Economics 210C - Macroeconomics Spring 2018-19

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MIU Exercises - Solutions

1. Money growth and real activity

a. The Lagrangian for the household's problem is

$$\mathcal{L} = \mathbb{E}_0 \sum\nolimits_{t=0}^{\infty} \beta^t \Bigg(\ln C_t + \omega \ln \frac{M_t^d}{P_t} - \chi \frac{L_t^{1+1/\eta}}{1+1/\eta}$$

$$+\lambda_t \big((W_t L_t + R_t K_{t-1} + (1-\delta) K_{t-1}) + \frac{M_{t-1}^d}{P_t} + R_{t-1}^n \frac{B_{t-1}^d}{P_t}$$

$$-C_t + K_t + T_t - \frac{M_t^d}{P_t} - \frac{B_t^d}{P_t}) \Bigg).$$

The first order necessary conditions for C_t, L_t, K_t, M_t^d, B_t^d and Λ_t are, after canceling terms and rearranging:

$$\frac{1}{C_t} = \lambda_t,$$

$$\chi L_t^{1/\eta} = \lambda_t W_t,$$

$$\beta \mathbb{E}_t \lambda_{t+1} \left(R_{t+1} + 1 - \delta \right) = \lambda_t,$$

$$\frac{\omega}{M_t^d} + \beta \mathbb{E}_t \lambda_{t+1} \frac{1}{P_{t+1}} = \lambda_t \frac{1}{P_t},$$

$$\beta \mathbb{E}_t \lambda_{t+1} \frac{R_t^n}{P_{t+1}} = \lambda_t \frac{1}{P_t},$$

together with the household's wealth constraint.

The firm's profit maximization problem is

$$\max_{L_t, K_{t-1}} \Big(K_{t-1}^{\alpha} \left(X_t L_t \right)^{1-\alpha} - R_t K_{t-1} + W_t L_t \Big).$$

First order necessary conditions are

$$K_{t-1}^{\alpha} X_t^{1-\alpha} L_t^{-\alpha} = W_t, \tag{1}$$

$$\alpha K_{t-1}^{\alpha-1}\left(X_tL_t\right)^{1-\alpha}=R_t. \tag{2}$$

Money growth satisfies

$$\frac{M_t}{M_{t-1}}=\psi_t.$$

In equilibrium, all factor and asset markets clear. Substituting (1), (2), the firm's maximized profit function, and the government budget constraint into the household's necessary conditions gives equilibrium conditions for the variables C_t, L_t, K_t, $m_t=M_t/P_t$, R_t^n,Λ_t and $\pi_t=P_t/P_{t-1}$ (note that $B_t=0$ for all t):

$$\frac{1}{C_t}=\lambda_t,$$

$$\chi L_t^{1/\eta}=\lambda_t\left(1-\alpha\right)K_{t-1}^{\alpha}X_t^{1-\alpha}L_t^{-\alpha},$$

$$\beta\mathbb{E}_t\lambda_{t+1}\left(\alpha K_t^{\alpha-1}X_{t+1}^{1-\alpha}L_{t+1}^{1-\alpha}+1-\delta\right)=\lambda_t,$$

$$\frac{\omega}{m_t}+\beta\mathbb{E}_t\lambda_{t+1}\frac{1}{\pi_{t+1}}=\lambda_t,$$

$$\beta\mathbb{E}_t\lambda_{t+1}\frac{R_t^n}{\pi_{t+1}}=\lambda_t,$$

$$K_{t-1}^{\alpha}\left(X_tL_t\right)^{1-\alpha}+\left(1-\delta\right)K_{t-1}=C_t+G_t,$$

$$\pi_t=\frac{m_{t-1}}{m_t}\psi_t.$$

b. Sustituting the detrended variables and the X_t process into the equilibrium conditions from part a gives

$$\frac{1}{C_t^c}=\lambda_t^c,$$

$$\chi L_t^{1/\eta}=\lambda_t^c\left(1-\alpha\right)\left(K_{t-1}^c\right)^{\alpha}L_t^{-\alpha}e^{-\alpha(\mu+\zeta_t)},$$

$$\beta\mathbb{E}_t\lambda_{t+1}^c\left(\alpha\left(K_t^c\right)^{\alpha-1}L_{t+1}^{1-\alpha}e^{-\alpha(\mu+\zeta_{t+1})}+\left(1-\delta\right)e^{-(\mu+\zeta_{t+1})}\right)=\lambda_t^c,$$

$$\frac{\omega}{m_t^c}+\beta\mathbb{E}_t\lambda_{t+1}^ce^{-(\mu+\zeta_{t+1})}\frac{1}{\pi_{t+1}}=\lambda_t^c,$$

$$\beta \mathbb{E}_t \lambda_{t+1}^c e^{-(\mu+\zeta_{t+1})} \frac{R_t^n}{\pi_{t+1}} = \lambda_t^c,$$

$$\left(K_{t-1}^c\right)^\alpha L_t^{1-\alpha} e^{-\alpha(\mu+\zeta_t)} + (1-\delta) K_{t-1}^c e^{-(\mu+\zeta_t)} = C_t^c + K_t^c + G_t^c,$$

