Phase Diagram Analysis of the Perfect Foresight Dynamics of the RBC Model

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1 Introduction

This note shows how to construct rigorously a phase diagram for analyzing the perfect foresight dynamics of a discrete-time RBC model. The construction begins by converting the necessary conditions for a solution to continuous time. After checking that it is possible to define the curves $\dot{\lambda} = 0$ and $\dot{K} = 0$ needed for the phase diagram, the analysis proceeds by totally log-differentiating these curves. Total log-differentiation is mathematically equivalent in its essentials to log-linearizing around a steady state, and as a consequence it provides a familiar and tractable procedure for studying phase diagrams.

Section 2 shows how to convert the necessary conditions to continuous time, and section 3 considers the definitions of the $\dot{\lambda} = 0$ and $\dot{K} = 0$ curves. Analysis of the curves via total log-differentiation is developed in Section 4. Section 5 provides an illustrative example of the entire construction, and section 6 briefly summarizes the procedures of conversion to continuous time and total log-differentiation.

2 Perfect foresight dynamics in continuous time

Consider a version of the Real Business Cycle model in which the period length is $\Delta > 0$. The social planner problem is given by

$$\max_{\{C_t, L_t, K_t\}} \mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} e^{-r\Delta t} \left(U_t(C_t) - V(L_t) \right) \Delta$$

$$\text{s.t.} \quad F(Z_t, K_{t-\Delta}, L_t)\Delta + (1 - \delta\Delta)K_{t-\Delta} = K_t + C_t\Delta + G_t\Delta,$$

where $0 < \beta, \delta < 1$, and $\{Z_t\}$ and $\{G_t\}$ are exogenous stationary processes. The functions U, V and F are assumed to satisfy standard regularity conditions. Necessary conditions for an interior solution of this problem are given by

$$U_C(C_t) = \lambda_t, \tag{1}$$

$$V_L(L_t) = \lambda_t F_L(Z_t, K_{t-\Delta}, L_t), \tag{2}$$

$$e^{-r\Delta}\mathbb{E}_t \lambda_{t+\Delta} \left(F_K(Z_{t+\Delta}, K_t, L_{t+\Delta})\Delta + 1 - \delta\Delta\right) = \lambda_t, \tag{3}$$

$$F(Z_t, K_{t-\Delta}, L_t)\Delta + (1 - \delta\Delta)K_{t-\Delta} = K_t + C_t\Delta + G_t\Delta, \tag{4}$$

where λ_t is the Lagrange multiplier on the resource constraint.

If the exogenous variables follow known deterministic paths, then the planner faces no uncertainty, and rational expectations are equivalent to perfect foresight. In this case, conditions (3) and (4) may be rewritten as

$$\frac{\lambda_{t+\Delta} - \lambda_t}{\Delta} = \lambda_{t+\Delta}\left(\frac{1 - e^{-r\Delta}}{\Delta} - e^{-r\Delta}\left(F_K(Z_{t+\Delta}, K_t, L_{t+\Delta}) - \delta\right)\right), \tag{5}$$

$$\frac{K_t - K_{t-\Delta}}{\Delta} = F(Z_t, K_{t-\Delta}, L_t) - \delta K_{t-\Delta} - C_t - G_t. \tag{6}$$

Taking the limit as $\Delta \to 0$ of (1)-(2) and (5)-(6) gives the following continuous time system:

$$U_C(C_t) = \lambda_t, \tag{7}$$

$$V_L(L_t) = \lambda_t F_L(Z_t, K_t, L_t), \tag{8}$$

$$\dot{\lambda}_t = \lambda_t\left(r + \delta - F_K(Z_t, K_t, L_t)\right), \tag{9}$$

$$\dot{K}_t = F(Z_t, K_t, L_t) - \delta K_t - C_t - G_t. \tag{10}$$

These equations characterize the perfect foresight dynamics of the model in continuous time.

