# Notes on Trend and Cycle Components

Garey Ramey

University of California, San Diego

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## 1 Deterministic trend and stochastic components

Let $\{Y_t\}$ be a random process that takes only strictly positive values. Define the deterministic trend component $\{y_t^{tr}\}$ as the unconditional mean of $\ln Y_t$ for each $t$:

$$y_t^{tr} = \mathbb{E} \ln Y_t,$$

and define the stochastic component $\{y_t^s\}$ as the deviation of $\ln Y_t$ from its unconditional mean for each $t$:

$$y_t^s = \ln Y_t - \mathbb{E} \ln Y_t.$$

This decomposes the process $\{\ln Y_t\}$ as

$$\ln Y_t = y_t^{tr} + y_t^s.$$

Note that $\{y_t^s\}$ is a zero-mean process that captures the stochastic behavior of $\ln Y_t$.

The stochastic component approximates the percentage deviations of $Y_t$ from the deterministic trend $Y_t^{tr} = \exp(y_t^{tr})$. To see this, consider the first-order Taylor expansion of $\ln Y_t$ around $y_t^{tr} = \ln Y_t^{tr}$:

$$\ln Y_t \simeq y_t^{tr} + \frac{1}{Y_t^{tr}}(Y_t - Y_t^{tr}),$$

which implies

$$y_t^s = \ln Y_t - y_t^{tr} \simeq \frac{Y_t - Y_t^{tr}}{Y_t^{tr}}.$$

This is a good approximation for aggregate output series, given that values of $y_t^s$ are small empirically.

## 2 Cyclical and permanent stochastic components

Business cycles are often analyzed by first fitting the logarithm of GDP to a deterministic trend, and then treating the residuals as the stationary stochastic component. Nelson and Plosser (*Journal of Monetary Economics*, 1982) showed that estimated stochastic components might be nonstationary in practice. This means deviations from the deterministic trend have permanent effects on the level of GDP.

As an example, suppose $\{y_t^s\}$ is given by

$$y_t^s = \rho y_{t-1}^s + \varepsilon_t, \tag{1}$$

where $0 < \rho \leq 1$ and $\{\varepsilon_t\}$ is a white noise process. In this case, the effect of an innovation $\varepsilon_1$ on $y_t^s$ is

$$y_t^s = \rho^{t-1}\varepsilon_1.$$

If $\rho < 1$, then (1) is a stationary $AR(1)$ process. Since $\lim_{t\to 0} \rho^{t-1}\varepsilon_1 = 0$, it follows that $\varepsilon_1$ has no long-run effect on $\ln Y_t$, and thus it makes sense to view $y_t^s$ as measuring a temporary departure from the trend. On the other hand, if $\rho = 1$, then $y_t^s = \varepsilon_1$ for all $t \geq 1$, and it follows that $\varepsilon_1$ has a <u>permanent</u> effect on $\ln Y_t$. In this case, $y_t^s$ reflects the stochastic behavior of the trend itself, rather than temporary departures from the trend.

This issue can be handled by decomposing $y_t^s$ into separate stationary and nonstationary components:

$$y_t^s \;=\; y_t^c + y_t^p.$$

Here $\{y_t^c\}$ is a zero-mean stationary series, called the <u>cyclical</u> or <u>transitory</u> or <u>trend-reverting</u> component, whereas $\{y_t^p\}$ is a zero-mean nonstationary series, called the <u>permanent</u> or <u>random walk</u> component, or the <u>stochastic trend</u>. $\{y_t^p\}$ is further assumed to be <u>difference stationary</u>, meaning that the series of first differences $\{\Delta y_t^p\}$ is stationary. Using the Wold Decomposition Theorem, we can represent these components as $MA(\infty)$ processes:

$$y_t^c \;=\; \sum\nolimits_{j=0}^{\infty} \psi_j^c \varepsilon_{t-j}^c, \qquad \Delta y_t^p \;=\; \sum\nolimits_{j=0}^{\infty} \psi_j^p \varepsilon_{t-j}^p,$$

where $\{\psi_j^c\}$ and $\{\psi_j^p\}$ are square summable sequences such that $\psi_0^c = \psi_0^p = 1$, and $\{\varepsilon_t^c\}$ and $\{\varepsilon_t^p\}$ are white noise processes. The effect of an innovation to the cyclical component is given by $y_t^c = \psi_{t-1}^c \varepsilon_1^c$, which has a temporary effect since $\lim_{t\to\infty} \psi_{t-1}^c = 0$. For an innovation to the permanent component, we have

$$y_t^p = \sum\nolimits_{j=1}^{t} \psi_{j-1}^p \varepsilon_1^p,$$

and the effect is permanent as long as

$$\sum\nolimits_{j=0}^{\infty} \psi_j^p \neq 0,$$

which is true when $\{y_t^p\}$ is nonstationary.

