\tau_t) \alpha Z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} + 1 - \delta \right),$$

and hence

$$\Delta \lambda_t = \lambda_t \left(1 - \beta \left((1-\tau_t) \alpha Z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} + 1 - \delta \right) \right). \tag{10}$$

Using (9), we have

$$\Delta K_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1} - C_t - G_t. \tag{11}$$

d. Necessary condition (1) is unchanged. The continuous time approximation of (7) is

$$L_t^{1/\eta} = \lambda_t (1-\alpha) Z_t K_t^{\alpha} L_t^{-\alpha}. \tag{12}$$

Note that K_{t-1} has been replaced by K_t, reflecting the fact that period length has been taken to zero.

Next, write (10) as

$$\Delta \lambda_t = \lambda_t \beta \left(\frac{1-\beta}{\beta} - \left((1-\tau_t) \alpha Z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} - \delta \right) \right). \tag{13}$$

The continuous time approximation of (13) is given by

$$\dot{\lambda}_t = \lambda_t \left(r - \left((1-\tau_t) \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta \right) \right), \tag{14}$$

where r satisfies $e^{-r} = \beta$. Finally, (11) becomes

$$\dot{K}_t = Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_t - C_t - G_t. \tag{15}$$

The continuous time model consists of (1), (12), (14) and (15).

e. The total log-derivatives of (1) and (12) are

$$-d\ln C_t = d\ln\lambda_t, \tag{16}$$

$$\frac{1}{\eta}d\ln L_t = d\ln\lambda_t + d\ln Z_t + \alpha\, d\ln K_t - \alpha\, d\ln L_t. \tag{17}$$

Totally log-differentiating (14) at points such that $\dot{\lambda}_t = 0$ gives

$$0 = \frac{\tau_t}{1-\tau_t}d\ln\tau_t - d\ln Z_t + (1-\alpha)\, d\ln K_t - (1-\alpha)\, d\ln L_t. \tag{18}$$

Finally, totally log-differentiating (15) at points such that $\dot{K}_t = 0$ gives

$$0 = Y_t\left(d\ln Z_t + \alpha\, d\ln K_t + (1-\alpha)d\ln L_t\right) \tag{19}$$

$$-\delta K_t d\ln K_t - C_t d\ln C_t - G_t d\ln G_t,$$

where $Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$.

To calculate the partial log-derivatives of the $\dot{\lambda} = 0$ curve, first solve (17) for $d\ln L_t$:

$$d\ln L_t = \frac{1}{1/\eta+\alpha}\left(d\ln\lambda_t + d\ln Z_t + \alpha\, d\ln K_t\right). \tag{20}$$

Substitute into (18) and rearrange:

$$\frac{1-\alpha}{1/\eta+\alpha}d\ln\lambda_t = (1-\alpha)\left(1-\frac{\alpha}{1/\eta+\alpha}\right)d\ln K_t$$

$$-\left(1+\frac{1-\alpha}{1/\eta+\alpha}\right)d\ln Z_t + \frac{\tau_t}{1-\tau_t}d\ln\tau_t.$$

The partial log-derivatives are calculated using the preceding equation:

$$\left.\frac{\partial\ln\lambda_t}{\partial\ln K_t}\right|_{\dot{\lambda}_t=0} = \left(\frac{1-\alpha}{1/\eta+\alpha}\right)^{-1}(1-\alpha)\left(1-\frac{\alpha}{1/\eta+\alpha}\right) = \frac{1}{\eta} > 0,$$

$$\left.\frac{\partial\ln\lambda_t}{\partial\ln Z_t}\right|_{\dot{\lambda}_t=0} = -\frac{1/\eta+1}{1-\alpha} < 0,$$

$$\left.\frac{\partial \ln \lambda_t}{\partial \ln \tau_t}\right|_{\dot{\lambda}_t=0} = \frac{1/\eta + \alpha}{1-\alpha} \frac{\tau_t}{1-\tau_t} > 0. \tag{21}$$

Finally, to calculate the partial log-derivatives of the $\dot{K}_t = 0$ curve, substitute (16) and (20) into (19) and rearrange:

$$(C_t + \Theta_t)\, d \ln \lambda_t = -\left(\alpha\left(Y_t + \Theta_t\right) - \delta K_t\right) d \ln K_t - \left(Y_t + \Theta_t\right) d \ln Z_t + G_t d \ln G_t,$$

where

$$\Theta_t = Y_t \frac{1-\alpha}{1/\eta + \alpha}.$$

In the steady state, we have $\dot{\lambda}_t = 0$, and hence

$$\alpha Y_t - \delta K_t = \left(\alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta\right) K_t$$

$$\geq \left((1-\tau_t)\alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} - \delta\right) K_t = r K_t > 0.$$

Hence the coefficient on $d \ln K_t$ is strictly positive in the neighborhood of the steady state, which implies

$$\left.\frac{\partial \ln \lambda_t}{\partial \ln K_t}\right|_{\dot{K}_t=0} = -\frac{\alpha\left(Y_t + \Theta_t\right) - \delta K_t}{C_t + \Theta_t} < 0.$$

Finally,

$$\left.\frac{\partial \ln \lambda_t}{\partial \ln Z_t}\right|_{\dot{K}_t=0} = -\frac{Y_t + \Theta_t}{C_t + \Theta_t} < 0,$$

$$\left.\frac{\partial \ln \lambda_t}{\partial \ln G_t}\right|_{\dot{K}_t=0} = \frac{G_t}{C_t + \Theta_t} > 0.$$

f. Figure 1 shows the canonical phase diagram for the Ramsey growth and RBC models. The steady state is at the point (K, λ). The existence of the $\dot{\lambda}_t = 0$ and $\dot{K}_t = 0$ curves, and the directions of adjustment for points away from the steady state, can be checked using conventional total differentiation of equations (1), (12), (14) and (15). The figure also depicts the saddlepoint path, which can be shown to be unique.

