Notes on Linear Rational Expectations Equilibria

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1 General model

Consider the following linear first-order expectational difference system:

$$A_1(\mu) \begin{bmatrix} E_t X_{t+1} \\ Y_{t+1} \end{bmatrix} = A_0(\mu) \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + B_0(\mu)\varepsilon_{t+1}. \tag{1}$$

$X_t = m \times 1$ vector of expectational variables

$Y_t = n \times 1$ vector of nonexpectational variables

$\varepsilon_t = d \times 1$ vector of exogenous shocks

$\mu =$ vector of underlying model parameters

$A_1(\mu), A_0(\mu) = (m+n) \times (m+n)$ matrices of coefficients

$B_0(\mu) = (m+n) \times d$ matrices of coefficients

Suppress dependence on μ. Sims (*Computational Economics*, 2001) proposes to solve the model by replacing the expectation terms with realized forecast errors:

$\eta_{t+1} = X_{t+1} - E_t X_{t+1} = m \times 1$ vector of realized forecast errors

The system may be rewritten as follows:

$$A_1 \begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix} = A_0 \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + B_0\varepsilon_{t+1} + A_1 \begin{bmatrix} \eta_{t+1} \\ 0_{n\times 1} \end{bmatrix}. \tag{2}$$

We will solve the model under given assumptions. Sims' method holds more generally.

Assumption 1 *A_1 is invertible.*

Under this assumption we can work with the following reduced-form system:

$$\begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + B\varepsilon_{t+1} + \begin{bmatrix} \eta_{t+1} \\ 0_{n\times 1} \end{bmatrix}, \tag{3}$$

where

$$A = A_1^{-1}A_0, \qquad B = A_1^{-1}B_0.$$

Let the eigenvalues of A be denoted by λ_i, $i = 1, ..., m+n$, ordered so that $|\lambda_1| \geq |\lambda_2| ... \geq |\lambda_{m+n}|$. Recall that the $(m+n)\times 1$ vector v_i is an eigenvector associated with λ_i if $Av_i = \lambda_i v_i$. Define the following $(m+n)\times(m+n)$ matrices:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{m+n} \end{bmatrix}, \qquad V = \begin{bmatrix} v_1 & v_2 & \cdots & v_{m+n} \end{bmatrix}.$$

Observe that the i^{th} column of the matrix V contains an eigenvector associated with the i^{th} element of the diagonal of Λ.

Assumption 2 *The eigenvalues of A satisfy $|\lambda_i| \neq 1$.*

Assumption 3 *The matrix V is invertible.*

2 Solving for the REE

a Set up decoupled system

Assumption 3 allows the matrix A to be expressed as:

$$A = V\Lambda V^{-1}.$$

Substitute into (3) and premultiply by V^{-1} to obtain:

$$V^{-1}\begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix} = \Lambda V^{-1}\begin{bmatrix} X_t \\ Y_t \end{bmatrix} + V^{-1}B\varepsilon_{t+1} + V^{-1}\begin{bmatrix} \eta_{t+1} \\ 0_{n\times 1} \end{bmatrix}. \tag{4}$$

A unique nonexplosive rational expectations solution exists under the following assumption:

Assumption 4 *The eigenvalues of* A *satisfy* $|\lambda_m| > 1 > |\lambda_{m+1}|$.

Thus, the number of explosive eigenvalues is exactly equal to the number of expectational variables. Intuitively, the solution method uses forward solutions for the m expectational variables to tie down the m explosive roots (as in Blanchard and Kahn (*Econometrica*, 1980)). Sims' approach does this by determining the appropriate expectation errors η_{t+1}.

In view of Assumption 4, (4) can be decoupled into separate "unstable" and "stable" subsystems by partitioning Λ and V^{-1} in an appropriate manner. Define the matrices Λ_U and Λ_S by

$$\Lambda_U = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{bmatrix}, \qquad \Lambda_S = \begin{bmatrix} \lambda_{m+1} & 0 & \cdots & 0 \\ 0 & \lambda_{m+2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

This allows Λ to be written as:

$$\Lambda = \begin{bmatrix} \Lambda_U & 0_{m\times n} \\ 0_{n\times m} & \Lambda_S \end{bmatrix}.$$