The dynamics of the system (7)-(10) may be analyzed using a phase diagram in the K_t-λ_t plane. To set up the diagram, begin by solving (7) and (8) for C_t and L_t:

$$C_t = \Phi^C(\lambda_t), \tag{11}$$

$$L_t = \Phi^L(\lambda_t, K_t, Z_t). \tag{12}$$

Note that Φ^C is the inverse of the marginal utility of consumption, whereas Φ^L depends on both the marginal disutility of labor and marginal product of labor. Substituting Φ^C and Φ^L into (9) and (10) gives

$$\begin{aligned}\dot{\lambda}_t &= \lambda_t \left(r + \delta - F_K(Z_t, K_t, \Phi^L(\cdot))\right) \qquad (13)\\ &= \Psi^\lambda(\lambda_t, K_t, Z_t),\end{aligned}$$

$$\begin{aligned}\dot{K}_t &= F(Z_t, K_t, \Phi^L(\cdot)) - \delta K_t - \Phi^C(\cdot) - G_t \qquad (14)\\ &= \Psi^K(\lambda_t, K_t, Z_t, G_t).\end{aligned}$$

Observe that the functions Ψ^λ and Ψ^K determine the dynamic behavior of K_t and λ_t for given perfectly-anticipated deterministic paths of Z_t and G_t.

3 Defining the $\dot{\lambda}_t = 0$ and $\dot{K}_t = 0$ curves

A phase diagram analysis in the K_t-λ_t plane involves tracing the paths of λ_t and K_t relative to a pair of curves along which $\dot{\lambda} = 0$ and $\dot{K} = 0$. The latter curves may be defined implicitly using (13) and (14):

$$0 = \Psi^\lambda(\bar{\lambda}^\lambda(K_t, Z_t), K_t, Z_t), \tag{15}$$

$$0 = \Psi^K(\bar{\lambda}^K(K_t, Z_t, G_t), K_t, Z_t, G_t), \tag{16}$$

where $\bar{\lambda}^\lambda$ and $\bar{\lambda}^K$ are functions indicating the $\dot{\lambda} = 0$ and $\dot{K} = 0$ curves, respectively. According to the Implicit Function Theorem, for $\bar{\lambda}^\lambda$ and $\bar{\lambda}^K$ to be well-defined, the functions

Ψ^λ and Ψ^K must be invertible with respect to λ_t at points where (15) and (16) hold. Consequently, this requirement must be checked as the first step in setting up the phase diagram.

For the $\dot{\lambda} = 0$ curve to be defined, we must have $\Psi^\lambda_\lambda \neq 0$ wherever $\Psi^\lambda = 0$. A sufficient condition for this may be obtained using (8) and (13). Begin by totally differentiating (8):

$$V_{LL} dL_t = F_L d\lambda_t + \lambda_t F_{LK} dK_t + \lambda_t F_{LL} dL_t + \lambda_t F_{LZ} dZ_t.$$

This yields

$$\Phi^L_\lambda = \frac{dL_t}{d\lambda_t} = \frac{F_L}{V_{LL} - \lambda_t F_{LL}}.$$

Now differentiate (13) with respect to λ_t:

$$\Psi^\lambda_\lambda = \frac{\Psi^\lambda}{\lambda_t} - \lambda_t F_{KL} \Phi^L_\lambda.$$

Substituting for Φ^L_λ and imposing $\Psi^\lambda = 0$ gives

$$\Psi^\lambda_\lambda = -\frac{\lambda_t F_{KL} F_L}{V_{LL} - \lambda_t F_{LL}}.$$

Under standard assumptions, $\Psi^\lambda_\lambda < 0$ will hold, and hence the $\dot{\lambda}_t = 0$ curve is well-defined by (15).

