

Notes on Trend and Cycle Components

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1 Deterministic trend and stochastic components

Let $\{Y_t\}$ be a random process, and put $y_t = \ln Y_t$. Define the deterministic trend component $\{y_t^{tr}\}$ as the unconditional mean for each t :

$$y_t^{tr} = E(y_t),$$

and define the stochastic component $\{y_t^s\}$ as the deviation from the unconditional mean for each t :

$$y_t^s = y_t - E(y_t).$$

This decomposes the process $\{y_t\}$ as

$$y_t = y_t^{tr} + y_t^s.$$

Note that $\{y_t^s\}$ is a zero-mean process that captures the stochastic behavior of y_t .

The stochastic component approximates the percentage deviations of Y_t from the deterministic trend $Y_t^{tr} = \exp(y_t^{tr})$. To see this, consider the first-order Taylor expansion of $\ln Y_t$ around $\ln Y_t^{tr}$:

$$\ln Y_t \simeq \ln Y_t^{tr} + \frac{1}{Y_t^{tr}}(Y_t - Y_t^{tr}),$$

which implies

$$y_t^s = \ln Y_t - \ln Y_t^{tr} \simeq \frac{Y_t - Y_t^{tr}}{Y_t^{tr}}.$$

This is a good approximation for aggregate output series, given that values of y_t^s are small empirically.

2 Transitory and permanent stochastic components

Business cycles are often analyzed by first fitting the logarithm of GDP to a deterministic trend, and then treating the residuals as the stationary stochastic component. Nelson and Plosser (*Journal of Monetary Economics*, 1982) showed that estimated stochastic components might be nonstationary in practice. This means deviations from the deterministic trend have permanent effects on the level of GDP.

As an example, suppose $\{y_t^s\}$ is given by

$$y_t^s = \rho y_{t-1}^s + \varepsilon_t, \quad (1)$$

where $0 < \rho \leq 1$ and $\{\varepsilon_t\}$ is a white noise process. In this case, the effect of an innovation ε_1 on y_t^s is

$$y_t^s = \rho^{t-1} \varepsilon_1.$$

If $\rho < 1$, then (1) is a stationary $AR(1)$ process. Since $\lim_{t \rightarrow \infty} \rho^{t-1} \varepsilon_1 = 0$, it follows that ε_1 has no long-run effect on y_t , and thus it makes sense to view y_t^s as measuring a temporary departure from the trend. On the other hand, if $\rho = 1$, then $y_t^s = \varepsilon_1$ for all $t \geq 1$, and it follows that ε_1 has a permanent effect on y_t . In this case, y_t^s reflects the stochastic behavior of the trend itself, rather than temporary departures from the trend.

This issue can be handled by decomposing y_t^s into separate stationary and nonstationary components:

$$y_t^s = y_t^c + y_t^p.$$

Here $\{y_t^c\}$ is a zero-mean stationary series, called the transitory or cyclical or trend-reverting component, whereas $\{y_t^p\}$ is a zero-mean nonstationary series, called the permanent or random walk component, or the stochastic trend. $\{y_t^p\}$ is further assumed to be difference stationary, meaning that the series of first differences $\{\Delta y_t^p\}$ is stationary. Using the Wold Decomposition Theorem, we can represent these components as $MA(\infty)$ processes:

$$y_t^c = \sum_{j=0}^{\infty} \psi_j^c \varepsilon_{t-j}^c, \quad \Delta y_t^p = \sum_{j=0}^{\infty} \psi_j^p \varepsilon_{t-j}^p,$$

where $\{\psi_j^c\}$ and $\{\psi_j^p\}$ are square summable sequences such that $\psi_0^c = \psi_0^p = 1$, and $\{\varepsilon_t^c\}$ and $\{\varepsilon_t^p\}$ are white noise processes. The effect of an innovation to the transitory component is given by $y_t^c = \psi_{t-1}^c \varepsilon_1^c$, which has a temporary effect since $\lim_{t \rightarrow \infty} \psi_{t-1}^c = 0$. For an innovation to the permanent component, we have

$$y_t^p = \sum_{j=1}^t \psi_{j-1}^p \varepsilon_1^p,$$

and the effect is permanent as long as

$$\sum_{j=0}^{\infty} \psi_j^p \neq 0,$$

which is true when $\{y_t^p\}$ is nonstationary.