Next, partition V^{-1} as:

$$V^{-1} = \begin{bmatrix} \Omega_{UU} & \Omega_{US} \\ \Omega_{SU} & \Omega_{SS} \end{bmatrix},$$

where Ω_{UU} is $m \times m$ and Ω_{SS} is $n \times n$. Define transformed variables by

$$\begin{bmatrix} \widetilde{X}_t \\ \widetilde{Y}_t \end{bmatrix} = V^{-1}\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} \Omega_{UU}X_t + \Omega_{US}Y_t \\ \Omega_{SU}X_t + \Omega_{SS}Y_t \end{bmatrix}. \tag{5}$$

(4) may be written as:

$$\begin{bmatrix} \widetilde{X}_{t+1} \\ \widetilde{Y}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_U & 0_{m\times n} \\ 0_{n\times m} & \Lambda_S \end{bmatrix} \begin{bmatrix} \widetilde{X}_t \\ \widetilde{Y}_t \end{bmatrix} + \begin{bmatrix} \Omega_{UU} & \Omega_{US} \\ \Omega_{SU} & \Omega_{SS} \end{bmatrix} B\varepsilon_{t+1} + \begin{bmatrix} \Omega_{UU} \\ \Omega_{SU} \end{bmatrix} \eta_{t+1}. \quad (6)$$

b Solve for X_t

The first m rows of (6) are:

$$\widetilde{X}_{t+1} = \Lambda_U \widetilde{X}_t + \begin{bmatrix} \Omega_{UU} & \Omega_{US} \end{bmatrix} B\varepsilon_{t+1} + \Omega_{UU}\eta_{t+1}. \quad (7)$$

This constitutes an unstable subsystem, since the diagonal elements of Λ_U lie outside of the unit circle. To obtain a nonexplosive REE we must have $\widetilde{X}_t = 0$ for all t. Because V is invertible, Ω_{UU} will also be invertible. Thus it follows from (5) that $\widetilde{X}_t = 0$ holds if and only if:

$$X_t = -\Omega_{UU}^{-1}\Omega_{US}Y_t. \quad (8)$$

c Solve for Y_{t+1}

In view of (7) and $\tilde{X}_t = 0$, η_{t+1} must satisfy:

$$\eta_{t+1} = -\Omega_{UU}^{-1} \begin{bmatrix} \Omega_{UU} & \Omega_{US} \end{bmatrix} B\varepsilon_{t+1} = - \begin{bmatrix} I_m & \Omega_{UU}^{-1}\Omega_{US} \end{bmatrix} B\varepsilon_{t+1}. \quad (9)$$

Substituting (9) and $\widetilde{X}_t = 0$ into (6) yields:

$$\begin{aligned} \begin{bmatrix} 0_{m\times 1} \\ \widetilde{Y}_{t+1} \end{bmatrix} &= \begin{bmatrix} \Lambda_U & 0_{m\times n} \\ 0_{n\times m} & \Lambda_S \end{bmatrix} \begin{bmatrix} 0_{m\times 1} \\ \widetilde{Y}_t \end{bmatrix} + \begin{bmatrix} \Omega_{UU} & \Omega_{US} \\ \Omega_{SU} & \Omega_{SS} \end{bmatrix} B\varepsilon_{t+1} \\ &\quad - \begin{bmatrix} \Omega_{UU} \\ \Omega_{SU} \end{bmatrix} \begin{bmatrix} I_m & \Omega_{UU}^{-1}\Omega_{US} \end{bmatrix} B\varepsilon_{t+1} \\ &= \begin{bmatrix} \Lambda_U & 0_{m\times n} \\ 0_{n\times m} & \Lambda_S \end{bmatrix} \begin{bmatrix} 0_{m\times 1} \\ \widetilde{Y}_t \end{bmatrix} + \begin{bmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & \Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US} \end{bmatrix} B\varepsilon_{t+1}. \end{aligned} \quad (10)$$

Note that the last n rows of (10) constitute a stable n-equation system, since the diagonal elements of Λ_S lie inside the unit circle. For given initial condition $\widetilde{Y}_0$, (10) determines a unique nonexplosive solution path of $\widetilde{Y}_{t+1}$ for any path of shocks ε_{t+1}.