Similarly, for the $\dot{K}_t = 0$ curve, we must have $\Psi^K_\lambda \neq 0$ at points such that $\Psi^K = 0$. This property may be established using (7), (8) and (14). Totally differentiating (7) yields

$$U_{CC} dC_t = d\lambda_t,$$

and hence

$$\Phi^C_\lambda = \frac{dC_t}{d\lambda_t} = \frac{1}{U_{CC}}.$$

Thus, differentiating (14) with respect to λ_t gives

$$\Psi^K_\lambda = F_L \Phi^L_\lambda - \Phi^C_\lambda = \frac{\lambda_t F_{KL} F_L^2}{V_{LL} - \lambda_t F_{LL}} - \frac{1}{U_{CC}},$$

and we have $\Psi_\lambda^K > 0$ under standard assumptions. It follows that the $\dot{K}_t = 0$ curve is well-defined by (16).

4 Total log-differentiation

After verifying that the two curves are well-defined, the analysis proceeds by calculating partial derivatives of the curves. This is customarily done via total differentiation of the equations (15) and (16) that define the curves. In the present context, however, it is convenient to express the total derivatives in terms of natural logarithms, rather than levels, since the functional forms used in macroeconomic models typically have constant elasticities.[1]

Let us first consider the log-derivative of the function Φ^C, which is defined implicitly by (1). For given strictly-positive values C_t, C_t', λ_t and λ_t', define

$$d\ln C_t = \ln C_t' - \ln C_t, \quad d\ln\lambda_t = \ln\lambda_t' - \ln\lambda_t,$$

which is equivalent to

$$C_t' = C_t e^{d\ln C_t}, \quad \lambda_t' = \lambda_t e^{d\ln\lambda_t}.$$

Thus, (1) evaluated at the point (C_t', λ_t') may be written as

$$U_C(C_t e^{d\ln C_t}) = \lambda_t e^{d\ln\lambda_t}.$$

A first-order approximation around $d\ln C_t = d\ln\lambda_t = 0$ yields

$$U_{CC}(C_t)C_t\, d\ln C_t = \lambda_t\, d\ln\lambda_t. \tag{17}$$

This is the total log-derivative of (1) at the point (C_t, λ_t). Using (1) to substitute for λ_t, we can write

$$\frac{d\ln\Phi^C}{d\ln\lambda_t} = \frac{\lambda_t}{U_{CC}(C_t)C_t} = \left(\frac{U_{CC}(C_t)C_t}{U_C(C_t)}\right)^{-1}. \tag{18}$$

[1] Log-differentiation cannot be used in the analysis of the preceding section, however, since the log-derivatives of Ψ^λ or Ψ^K do not exist at points where $\Psi^\lambda = 0$ or $\Psi^K = 0$.

Observe that the log-derivative of Φ^C depends on the elasticity of the marginal utility of consumption with respect to consumption. This is often a constant parameter in applications, making the expression very simple.

It is interesting to note that the calculation of (17) is mathematically equivalent to log-linearizing around a steady state equilibrium, except for two distinctions. First, the necessary condition (1) may be totally log-differentiated at <u>any</u> strictly positive values, not just steady state values. Second, the variables $d\ln C_t$ and $d\ln\lambda_t$ are log-differentials, not log-deviations from the steady state. These log-differentials become infinitesimally small when partial log-derivatives are constructed.

Equation (2) can be totally log-differentiated in the same manner:

$$\frac{V_{LL}L_t}{V_L}d\ln L_t = d\ln\lambda_t + \frac{F_{LK}K_t}{F_L}d\ln K_t + \frac{F_{LL}L_t}{F_L}d\ln L_t + \frac{F_{LZ}Z_t}{F_L}d\ln Z_t,$$

where the derivatives of V are evaluated at L_t, and the derivatives of F are evaluated at (K_t, L_t, Z_t). Using this equation, the partial log-derivatives of Φ^L are calculated as

$$\frac{\partial\ln\Phi^L}{\partial\ln\lambda_t} = \frac{1}{D}, \quad \frac{\partial\ln\Phi^L}{\partial\ln K_t} = \frac{1}{D}\frac{F_{LK}K_t}{F_L}, \quad \frac{\partial\ln\Phi^L}{\partial\ln Z_t} = \frac{1}{D}\frac{F_{LZ}Z_t}{F_L}, \tag{19}$$