Suppose τ_t rises, while Z_t and G_t remain constant. Since the partial log-derivative (21) is strictly positive, it follows that λ_t must increase at every level of K_t in order to maintain $\dot{\lambda}_t = 0$. This means that the $\dot{\lambda}_t = 0$ curve shifts upward. The $\dot{K}_t = 0$ curve is unaffected, since τ_t does not enter equation (15).

g. Figure 2 shows the shift in the $\dot{\lambda}_t = 0$ curve that occurs at time t_0, along with the ensuing perfect foresight path. The larger arrows depict directions of adjustment during the interval $[t_0, t_1)$.

At t_0, the economy jumps immediately to point A, with $\lambda_{t_0} > \lambda$ and $K_{t_0} = K$. Consumption falls and labor supply rises, as the household anticipates building up capital during the period of low tax rates. Wages must correspondingly fall, to clear the labor market. As the economy adjusts from A to B, capital at first rises, and then falls, in anticipation of the upcoming tax rate increase. λ_t falls steadily, so that consumption gradually rises. The paths of labor input and wages are unclear, since the adjustments of λ_t and K_t have offsetting effects on the labor market equation (12).

At time t_1, the economy hits point B, and then it subsequently follows the original saddlepoint path back to the initial steady state, at point C. As λ_t rises, consumption falls.

Overall, the household takes advantage of the temporary tax cut by building up capital. After an initial buildup phase, consumption is higher, and it remains higher for an interval following the reversal of the tax cut. Labor input may rise or fall along the transition path, due to conflicting wealth and substitution effects.

Figure 1

Canonical Phase Diagram

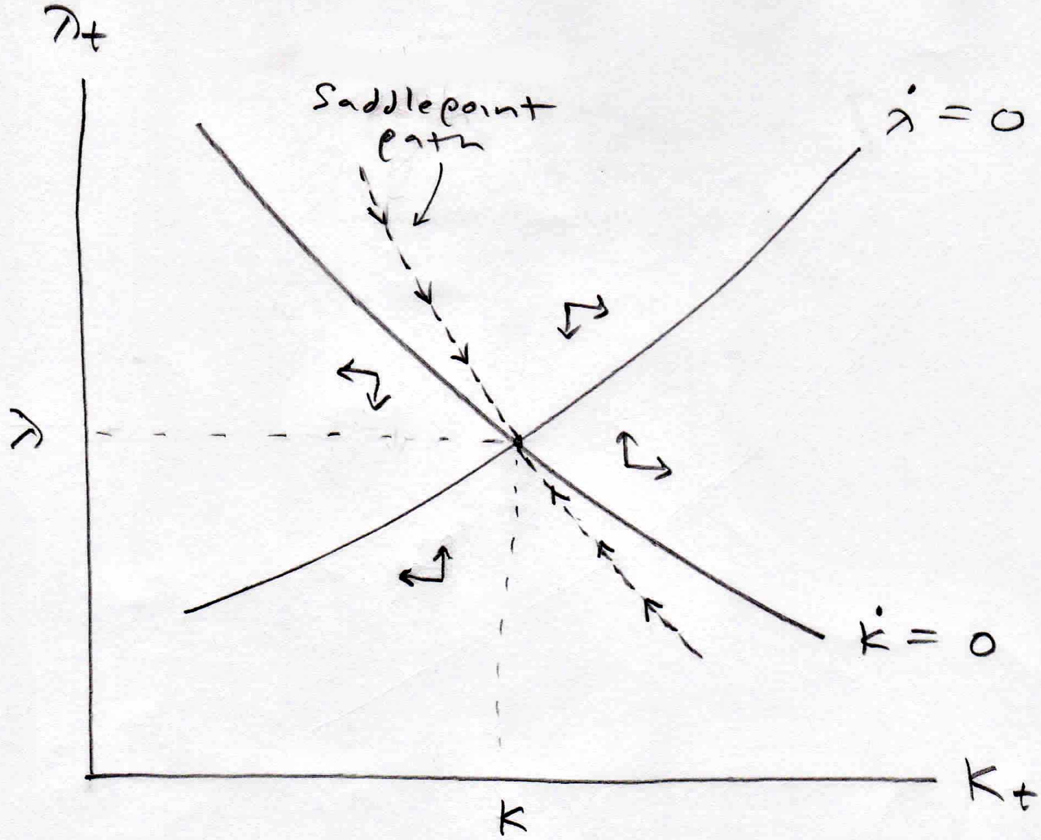

Figure 2

Temporary Cut in Capital Income Tax