# Notes on Perfect Foresight Dynamics of RBC Model

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## 1 Model and equilibrium conditions

A standard Real Business Cycle model may be expressed as the following social planner problem:

$$\max \mathbb{E}_t \sum\nolimits_{s=0}^{\infty} \beta^s \left( \ln C_{t+s} - \frac{L_{t+s}^{1+1/\eta}}{1+1/\eta} \right)$$

$$\text{s.t.} \quad Z_{t+s} K_{t+s-1}^{\alpha} L_{t+s}^{1-\alpha} + (1-\delta) K_{t+s-1} = K_{t+s} + C_{t+s} + G_{t+s},$$

$$Z_{t+s} = Z_{t+s-1}^{\rho} e^{\varepsilon_{t+s}},$$

$$G_{t+s} = G^{1-\gamma} G_{t+s-1}^{\gamma} e^{\nu_{t+s}},$$

$$K_0,\ Z_0 \text{ and } G_0 \text{ given},$$

where $0 < \beta, \alpha, \delta, \rho, \gamma < 1$, $\eta, G > 0$, and $\{\varepsilon_t\}$, $\{\nu_t\}$ are white noise processes.

Necessary conditions for a maximum include, for $t = 1, 2, ...$,

$$\frac{1}{C_t} - \lambda_t = 0,$$

$$-L_t^{1/\eta} + \lambda_t (1-\alpha) Z_t K_{t-1}^{\alpha} L_t^{-\alpha} = 0,$$

$$-\lambda_t + \mathbb{E}_t \beta \lambda_{t+1} (\alpha Z_{t+1} K_t^{\alpha-1} L_{t+1}^{1-\alpha} + 1 - \delta) = 0,$$

$$Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} + (1-\delta) K_{t-1} = K_t + C_t + G_t.$$

These equations determine paths of $C_t$, $L_t$, $\lambda_t$ and $K_t$ for the given exogenous processes $\{Z_t\}$ and $\{G_t\}$.

## 2 Perfect foresight dynamics

Suppose $\{\varepsilon_t\}$ and $\{\nu_t\}$ have zero variance. Then the expected and realized future values of the variables are equivalent, and the expectation operator may be dropped from the equilibrium conditions. In this case, the system exhibits deterministic dynamics that may be expressed in terms of the endogenous variables $K_{t-1}$, $\lambda_t$ and exogenous variables $Z_t, G_t$.

The law of motion for $\lambda_t$ is given by

$$\begin{aligned}
\Delta\lambda_t = \lambda_t - \lambda_{t-1} &= \lambda_t - \beta\lambda_t \left(\alpha Z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha} + 1 - \delta\right) \\
&= \beta\lambda_t \left(\frac{1-\beta}{\beta} + \delta - \alpha Z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha}\right) \\
&= \beta\lambda_t \left(R + \delta - \alpha Z_t K_{t-1}^{\alpha-1} L_t^{1-\alpha}\right),
\end{aligned}$$

where $R = (1-\beta)/\beta$ is the steady-state net return on capital investment.

The law of motion for $K_t$ is given by

$$\begin{aligned}
\Delta K_t = K_t - K_{t-1} &= Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1} - C_t - G_t \\
&= Z_t K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1} - \lambda_t^{-1} - G_t.
\end{aligned}$$

The second equilibrium condition gives

$$L_t = \left(\lambda_t(1-\alpha) Z_t K_{t-1}^{\alpha}\right)^{1/(\alpha+1/\eta)},$$

which may be used to eliminate $L_t$ from the laws of motion. Note that the partial derivatives of $L_t$ are

$$\begin{aligned}
\frac{\partial L_t}{\partial K_{t-1}} &= \frac{\alpha}{\alpha + 1/\eta} L_t K_{t-1}^{-1} > 0, \\
\frac{\partial L_t}{\partial \lambda_t} &= \frac{1}{\alpha + 1/\eta} L_t \lambda_t^{-1} > 0, \\
\frac{\partial L_t}{\partial Z_t} &= \frac{1}{\alpha + 1/\eta} L_t Z_t^{-1} > 0.
\end{aligned}$$

## 3 $\Delta\lambda_t = 0$ equation

The partial derivative of the $\Delta\lambda_t$ equation with respect to $K_{t-1}$ is calculated as follows.