$$\pi_t = \frac{m_{t-1}^c}{m_t^c} e^{-(\mu+\zeta_t)} \psi_t.$$

c. Define $\kappa^c = K^c/L$. Then the nonstochastic steady state solution is determined by

$$\frac{1}{C^c} = \lambda^c,$$

$$\chi L^{1/\eta} = \lambda^c (1-\alpha) (\kappa^c)^\alpha e^{-\alpha\mu},$$

$$\beta \left(\alpha (\kappa^c)^{\alpha-1} e^{-\alpha\mu} + (1-\delta) e^{-\mu}\right) = 1,$$

$$\frac{\omega}{\lambda^c m^c} + \beta e^{-\mu} \frac{1}{\pi} = 1,$$

$$\beta e^{-\mu} \frac{R^n}{\pi} = 1,$$

$$(\kappa^c)^\alpha e^{-\alpha\mu} + (1-\delta) \kappa^c e^{-\mu} = \frac{C^c}{L} + \kappa^c + \frac{G^c}{L},$$

$$\pi = e^{-\mu} \psi.$$

Thus,

$$\kappa^c = \left(\frac{\alpha e^{-\alpha\mu}}{\beta^{-1} - (1-\delta) e^{-\mu}}\right)^{\frac{1}{1-\alpha}},$$

$$R^n = \frac{\psi}{\beta}.$$

C^c, L and m^c are determined by

$$\chi C^c L^{1/\eta} = (1-\alpha) (\kappa^c)^\alpha e^{-\alpha\mu},$$

$$(\kappa^c)^\alpha e^{-\alpha\mu} + (1-\delta) \kappa^c e^{-\mu} = \frac{C^c}{L} + \kappa^c + \frac{G^c}{L},$$

$$m^c = \frac{\omega\psi}{\psi - \beta} C^c.$$

d. Log-linearized equilibrium conditions:

$$-\hat{c}_t^c = \hat{\lambda}_t^c,$$

$$\frac{1}{\eta}\hat{l}_t = \hat{\lambda}_t^c + \alpha \hat{k}_{t-1}^c - \alpha \hat{l}_t - \alpha \zeta_t,$$

$$E_t \hat{\lambda}_{t+1}^c + \left(1 - \beta\left(1-\delta\right)e^{-\mu}\right)\left(-(1-\alpha)\hat{k}_t^c + (1-\alpha)E_t\hat{l}_{t+1} - \alpha E_t \zeta_{t+1}\right)$$
$$-\beta\left(1-\delta\right)e^{-\mu}E_t\zeta_{t+1} = \hat{\lambda}_t^c,$$

$$-\frac{\psi-\beta}{\psi}\hat{m}_t^c + \frac{\beta}{\psi}\left(\mathbb{E}_t\hat{\lambda}_{t+1}^c - \mathbb{E}_t\hat{\pi}_{t+1}\right) = \hat{\lambda}_t^c,$$

$$\mathbb{E}_t\hat{\lambda}_{t+1}^c + \hat{r}_t^n - \mathbb{E}_t\hat{\pi}_{t+1} = \hat{\lambda}_t^c,$$

$$\left(\kappa^c e^{-\mu}\right)^\alpha \left(\alpha\left(\hat{k}_{t-1}^c - \zeta_t\right) + (1-\alpha)\,\hat{l}_t\right) + (1-\delta)\kappa^c e^{-\mu}\left(\hat{k}_{t-1}^c - \zeta_t\right)$$
$$= \frac{C^c}{L}\hat{c}_t^c + \kappa^c \hat{k}_t^c + \frac{G^c}{L}\hat{g}_t^c,$$

$$\hat{\pi}_t = \hat{m}_{t-1}^c - \hat{m}_t^c + \zeta_t + \hat{\psi}_t.$$

These equations determine the endogenous variables $\hat{c}_t^c$, $\hat{l}_t$, $\hat{k}_t^c$, $\hat{m}_t^c$, $\hat{r}_t^n$, $\hat{\lambda}_t^c$ and $\hat{\pi}_t$ given exogenous variables ζ_t, $\hat{g}_t^c$ and $\hat{\psi}_t$. Observe that steady state values of endogenous variables appear only in the next-to-last equation.

e. The money growth equation implies

$$\mathbb{E}_t\hat{\pi}_{t+1} = \hat{m}_t^c - \mathbb{E}_t\hat{m}_{t+1}^c + \rho\hat{\psi}_t.$$

Substitute into money demand equation and rearrange:

$$-\hat{m}_t^c + \frac{\beta}{\psi}\left(\mathbb{E}_t\hat{\lambda}_{t+1}^c - \mathbb{E}_t\hat{m}_{t+1}^c + \rho\hat{\psi}_t\right) = \hat{\lambda}_t^c.$$