3 Incorporating trends into the RBC model

The RBC model can incorporate deterministic and stochastic trends via labor-augmenting technological progress. For example, Christiano and Eichenbaum (*American Economic Review*, 1992) consider the following specification:

$$\begin{aligned} \max_{\{C_t, L_t, K_t\}} \quad & E_0 \sum_{t=0}^{\infty} \beta^t (\ln C_t + \theta(1 - L_t)), \\ \text{s.t.} \quad & Y_t + (1 - \delta)K_{t-1} = C_t + K_t + G_t, \\ & Y_t = K_{t-1}^{\alpha} (X_t L_t)^{1-\alpha}. \end{aligned}$$

$\{X_t\}$ and $\{G_t\}$ are exogenous random processes given by

$$\begin{aligned} X_t &= X_{t-1} e^{\mu + \zeta_t}, \\ \frac{G_t}{X_t} &= g^{1-\gamma} \left(\frac{G_{t-1}}{X_{t-1}} \right)^{\gamma} e^{\nu_t}, \end{aligned} \tag{2}$$

where $\{\zeta_t\}$ is a zero-mean stationary process, $\{\nu_t\}$ is a white noise process, and $0 < \gamma < 1$. In this model, the trend is driven by technology X_t , while government spending G_t introduces transitory departures from the trend.

Detrended output is defined as

$$Y_t^d = \frac{Y_t}{X_t}. \quad (3)$$

Take logs of (2) and (3) and rearrange:

$$\begin{aligned} y_t &= y_t^d + \ln X_t \\ &= \hat{y}_t + y^d + \ln X_0 + t\mu + \sum_{j=1}^t \zeta_j, \end{aligned}$$

where $y_t^d = \ln Y_t^d$, y^d is the nonstochastic steady state value of y_t^d , and \hat{y}_t expresses y_t^d in terms of deviation from the steady state:

$$\hat{y}_t = y_t^d - y^d.$$

The deterministic trend and stochastic components are given by

$$\begin{aligned} y_t^{tr} &= E(y_t) = y^d + \ln X_0 + t\mu, \\ y_t^s &= y_t - E(y_t) = \hat{y}_t + \sum_{j=1}^t \zeta_j. \end{aligned}$$

The transitory and permanent components may be defined as

$$\begin{aligned} y_t^c &= \hat{y}_t, \\ y_t^p &= \sum_{j=1}^t \zeta_j. \end{aligned}$$

Note that $\{\hat{y}_t\}$ is a stationary process, as a consequence of restrictions imposed by the rational expectations equilibrium, while $\{y_t^p\}$ is clearly difference stationary.

4 Decomposition via long-run conditional forecasts

Beveridge and Nelson (*Journal of Monetary Economics*, 1981) define the permanent component as the long-run conditional forecast of the stochastic component:

$$y_t^p = \lim_{k \rightarrow \infty} E_t y_{t+k}^s. \quad (4)$$

(4) is called the Beveridge-Nelson trend. It captures the idea that the stochastic trend is the amount by which y_t is expected to depart permanently from the deterministic trend. The implied transitory component is

$$y_t^c = y_t^s - \lim_{k \rightarrow \infty} E_t y_{t+k}^s.$$

Thus, y_t^c measures the amount by which y_t^s is expected to decline over the long run.

The transitory component may also be expressed in terms of the forecasted long-run relationship to the deterministic trend. We can write

$$-y_t^c = \lim_{k \rightarrow \infty} E_t (y_{t+k} - (y_t + (y_{t+k}^{tr} - y_t^{tr}))).$$

Thus, $-y_t^c$ shows the amount by which the conditional forecast departs from the deterministic trend over the infinite horizon. For example, suppose the deterministic trend is log-linear:

$$y_t^{tr} = y_0^{tr} + \mu t,$$

Then the negative of the cyclical component is

$$-y_t^c = \lim_{k \rightarrow \infty} E_t (y_{t+k} - (y_t + \mu k)).$$

Empirical estimates of the Beveridge-Nelson trend can be obtained by constructing long-horizon forecasts from an estimated forecasting model; see, e.g., Rotemberg and Woodford (*American Economic Review*, 1996).

In the RBC model of the preceding section, defining the permanent component to be the Beveridge-Nelson trend gives

$$\begin{aligned} y_t^p &= \lim_{k \rightarrow \infty} E_t (y_{t+k}^s) \\ &= \sum_{j=1}^t \zeta_j + \lim_{k \rightarrow \infty} E_t \sum_{j=1}^k \zeta_{t+j}, \end{aligned}$$

and the implied cyclical component is

$$y_t^c = y_t^s - y_t^p = \hat{y}_t - \lim_{k \rightarrow \infty} E_t \sum_{j=1}^k \zeta_{t+j}.$$