where

$$D = \frac{V_{LL}L_t}{V_L} - \frac{F_{LL}L_t}{F_L}.$$

Next consider the function $\bar{\lambda}^\lambda$, defined implicitly by (15). Total log-differentiation of (15), using (13), gives

$$0 = (r+\delta-F_K)\,\lambda_t d\ln\bar{\lambda}^\lambda - \bar{\lambda}^\lambda F_{KZ}Z_t d\ln Z_t - \bar{\lambda}^\lambda F_{KK}K_t d\ln K_t$$

$$-\bar{\lambda}^\lambda F_{KL}\Phi^L\left(\frac{\partial\ln\Phi^L}{\partial\ln\lambda_t}d\ln\bar{\lambda}^\lambda + \frac{\partial\ln\Phi^L}{\partial\ln K_t}d\ln K_t + \frac{\partial\ln\Phi^L}{\partial\ln Z_t}d\ln Z_t\right).$$

Moreover, the first coefficient is zero, since (13) and (15) imply

$$(r+\delta-F_K)\,\bar{\lambda}^\lambda = \Psi^\lambda = 0.$$

Substituting from (12) and (19) and rearranging leads to

$$0 = \frac{1}{D}\frac{F_{KL}L_t}{F_K}d\ln\bar{\lambda}^{\lambda} + \left(\frac{F_{KK}K_t}{F_K} + \frac{1}{D}\frac{F_{KL}L_t}{F_K}\frac{F_{LK}K_t}{F_L}\right)d\ln K_t$$

$$+\left(\frac{F_{KZ}Z_t}{F_K} + \frac{1}{D}\frac{F_{KL}L_t}{F_K}\frac{F_{LZ}Z_t}{F_L}\right)d\ln Z_t.$$

This expression can be used to calculate the partial log-derivatives $\partial\ln\bar{\lambda}^{\lambda}/\partial\ln K_t$ and $\partial\ln\bar{\lambda}^{\lambda}/\partial\ln Z_t$.

Finally, to obtain partial log-derivatives of the function $\bar{\lambda}^K$, totally log-differentiate (16), using (14):

$$0 = F_Z Z_t d\ln Z_t + F_K K_t d\ln K_t$$

$$+F_L\Phi^L\left(\frac{\partial\ln\Phi^L}{\partial\ln\lambda_t}d\ln\bar{\lambda}^{\lambda} + \frac{\partial\ln\Phi^L}{\partial\ln K_t}d\ln K_t + \frac{\partial\ln\Phi^L}{\partial\ln Z_t}d\ln Z_t\right)$$

$$-\delta K_t d\ln K_t - \Phi^C\frac{\partial\ln\Phi^C}{\partial\ln\lambda_t}d\ln\bar{\lambda}^{\lambda} - G_t d\ln G_t.$$

Substituting from (11), (12), (18) and (19) and rearranging gives

$$0 = \left(\frac{1}{D}\frac{F_L L_t}{F} - \frac{C_t}{F}\left(\frac{U_{CC}(C_t)C_t}{U_C(C_t)}\right)^{-1}\right)d\ln\bar{\lambda}^{\lambda}$$

$$+\left(\frac{F_K K_t}{F} + \frac{1}{D}\frac{F_L L_t}{F}\frac{F_{LK}K_t}{F_L} - \delta\frac{K_t}{F}\right)d\ln K_t$$

$$+\left(\frac{F_Z Z_t}{F} + \frac{1}{D}\frac{F_L L_t}{F}\frac{F_{LZ}Z_t}{F_L}\right)d\ln Z_t - \frac{G_t}{F}d\ln G_t.$$

This expression yields $\partial\ln\bar{\lambda}^K/\partial\ln K_t$, $\partial\ln\bar{\lambda}^K/\partial\ln Z_t$ and $\partial\ln\bar{\lambda}^K/\partial\ln G_t$.