$$\frac{\partial}{\partial K_{t-1}}\Delta\lambda_t = -\beta\lambda_t Z_t\left((\alpha-1)K_{t-1}^{\alpha-2}L_t^{1-\alpha} + (1-\alpha)K_{t-1}^{\alpha-1}L_t^{-\alpha}\frac{\partial L_t}{\partial K_{t-1}}\right)$$

$$= -\beta\lambda_t Z_t K_{t-1}^{\alpha-2}L_t^{1-\alpha}\left((\alpha-1) + (1-\alpha)\frac{\alpha}{\alpha+1/\eta}\right)$$

$$= \frac{\beta(1-\alpha)\,(1/\eta)}{\alpha+1/\eta}\lambda_t\frac{Y_t}{K_{t-1}^2} > 0,$$

where $Y_t = Z_t K_{t-1}^{\alpha}L_t^{1-\alpha}$. Similarly,

$$\frac{\partial}{\partial\lambda_t}\Delta\lambda_t = \frac{\Delta\lambda_t}{\lambda_t} - \frac{\beta\alpha(1-\alpha)}{\alpha+1/\eta}\frac{Y_t}{K_t},$$

$$\frac{\partial}{\partial Z_t}\Delta\lambda_t = -\frac{\beta\alpha\,(1+1/\eta)}{\alpha+1/\eta}\lambda_t\frac{Y_t}{Z_t K_t} < 0.$$

Note that $\partial\Delta\lambda_t/\partial\lambda_t < 0$ if $\Delta\lambda_t \leq 0$.

Totally differentiating the $\Delta\lambda_t$ equation gives

$$d\left(\Delta\lambda_t\right) = \left(\frac{\partial}{\partial K_{t-1}}\Delta\lambda_t\right)dK_{t-1} + \left(\frac{\partial}{\partial\lambda_t}\Delta\lambda_t\right)d\lambda_t + \left(\frac{\partial}{\partial Z_t}\Delta\lambda_t\right)dZ_t.$$

Thus,

$$\left.\frac{\partial\lambda_t}{\partial K_{t-1}}\right|_{\Delta\lambda_t=0} = -\left(\frac{\partial}{\partial K_{t-1}}\Delta\lambda_t\right)\left(\frac{\partial}{\partial\lambda_t}\Delta\lambda_t\right)^{-1} > 0.$$

This means the $\Delta\lambda_t = 0$ curve is upward-sloping in the $K_{t-1}$-$\lambda_t$ plane. Moreover, since $\partial\Delta\lambda_t/\partial K_{t-1} > 0$, it follows that $\Delta\lambda_t > 0$ when $K_{t-1}$ is increased from the $\Delta\lambda_t = 0$ curve, and $\Delta\lambda_t < 0$ when $K_{t-1}$ is decreased from the curve.

Finally, we have

$$\left.\frac{\partial\lambda_t}{\partial Z_t}\right|_{\Delta\lambda_t=0} = -\left(\frac{\partial}{\partial Z_t}\Delta\lambda_t\right)\left(\frac{\partial}{\partial\lambda_t}\Delta\lambda_t\right)^{-1} < 0,$$

$$\left.\frac{\partial\lambda_t}{\partial G_t}\right|_{\Delta\lambda_t=0} = 0.$$

This means an increase in $Z_t$ shifts the $\Delta\lambda_t = 0$ curve to lower levels of $\lambda_t$ for each $K_{t-1}$, while changes in $G_t$ have no effect on the curve.

## 4 $\Delta K_t = 0$ equation

The partial derivatives of the $\Delta K_t$ equation are

$$\frac{\partial}{\partial K_{t-1}} \Delta K_t = \alpha \left( \frac{1+1/\eta}{\alpha+1/\eta} \right) \frac{Y_t}{K_{t-1}} - \delta,$$

$$\frac{\partial}{\partial \lambda_t} \Delta K_t = \lambda_t^{-1} \left( \frac{1-\alpha}{\alpha+1/\eta} Y_t + \lambda_t^{-1} \right) > 0,$$

$$\frac{\partial}{\partial Z_t} \Delta K_t = \frac{1/\eta+1}{\alpha+1/\eta} \frac{Y_t}{Z_t} > 0,$$

$$\frac{\partial}{\partial G_t} \Delta K_t = -1 < 0.$$

Note that

$$\alpha \left( \frac{1+1/\eta}{\alpha+1/\eta} \right) \frac{Y_t}{K_{t-1}} - \delta > \alpha \frac{Y_t}{K_{t-1}} - \delta.$$

In the neighborhood of the steady state, we have

$$\alpha \frac{Y_t}{K_{t-1}} - \delta \cong \alpha \frac{Y}{K} - \delta = \alpha \kappa^{\alpha-1} - \delta = R > 0,$$

and hence $\partial \Delta K_t / \partial K_{t-1} > 0$ if $Y_t / K_{t-1}$ lies either above, or below but close to, the steady-state value $Y/K$. Moreover,

$$\frac{\partial}{\partial K_{t-1}} \frac{Y_t}{K_{t-1}} = -\frac{(1-\alpha)(1/\eta)}{\alpha+1/\eta} Z_t K_{t-1}^{\alpha-2} L_t^{1-\alpha} < 0.$$

