Notes on Trend and Cycle Components

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1 Definitions

Suppose the series $\{Y_t\}$ has a trend component $\{Y_t^{tr}\}$. Write

$$Y_t = Y_t^{tr}\frac{Y_t}{Y_t^{tr}} = Y_t^{tr} \cdot \widehat{Y}_t,$$

which defines the cycle component as

$$\widehat{Y}_t = \frac{Y_t}{Y_t^{tr}}.$$

Take logs:

$$\ln Y_t = \ln Y_t^{tr} + \ln \widehat{Y}_t.$$

Now define

$$y_t = \ln Y_t, \quad y_t^{tr} = \ln Y_t^{tr},$$

$$\hat{y}_t = \ln \widehat{Y}_t = \ln Y - \ln Y_t^{tr}.$$

Then we may write:

$$y_t = y_t^{tr} + \hat{y}_t.$$

Note that $\hat{y}_t$ approximates the percentage deviation of output from trend. To see this, consider the Taylor expansion around the trend component:

$$\ln Y_t \simeq \ln Y_t^{tr} + \left.\frac{d\ln Y_t}{dY_t}\right|_{Y_t=Y_t^{tr}} \cdot (Y_t - Y_t^{tr}) = \ln Y_t^{tr} + \frac{1}{Y_t^{tr}}(Y_t - Y_t^{tr}),$$

which implies

$$\ln Y_t - \ln Y_t^{tr} = \hat{y}_t \simeq \frac{Y_t - Y_t^{tr}}{Y_t^{tr}}.$$

This is a good approximation given that values of $\hat{y}_t$ are small empirically.

2 Modeling the trend component

a Polynomial trend

The trend component is assumed to be a polynomial in t:

$$y_t^{tr} = \sum_{k=0}^{K} \mu_k t^k.$$

Example: Log-linear trend ($K = 1$):

$$y_t^{tr} = \mu_0 + \mu_1 t \quad \Rightarrow \quad Y_t^{tr} = Y_0 \gamma^t,$$

where

$$Y_0 = e^{\mu_0}, \quad \gamma = e^{\mu_1}.$$

Note that a polynomial can be used to approximate any deterministic trend. Moreover, estimation and inference are straightforward.

b Hodrick-Prescott (HP) filter

For a given series $\{y_t\}_{t=1}^T$, the trend component $\{y_t^{tr}\}_{t=1}^T$ is chosen to solve

$$\min_{\{y_t^{tr}\}_{t=1}^T} \sum_{t=1}^{T} [(y_t - y_t^{tr})^2 + \lambda((y_{t+1}^{tr} - y_t^{tr}) - (y_t^{tr} - y_{t-1}^{tr}))^2].$$

The first term captures how closely trend tracks the data, while the second term captures smoothness of the trend in terms of second differences. The smoothing parameter λ governs the weight assigned to tracking vs. smoothness in the minimization problem. As λ rises, the trend becomes smoother, and correspondingly more variations are assigned to the cycle component. As $\lambda \to \infty$,positive second differences are not allowed, and y_t^{tr} becomes linear in t.

The choice $\lambda = 1600$ is reasonable for business cycle analysis using quarterly data, as it works to remove cyclical movements with periods in excess of 4-6 years.

The HP filter is very flexible, and takes no stand on the form of the trend. It can induce spurious volatility into the cycle component, however.

c Stochastic trend

The trend component is assumed to be a random process.

Example: Suppose $\{y_t\}$ is a random walk with drift:

$$y_t = y_{t-1} + \gamma + \varepsilon_t,$$

where ε_t is white noise. This is a "unit root" specification. Define:

$$y_t^{tr} = y_{t-1} + \gamma, \quad \hat{y}_t = \varepsilon_t.$$

In this case the trend is the linear projection, and the cycle is the innovation. Moreover, innovations to output have a permanent effect on the trend.

To obtain a stationary representation of $\{y_t\}$, take the first difference:

$$\Delta y_t = y_t - y_{t-1} = \gamma + \varepsilon_t.$$

Note that Δy_t approximates the growth rate of output:

$$\Delta y_t = \ln Y_t - \ln Y_{t-1} \simeq \frac{Y_t - Y_{t-1}}{Y_{t-1}}.$$

Thus, for a unit root specification, we consider output growth rates rather than log levels.

There are numerous other approaches to modeling trend and cycle components, e.g., state-space and frequency domain methods.