The partial log-derivatives are useful for analyzing the $\dot{\lambda}_t = 0$ and $\dot{K}_t = 0$ curves, which are represented here by the functions $\bar{\lambda}^{\lambda}$ and $\bar{\lambda}^K$. For example, we can write

$$\frac{\partial\ln\bar{\lambda}^{\lambda}}{\partial\ln K_t} = \frac{\partial\bar{\lambda}^{\lambda}/\bar{\lambda}^{\lambda}}{\partial K_t/K_t}.$$

Thus, the slope of $\bar{\lambda}^{\lambda}$ in the K_t-λ_t plane satisfies

$$\frac{\partial\bar{\lambda}^{\lambda}}{\partial K_t} = \frac{\partial \ln \bar{\lambda}^{\lambda}}{\partial \ln K_t}\frac{\bar{\lambda}^{\lambda}}{K_t}.$$

In particular, $\partial\bar{\lambda}^{\lambda}/\partial K_t$ has the same sign as $\partial \ln \bar{\lambda}^{\lambda}/\partial \ln K_t$, so that calculating the partial log-derivative is sufficient for determining whether the $\dot{\lambda}_t = 0$ locus is upward- or downward-sloping. Similarly, knowing the sign of $\partial \ln \bar{\lambda}^{\lambda}/\partial \ln Z_t$ is sufficient for determining whether an increase in Z_t shifts the locus upward or downward.

5 Example

Consider the following parameterization of the model:

$$\max_{\{C_t, L_t, K_t\}} \mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} e^{-r\Delta t}\left(\ln C_t + \frac{L_t^{1+1/\eta}}{1+1/\eta}\right)\Delta$$

$$\text{s.t.} \quad Z_t K_{t-\Delta}^{\alpha} L_t^{1-\alpha}\Delta + (1-\delta\Delta)K_{t-\Delta} = K_t + C_t\Delta + G_t\Delta,$$

where $\eta > 0$ and $0 < \alpha < 1$. Necessary conditions for the solution in continuous time are

$$\frac{1}{C_t} = \lambda_t, \tag{20}$$

$$L_t^{1/\eta} = \lambda_t(1-\alpha)Z_t K_t^{\alpha} L_t^{-\alpha}, \tag{21}$$

$$\dot{\lambda}_t = \lambda_t\left(r + \delta - \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha}\right), \tag{22}$$

$$\dot{K}_t = Z_t K_t^{\alpha} L_t^{1-\alpha} - \delta K_t - C_t - G_t. \tag{23}$$

a $\dot{\lambda}_t = 0$ and $\dot{K}_t = 0$ curves

The partial derivatives Φ_{λ}^{C} and Φ_{λ}^{L} can be obtained by implicitly differentiating (20) and (21):

$$-\frac{1}{C_t^2}dC_t = d\lambda_t,$$

$$\frac{1}{\eta} L_t^{1/\eta-1} dL_t = (1-\alpha) Z_t K_t^\alpha L_t^{-\alpha} d\lambda_t - \lambda_t \alpha (1-\alpha) Z_t K_t^\alpha L_t^{-\alpha-1} dL_t.$$

Rearranging gives

$$\Phi_\lambda^C = \frac{dC_t}{d\lambda_t} = -C_t^2 < 0,$$

$$\Phi_\lambda^L = \frac{dL_t}{d\lambda_t} = \frac{(1-\alpha) Z_t K_t^\alpha, L_t^{-\alpha}}{\frac{1}{\eta} L_t^{1/\eta-1} + \lambda_t \alpha (1-\alpha) Z_t K_t^\alpha L_t^{-\alpha-1}} > 0.$$

Thus, using (22) and (23),

$$\Psi_\lambda^\lambda = \frac{\Psi^\lambda}{\lambda_t} - \lambda_t \alpha (1-\alpha) Z_t K_t^{\alpha-1} L_t^{-\alpha} \Phi_\lambda^L,$$

$$\Psi_\lambda^K = (1-\alpha) Z_t K_t^\alpha L_t^{-\alpha} \Phi_\lambda^L - \Phi_\lambda^C.$$

It follows that $\Psi_\lambda^\lambda < 0$ at points such that $\Psi^\lambda = 0$, and $\Psi_\lambda^K > 0$ holds globally. This means the functions $\bar{\lambda}^\lambda$ and $\bar{\lambda}^K$ are well-defined.