To recover Y_t, partition B and V as

$$B = \begin{bmatrix} B_U \\ B_S \end{bmatrix}, \quad V = \begin{bmatrix} V_{UU} & V_{US} \\ V_{SU} & V_{SS} \end{bmatrix},$$

where B_U is $m \times 1$, B_S is $n \times 1$, V_{UU} is $m \times m$ and V_{SS} is $n \times n$. Using (10):

$$\begin{aligned}
\begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix} &= V \begin{bmatrix} 0_{m\times 1} \\ \widetilde{Y}_{t+1} \end{bmatrix} = \begin{bmatrix} V_{UU} & V_{US} \\ V_{SU} & V_{SS} \end{bmatrix} \left(\begin{bmatrix} \Lambda_U & 0_{m\times n} \\ 0_{n\times m} & \Lambda_S \end{bmatrix} \begin{bmatrix} 0_{m\times 1} \\ \widetilde{Y}_t \end{bmatrix} \right. \\
&\qquad \left. + \begin{bmatrix} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & \Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US} \end{bmatrix} B\varepsilon_{t+1} \right) \\
&= \begin{bmatrix} V_{UU}\Lambda_U & V_{US}\Lambda_S \\ V_{SU}\Lambda_U & V_{SS}\Lambda_S \end{bmatrix} \begin{bmatrix} 0_{m\times 1} \\ \widetilde{Y}_t \end{bmatrix} \\
&\qquad + \begin{bmatrix} 0_{m\times m} & V_{US}(\Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US}) \\ 0_{n\times m} & V_{SS}(\Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US}) \end{bmatrix} \begin{bmatrix} B_U \\ B_S \end{bmatrix} \varepsilon_{t+1} \\
&= \begin{bmatrix} V_{US} \\ V_{SS} \end{bmatrix} (\Lambda_S \widetilde{Y}_t + (\Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US}) B_S \varepsilon_{t+1}).
\end{aligned}$$

Substitute from (5) and (8):

$$\begin{aligned}
Y_{t+1} &= V_{SS}\Lambda_S(\Omega_{SU}X_t + \Omega_{SS}Y_t) + V_{SS}(\Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US})B_S\varepsilon_{t+1} \\
&= V_{SS}\Lambda_S(-\Omega_{SU}\Omega_{UU}^{-1}\Omega_{US} + \Omega_{SS})Y_t + V_{SS}(\Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US})B_S\varepsilon_{t+1} \\
&= V_{SS}\Lambda_S V_{SS}^{-1} Y_t + B_S\varepsilon_{t+1},
\end{aligned}$$

where $V_{SS} = (\Omega_{SS} - \Omega_{SU}\Omega_{UU}^{-1}\Omega_{US})^{-1}$ follows from a standard formula for inverting partitioned matrices (see Hayashi, *Econometrica*, p. 673). The same formula also gives $-\Omega_{UU}^{-1}\Omega_{US} = V_{US}V_{SS}^{-1}$.

The following proposition summarizes the solution.

Proposition *Under Assumptions 1-4, the model (1) possesses a unique nonexplosive REE, given by:*

$$X_t = V_{US}V_{SS}^{-1}Y_t, \qquad Y_{t+1} = V_{SS}\Lambda_S V_{SS}^{-1}Y_t + B_S\varepsilon_{t+1}.$$

d Check the solution

Put $\Phi_U = V_{US}V_{SS}^{-1}$ and $\Phi_S = V_{SS}\Lambda_S V_{SS}^{-1}$. The reduced form of (1) is

$$\begin{bmatrix} E_t X_{t+1} \\ Y_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + B\varepsilon_{t+1}. \tag{11}$$

The solution is checked by substituting $X_t = \Phi_U Y_t$ and $Y_{t+1} = \Phi_S Y_t + B_S\varepsilon_{t+1}$ into (11).

Substitute for X_{t+1} and Y_{t+1} in the left-hand side of (3):

$$\begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix} = \begin{bmatrix} \Phi_U \\ I_n \end{bmatrix} Y_{t+1} = \begin{bmatrix} \Phi_U\Phi_S \\ \Phi_S \end{bmatrix} Y_t + \begin{bmatrix} \Phi_U B_S \\ B_S \end{bmatrix} \varepsilon_{t+1},$$

where I_n is the $n \times n$ identity matrix. Take expectation of the first m rows:

$$\begin{bmatrix} E_t X_{t+1} \\ Y_{t+1} \end{bmatrix} = \begin{bmatrix} \Phi_U\Phi_S \\ \Phi_S \end{bmatrix} Y_t + \begin{bmatrix} 0_{m\times d} \\ B_S \end{bmatrix} \varepsilon_{t+1}.$$

Substitute for X_t in the right-hand side of (11):

$$A \begin{bmatrix} X_t \\ Y_t \end{bmatrix} + B\varepsilon_{t+1} = A \begin{bmatrix} \Phi_U \\ I_n \end{bmatrix} Y_t + B\varepsilon_{t+1}.$$