Thus $\partial \Delta K_t / \partial K_{t-1} > 0$ for a range of low values of $K_{t-1}$ that includes the steady state value; i.e., there exists $\bar{K}_{t-1} > K$ such that $\partial \Delta K_t / \partial K_{t-1} > 0$ for $K_{t-1} < \bar{K}_{t-1}$, and $\partial \Delta K_t / \partial K_{t-1} < 0$ for $K_{t-1} > \bar{K}_{t-1}$.

Totally differentiating the $\Delta K_t$ equation gives

$$d\left(\Delta K_t\right) = \left( \frac{\partial}{\partial K_{t-1}} \Delta K_t \right) dK_{t-1} + \left( \frac{\partial}{\partial \lambda_t} \Delta K_t \right) d\lambda_t + \left( \frac{\partial}{\partial Z_t} \Delta K_t \right) dZ_t$$

$$+\left(\frac{\partial}{\partial G_t}\Delta K_t\right) dG_t.$$

Thus,

$$\left.\frac{\partial \lambda_t}{\partial K_{t-1}}\right|_{\Delta K_t=0} = -\left(\frac{\partial}{\partial K_{t-1}}\Delta K_t\right)\left(\frac{\partial}{\partial \lambda_t}\Delta K_t\right)^{-1} \begin{cases} < 0, & K_{t-1} < \bar{K}_{t-1}, \\ > 0, & K_{t-1} > \bar{K}_{t-1}, \end{cases}.$$

It follows that the $\Delta K_t = 0$ curve is downward-sloping for a low region of $K_{t-1}$, and upward-sloping for a high region of $K_{t-1}$. Since the steady state values satisfy $\Delta\lambda_t = \Delta K_t = 0$, the $\Delta K_t = 0$ curve must be downward-sloping at any point of intersection with the $\Delta\lambda_t = 0$ curve. This means the curves have a unique intersection.

Since $\partial \Delta K_t / \partial \lambda_t > 0$, it follows that $\Delta K_t > 0$ when $\lambda_t$ is increased from the $\Delta K_t = 0$ curve, and $\Delta K_t < 0$ when $\lambda_t$ is decreased from the curve.

Finally, we have

$$\left.\frac{\partial \lambda_t}{\partial Z_t}\right|_{\Delta K_t=0} = -\left(\frac{\partial}{\partial Z_t}\Delta K_t\right)\left(\frac{\partial}{\partial \lambda_t}\Delta K_t\right)^{-1} < 0,$$

$$\left.\frac{\partial \lambda_t}{\partial G_t}\right|_{\Delta K_t=0} = -\left(\frac{\partial}{\partial G_t}\Delta K_t\right)\left(\frac{\partial}{\partial \lambda_t}\Delta K_t\right)^{-1} > 0.$$

This means an increase in $Z_t$ shifts the $\Delta K_t = 0$ curve to lower levels of $\lambda_t$ for each $K_{t-1}$, while an increase in $G_t$ shifts the curve to higher levels of $\lambda_t$ for each $K_{t-1}$.

## 5 Labor market equilibrium

Labor supply and demand may be expressed as

$$L^s(W_t; \lambda_t) = (\lambda_t W_t)^\eta,$$

$$L^d(W_t; Z_t, K_{t-1}) = \left(\frac{(1-\alpha) Z_t K_{t-1}^\alpha}{W_t}\right)^{1/\alpha}.$$

where $W_t$ is the wage rate. Equating supply and demand determines the equilibrium wage $W_t^e$:

$$L^s(W_t^e; \lambda_t) = L^d(W_t^e; Z_t, K_{t-1}).$$

We have

$$\frac{\partial L^s}{\partial W_t},\ \frac{\partial L^s}{\partial \lambda_t} > 0.$$

Thus $L^s$ is upward-sloping in the $L_t$-$W_t$ plane, and an increase in $\lambda_t$ shifts $L^s$ to higher levels of $L_t$ for each $W_t$. Moreover,

$$\frac{\partial L^d}{\partial W_t} < 0 < \frac{\partial L^d}{\partial Z_t},\ \frac{\partial L^d}{\partial K_{t-1}}.$$

Thus $L^d$ is downward-sloping in the $L_t$-$W_t$ plane, and an increase in $Z_t$ or $G_t$ shifts $L^d$ to higher levels of $L_t$ for each $W_t$.