Moreover, these results establish the direction of motion at each point in the K_t-λ_t plane. It is easy to see that $\Psi_\lambda^\lambda > 0$ must hold whenever $\lambda_t < \bar{\lambda}^\lambda$, and $\Psi_\lambda^\lambda < 0$ holds whenever $\lambda_t > \bar{\lambda}^\lambda$. Hence we have $\dot{\lambda} > 0$ at points below the $\dot{\lambda} = 0$ curve, and $\dot{\lambda} < 0$ at points above the curve. Similarly, $\dot{K} < 0$ at points below the $\dot{K} = 0$ curve, and $\dot{K} > 0$ above the curve.

b Partial log-derivatives

To calculate the partial log-derivatives, begin by totally log-differentiating (20) and (21):

$$-d \ln C_t = d \ln \lambda_t,$$

$$\frac{1}{\eta} d \ln L_t = d \ln \lambda_t + d \ln Z_t + \alpha d \ln K_t - \alpha d \ln L_t.$$

Thus,

$$\frac{d \ln \Phi^C}{d \ln \lambda_t} = -1,$$

$$\frac{\partial \ln \Phi^L}{\partial \ln \lambda_t} = \frac{1}{1/\eta + \alpha}, \quad \frac{\partial \ln \Phi^L}{\partial \ln K_t} = \frac{\alpha}{1/\eta + \alpha}, \quad \frac{\partial \ln \Phi^L}{\partial \ln Z_t} = \frac{1}{1/\eta + \alpha}.$$

Next, totally log-differentiate (15) using (22):

$$0 = \left(r + \delta - \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha}\right) \bar{\lambda}^\lambda d \ln \bar{\lambda}^\lambda$$

$$+\bar{\lambda}^\lambda \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} \left(-d \ln Z_t + (1-\alpha) d \ln K_t - (1-\alpha) d \ln \Phi^L\right).$$

where $L_t = \Phi^L$. Moreover, using (22) we have

$$0 = \bar{\lambda}^\lambda \left(r + \delta - \alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha}\right).$$

Thus the expression becomes

$$0 = -d \ln Z_t + (1-\alpha) d \ln K_t - (1-\alpha) d \ln \Phi^L$$

$$= -d \ln Z_t + (1-\alpha) d \ln K_t$$

$$-(1-\alpha) \left(\frac{\partial \ln \Phi^L}{\partial \ln \lambda_t} d \ln \bar{\lambda}^\lambda + \frac{\partial \ln \Phi^L}{\partial \ln K_t} d \ln K_t + \frac{\partial \ln \Phi^L}{\partial \ln Z_t} d \ln Z_t\right).$$

Substituting for the partial log-derivatives of Φ^L and rearranging gives

$$0 = -\frac{1-\alpha}{1/\eta + \alpha} d \ln \bar{\lambda}^\lambda + (1-\alpha)\left(1 - \frac{\alpha}{1/\eta + \alpha}\right) d \ln K_t \tag{24}$$

$$-\left(1 + \frac{1-\alpha}{1/\eta + \alpha}\right) d \ln Z_t.$$

(24) gives the total log-derivative of the $\dot{\lambda} = 0$ curve. Using this expression, the partial log-derivatives of the curve are calculated as

$$\frac{\partial \ln \bar{\lambda}^\lambda}{\partial \ln K_t} = \frac{1}{\eta} > 0, \quad \frac{\partial \ln \bar{\lambda}^\lambda}{\partial \ln Z_t} = -\frac{1/\eta + 1}{1-\alpha} < 0.$$

This implies that the curve is upward-sloping in the K_t-λ_t plane, and the curve shifts down when Z_t rises.