Thus, (11) becomes

$$\begin{bmatrix} \Phi_U\Phi_S \\ \Phi_S \end{bmatrix} Y_t + \begin{bmatrix} 0_{m\times d} \\ B_S \end{bmatrix} \varepsilon_{t+1} = A \begin{bmatrix} \Phi_U \\ I_n \end{bmatrix} Y_t + B\varepsilon_{t+1}.$$

It follows that the solution satisfies (11) if and only if

$$\begin{bmatrix} \Phi_U\Phi_S \\ \Phi_S \end{bmatrix} = A \begin{bmatrix} \Phi_U \\ I_n \end{bmatrix}, \qquad \begin{bmatrix} 0_{m\times d} \\ B_S \end{bmatrix} = B.$$

3 Example: Asset pricing model

The equilibrium conditions for a standard asset pricing model are:

$$P_t = E_t \frac{\beta C_t}{C_{t+1}} \left(D_{t+1} + P_{t+1} \right),$$

$$C_t = D_t, \qquad D_{t+1} = D_t^{\rho} e^{\varepsilon_{t+1}},$$

where $\rho \in (0,1)$ and $E_t \varepsilon_{t+1} = 0$. Let the steady state solution be denoted by $\bar{P}, \bar{C}, \bar{D}$. Log linearizing around the steady state gives:

$$\widehat{p}_t = \widehat{c}_t - E_t \widehat{c}_{t+1} + \frac{\bar{D}}{\bar{D}+\bar{P}} E_t \widehat{d}_{t+1} + \frac{\bar{P}}{\bar{D}+\bar{P}} E_t \widehat{p}_{t+1},$$

$$\widehat{c}_t = \widehat{d}_t, \qquad \widehat{d}_{t+1} = \rho \widehat{d}_t + \varepsilon_{t+1},$$

where $\widehat{p}_t = \ln P_t - \ln \bar{P}$, etc. Use the second equation to eliminate $\widehat{c}_t$, substitute $E_t \widehat{d}_{t+1} = \rho \widehat{d}_t$ and rearrange:

$$\frac{\bar{P}}{\bar{D}+\bar{P}} E_t \widehat{p}_{t+1} = \widehat{p}_t - \left(1 - \frac{\bar{P}\rho}{\bar{D}+\bar{P}} \right) \widehat{d}_t,$$

$$\widehat{d}_{t+1} = \rho \widehat{d}_t + \varepsilon_{t+1}.$$

The system can be expressed in the form (1):

$$\begin{bmatrix} \frac{\bar{P}}{\bar{D}+\bar{P}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_t \widehat{p}_{t+1} \\ \widehat{d}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & -\left(1 - \frac{\bar{P}\rho}{\bar{D}+\bar{P}}\right) \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \widehat{p}_t \\ \widehat{d}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}.$$

Reduced form:

$$\begin{bmatrix} \widehat{p}_{t+1} \\ \widehat{d}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{\bar{D}+\bar{P}}{\bar{P}} & \rho - \frac{\bar{D}+\bar{P}}{\bar{P}} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \widehat{p}_t \\ \widehat{d}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1} + \begin{bmatrix} \eta_{t+1} \\ 0 \end{bmatrix}.$$

Eigenvalues and eigenvectors:

$$\Lambda = \begin{bmatrix} \frac{\bar{D}+\bar{P}}{\bar{P}} & 0 \\ 0 & \rho \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution:

$$\widehat{p}_t = V_{US} V_{SS}^{-1} \widehat{d}_t = \widehat{d}_t,$$

$$\widehat{d}_{t+1} = V_{SS}\Lambda_S V_{SS}^{-1}\widehat{d}_t + B_S\varepsilon_{t+1} = \rho\widehat{d}_t + \varepsilon_{t+1}.$$