## 3 Incorporating trends into the RBC model

The RBC model can incorporate deterministic and stochastic trends via labor-augmenting technological progress. For example, Christiano and Eichenbaum (*American Economic Review*, 1992) consider the following specification:

$$\max_{\{C_t, L_t, K_t\}} E_0 \sum\nolimits_{t=0}^{\infty} \beta^t \left(\ln C_t + \theta(1 - L_t)\right),$$

$$\text{s.t.} \quad Y_t + (1-\delta)K_{t-1} = C_t + K_t + G_t,$$

$$Y_t = K_{t-1}^{\alpha} \left(X_t L_t\right)^{1-\alpha}.$$

$\{X_t\}$ and $\{G_t\}$ are exogenous random processes given by

$$X_t = X_{t-1} e^{\mu + \zeta_t}, \tag{2}$$

$$\frac{G_t}{X_t} = g^{1-\gamma} \left(\frac{G_{t-1}}{X_{t-1}}\right)^{\gamma} e^{\nu_t},$$

where $\{\zeta_t\}$ is a zero-mean stationary process, $\{\nu_t\}$ is a white noise process, and $0 < \gamma < 1$. In this model, the trend is driven by technology $X_t$, while government spending $G_t$ introduces temporary departures from the trend.Detrended output is defined as

$$y_t = \frac{Y_t}{X_t}. \tag{3}$$

Let $\hat{y}_t$ denote the log deviation of $y_t$ from its nonstochastic steady state value $y$. Taking logs of (2) and (3) and rearranging gives

$$\ln Y_t = \ln y_t + \ln X_t = \hat{y}_t + \ln y + \ln X_t$$

$$= \hat{y}_t + \ln y + \ln X_0 + \mu t + \sum\nolimits_{j=1}^{t} \zeta_j.$$

The deterministic trend and stochastic components are given by

$$y_t^{tr} = \mathbb{E} \ln Y_t = \ln y + \ln X_0 + \mu t,$$

$$y_t^s = \ln Y_t - \mathbb{E} \ln Y_t = \hat{y}_t + \sum\nolimits_{j=1}^{t} \zeta_j.$$

The cyclical and permanent components may be defined as

$$y_t^c = \hat{y}_t,$$

$$y_t^p = \sum\nolimits_{j=1}^{t} \zeta_j.$$

Note that $\{\hat{y}_t\}$ is a stationary process, as a consequence of restrictions imposed by the rational expectations equilibrium, while $\{y_t^p\}$ is clearly difference stationary.

## 4 Decomposition via long-run conditional forecasts

Beveridge and Nelson (*Journal of Monetary Economics*, 1981) define the permanent component as the long-run conditional forecast of the stochastic component:

$$y_t^p = \lim_{k \to \infty} \mathbb{E}_t y_{t+k}^s. \qquad (4)$$

(4) is called the Beveridge-Nelson trend. It captures the idea that the stochastic trend is the amount by which $\ln Y_t$ is expected to depart permanently from the deterministic trend. The implied cyclical component is

$$y_t^c = y_t^s - \lim_{k \to \infty} \mathbb{E}_t y_{t+k}^s.$$

Thus, $y_t^c$ measures the amount by which $y_t^s$ is expected to decline over the long run.

The cyclical component may also be expressed in terms of the forecasted long-run relationship to the deterministic trend. We can write

$$-y_t^c = \lim_{k\to\infty} \mathbb{E}_t \left(\ln Y_{t+k} - \left(\ln Y_t + \left(y_{t+k}^{tr} - y_t^{tr}\right)\right)\right).$$

Thus, $-y_t^c$ shows the amount by which the conditional forecast departs from the deterministic trend over the infinite horizon. For example, suppose the deterministic trend is log-linear:

$$y_t^{tr} = y_0^{tr} + \mu t,$$

Then the negative of the cyclical component is

$$-y_t^c = \lim_{k\to\infty} \mathbb{E}_t \left(y_{t+k} - (\ln Y_t + \mu k)\right).$$

Empirical estimates of the Beveridge-Nelson trend can be obtained by constructing long-horizon forecasts from an estimated forecasting model; see, e.g., Rotemberg and Woodford (*American Economic Review*, 1996).

In the RBC model of the preceding section, defining the permanent component to be the Beveridge-Nelson trend gives

$$\begin{aligned} y_t^p &= \lim_{k\to\infty} \mathbb{E}_t(y_{t+k}^s) \\ &= \sum\nolimits_{j=1}^{t} \zeta_j + \lim_{k\to\infty} \mathbb{E}_t \sum\nolimits_{j=1}^{k} \zeta_{t+j}, \end{aligned}$$

and the implied cyclical component is

$$y_t^c = y_t^s - y_t^p = \hat{y}_t - \lim_{k\to\infty} \mathbb{E}_t \sum\nolimits_{j=1}^{k} \zeta_{t+j}.$$