Finally, totally log-differentiate (16) using (23):

$$0 = Y_t d\ln Z_t + \alpha Y_t d\ln K_t$$

$$+(1-\alpha)Y_t\left(\frac{\partial \ln \Phi^L}{\partial \ln \lambda_t} d\ln \bar{\lambda}^\lambda + \frac{\partial \ln \Phi^L}{\partial \ln K_t} d\ln K_t + \frac{\partial \ln \Phi^L}{\partial \ln Z_t} d\ln Z_t\right)$$

$$-\delta K_t d\ln K_t - C_t \frac{\partial \ln \Phi^C}{\partial \ln \lambda_t} d\ln \bar{\lambda}^\lambda - G_t d\ln G_t,$$

where Y_t is defined by

$$Y_t = Z_t K_t^\alpha L_t^{1-\alpha},$$

and $L_t = \Phi^L$, $C_t = \Phi^C$. Substituting for the partial log-derivatives of Φ^L and Φ^C and rearranging gives

$$0 = \left(\frac{(1-\alpha)Y_t}{1/\eta + \alpha} + C_t\right) d\ln \bar{\lambda}^\lambda + \left(\alpha Y_t \left(1 + \frac{1-\alpha}{1/\eta+\alpha}\right) - \delta K_t\right) d\ln K_t \tag{25}$$

$$+Y_t\left(1 + \frac{1-\alpha}{1/\eta+\alpha}\right) d\ln Z_t - G_t d\ln G_t.$$

(25) gives the total log-derivative of the $\dot{K} = 0$ curve, and it can be used to calculate the partial log-derivatives of the curve.

6 Summary

a Converting to continuous time

Discrete-time intertemporal Euler equations take the general form

$$\beta \mathbb{E}_t \lambda_{t+1} (1 + f_{t+1}) = \lambda_t.$$

Under perfect foresight, we can eliminate the expectation operator and write

$$\Delta\lambda_{t+1} = \lambda_{t+1} - \lambda_t \tag{26}$$

$$= \lambda_{t+1} - \beta\lambda_{t+1}(1 + f_{t+1})$$

$$= \lambda_{t+1}\beta\left(\frac{1-\beta}{\beta} - f_{t+1}\right).$$

To convert to continuous time, define r by

$$e^{-r} = \beta.$$

Then the continuous-time analog to (26) is

$$\dot{\lambda}_t = \lambda_t(r - f_t). \tag{27}$$

In other words, after forming (26), replace β and $(1-\beta)/\beta$ with 1 and r, respectively.

b Total log-differentiation

Consider the function $f(X_t, Y_t)$, and suppose $X_t, Y_t > 0$. At points (X_t', Y_t') sufficiently close to (X_t, Y_t), we can write

$$f(X_t', Y_t') = f(X_t e^{d\ln X_t}, Y_t e^{d\ln Y_t}), \tag{28}$$

where

$$d\ln X_t = \ln X_t' - \ln X_t, \quad d\ln Y_t = \ln Y_t' - \ln Y_t.$$

The total log-derivative of f at the point (X_t, Y_t) is obtained by taking a first-order approximation of (28) around $d\ln X_t = d\ln Y_t = 0$:

$$f(X_t, Y_t)d\ln f = f_X(X_t e^0, Y_t e^0)X_t e^0(d\ln X_t - 0) \tag{29}$$

$$+ f_Y(X_t e^0, Y_t e^0)Y_t e^0(d\ln Y_t - 0)$$

$$= f_X(X_t, Y_t) X_t \, d\ln X_t + f_Y(X_t, Y_t) Y_t \, d\ln Y_t.$$

Notice that (29) is not well-defined unless $f(X_t, Y_t) > 0$.

Alternatively, the usual total derivative of f at (X_t, Y_t) is given by

$$df = f_X(X_t, Y_t) dX_t + f_Y(X_t, Y_t) dY_t. \tag{30}$$

If $f(X_t, Y_t)$, X_t and Y_t are strictly positive, then we can write

$$d\ln f = \frac{df}{f(X_t, Y_t)}, \quad d\ln X_t = \frac{dX_t}{X_t}, \quad d\ln Y_t = \frac{dY_t}{Y_t}.$$

Substituting these expressions into (30) gives (29).