4 Example: RBC model

The equilibrium conditions for a standard version of the RBC model are:

$$\frac{\theta C_t}{1-H_t} = \alpha Z_t K_t^{1-\alpha} H_t^{\alpha-1},$$

$$E_t \frac{\beta(1+g)^{-1}C_t}{C_{t+1}} \left[(1-\alpha)Z_{t+1}K_{t+1}^{-\alpha}H_{t+1}^{\alpha} + 1 - \delta\right] = 1,$$

$$Z_t K_t^{1-\alpha} H_t^{\alpha} + (1-\delta)K_t = C_t + K_{t+1},$$

$$Z_{t+1} = Z_t^{\rho} e^{\varepsilon_{t+1}},$$

where $\theta, g > 0$, $\alpha, \beta, \delta, \rho \in (0,1)$, and $\{\varepsilon_t\}$ is a white noise process. Log-linearizing around the nonstochastic steady state values C, K, H and $Z = 1$ gives

$$\hat{c}_t + \frac{H}{1-H}\hat{h}_t = (\hat{z}_t + (1-\alpha)\hat{k}_t + (\alpha-1)\hat{h}_t,$$

$$\hat{c}_t - E_t\hat{c}_{t+1} + \beta(1+g)^{-1}(1-\alpha)\kappa^{-\alpha}\left(E_t\hat{z}_{t+1} - \alpha\hat{k}_{t+1} + \alpha E_t\hat{h}_{t+1}\right) = 0,$$

$$\kappa^{-\alpha}\left(\hat{z}_t + (1-\alpha)\hat{k}_t + \alpha\hat{h}_t\right) + (1-\delta)\hat{k}_t = \frac{C}{K}\hat{c}_t + \hat{k}_{t+1},$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \varepsilon_{t+1},$$

where

$$\kappa = \frac{K}{L} = \left(\frac{\beta(1-\alpha)}{1+g-\beta(1-\delta)}\right)^{1/\alpha}.$$

Substitute $E_t\hat{z}_{t+1} = \rho\hat{z}_t$ and rearrange:

$$0 = -\hat{c}_t - \psi_1\hat{h}_t + (1-\alpha)\hat{k}_t + \hat{z}_t,$$

$$E_t\hat{c}_{t+1} - \alpha\psi_2 E_t\hat{h}_{t+1} + \alpha\psi_2\hat{k}_{t+1} = \hat{c}_t + \rho\psi_2\hat{z}_t,$$

$$\hat{k}_{t+1} = -\frac{C}{K}\hat{c}_t + \kappa^{-\alpha}\alpha\hat{h}_t + \psi_3\hat{k}_t + \kappa^{-\alpha}\hat{z}_t,$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \varepsilon_{t+1},$$

where

$$\psi_1 = \frac{H}{1-H} + 1 - \alpha, \quad \psi_2 = \beta(1+g)^{-1}(1-\alpha)\kappa^{-\alpha},$$
$$\psi_3 = \kappa^{-\alpha}(1-\alpha) + 1 - \delta.$$

To obtain the reduced-form system, use the first equation to eliminate $\hat{h}_t$ from the second and third equations:

$$\left(1 + \frac{\alpha\psi_2}{\psi_1}\right) E_t\hat{c}_{t+1} + \alpha\psi_2\left(1 - \frac{1-\alpha}{\psi_1}\right)\hat{k}_{t+1} = \hat{c}_t + \rho\psi_2\left(1 + \frac{\alpha}{\psi_1}\right)\hat{z}_t,$$
$$\hat{k}_{t+1} = -\left(\frac{C}{K} + \frac{\alpha\kappa^{-\alpha}}{\psi_1}\right)\hat{c}_t + \left(\psi_3 + \frac{\alpha\,(1-\alpha)\,\kappa^{-\alpha}}{\psi_1}\right)\hat{k}_t + \kappa^{-\alpha}\left(1 + \frac{\alpha}{\psi_1}\right)\hat{z}_t.$$

Express the reduced system in matrix form:

$$\begin{bmatrix} 1 + \frac{\alpha\psi_2}{\psi_1} & \alpha\psi_2\left(1 - \frac{1-\alpha}{\psi_1}\right) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} E_t\hat{c}_{t+1} \\ \hat{k}_t \\ \hat{z}_{t+1} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & \rho\psi_2\left(1 + \frac{\alpha}{\psi_1}\right) \\ -\left(\frac{C}{K} + \frac{\alpha\kappa^{-\alpha}}{\psi_1}\right) & \psi_3 + \frac{\alpha(1-\alpha)\kappa^{-\alpha}}{\psi_1} & \kappa^{-\alpha}\left(1 + \frac{\alpha}{\psi_1}\right) \\ 0 & 0 & \rho \end{bmatrix}\begin{bmatrix} \hat{c}_t \\ \hat{k}_{t-1} \\ \hat{z}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\varepsilon_{t+1}.$$

The system may now be solved for the REE, as described above.