If $\rho = 0$, then we can write

$$-\hat{c}_t^c = \hat{\lambda}_t^c,$$

$$\frac{1}{\eta}\hat{l}_t = \hat{\lambda}_t^c + \alpha \hat{k}_{t-1}^c - \alpha \hat{l}_t^c - \alpha \zeta_t,$$

$$E_t \hat{\lambda}_{t+1} + \left(1 - \beta (1-\delta) e^{-\mu}\right) \left(-(1-\alpha)\hat{k}_t^c + (1-\alpha) E_t \hat{l}_{t+1} - \alpha E_t \zeta_{t+1}\right)$$

$$-\beta (1-\delta) e^{-\mu} E_t \zeta_{t+1} = \hat{\lambda}_t^c,$$

$$-\hat{m}_t^c + \frac{\beta}{\psi}\left(\mathbb{E}_t \hat{\lambda}_{t+1}^c - \mathbb{E}_t \hat{m}_{t+1}^c\right) = \hat{\lambda}_t^c,$$

$$\left(\alpha + (1-\delta)e^{-\mu}\frac{\kappa^c L}{y}\right)\left(\hat{k}_{t-1}^c - \zeta_t\right) + (1-\alpha)\hat{l}_t - \frac{\kappa^c L}{y}\hat{k}_t^c - \frac{c}{y}\hat{c}_t^c - \frac{g}{y}\hat{g}_t^c = 0.$$

These equations determine $\hat{c}_t^c$, $\hat{l}_t$, $\hat{k}_t^c$, $\hat{m}_t^c$ and $\hat{\lambda}_t^c$ completely separately from $\hat{\pi}_t$ and $\hat{\psi}_t$.

Intuitively, to a first order approximation, the monetary sector affects real variables only through expected inflation, which alters intertemporal tradeoffs across assets. In turn, expected inflation is determined by the path of real balances and expected money growth. If $\rho = 0$, then expected money growth is always zero, and hence intertemporal tradeoffs are unaffected by money growth.

f. The detrended equilibrium conditions are now given by

$$\frac{1}{C_t^c} = \lambda_t^c,$$

$$\chi L_t^{1/\eta} = \lambda_t^c (1-\alpha),$$

$$\frac{\omega}{m_t^c} + \beta \mathbb{E}_t \lambda_{t+1}^c e^{-(\mu+\zeta_{t+1})}\frac{1}{\pi_{t+1}} = \lambda_t^c,$$

$$\beta \mathbb{E}_t \lambda_{t+1}^c e^{-(\mu+\zeta_{t+1})}\frac{R_t^n}{\pi_{t+1}} = \lambda_t^c,$$

$$L_t = C_t^c + G_t^c,$$

$$\pi_t = \frac{m_{t-1}^c}{m_t^c} e^{-(\mu+\zeta_t)}\psi_t.$$

Log-linearizing around the steady state gives

$$-\hat{c}_t^c = \hat{\lambda}_t^c,$$

$$\frac{1}{\eta}\hat{l}_t = \hat{\lambda}_t^c,$$

$$-\frac{\psi-\beta}{\psi}\hat{m}_t^c + \frac{\beta}{\psi}\left(\mathbb{E}_t\hat{\lambda}_{t+1}^c - \mathbb{E}_t\hat{\pi}_{t+1}\right) = \hat{\lambda}_t^c,$$

$$\mathbb{E}_t\hat{\lambda}_{t+1}^c + \hat{r}_t^n - \mathbb{E}_t\hat{\pi}_{t+1} = \hat{\lambda}_t^c,$$

$$\hat{l}_t = \frac{C^c}{L}\hat{c}_t^c + \frac{G^c}{L}\hat{g}_t^c,$$

$$\hat{\pi}_t = \hat{m}_{t-1}^c - \hat{m}_t^c + \zeta_t + \hat{\psi}_t.$$

Thus, for any ρ, the variables $\hat{c}_t^c$, $\hat{l}_t$ and $\hat{\lambda}_t^c$ are determined by

$$-\hat{c}_t^c = \frac{1}{\eta}\hat{l}_t,$$

$$\hat{l}_t = \frac{C^c}{L}\hat{c}_t^c + \frac{G^c}{L}\hat{g}_t^c,$$

or

$$\hat{l}_t = \left(1 + \frac{c}{y}\frac{1}{\eta}\right)^{-1}\frac{g}{y}\hat{g}_t^c,$$

$$\hat{c}_t^c = -\frac{1}{\eta}\left(1 + \frac{c}{y}\frac{1}{\eta}\right)^{-1}\frac{g}{y}\hat{g}_t^c.$$

Observe that these equations do not depend on any nominal variable.

2. Friedman rule and superneutrality

a. The necessary conditions for an equilibrium are

$$U_{Ct} = \lambda_t,$$

$$-U_{Lt} = \lambda_t F_{Lt},$$

$$\beta \mathbb{E}_t \lambda_{t+1} \left(F_{K,t+1} + 1 - \delta \right)) = \lambda_t,$$

$$U_{mt} + \beta \mathbb{E}_t \lambda_{t+1} \frac{1}{\pi_{t+1}} = \lambda_t,$$

$$\beta \mathbb{E}_t \lambda_{t+1} \frac{R_t^n}{\pi_{t+1}} = \lambda_t,$$

$$F_t + (1-\delta) K_{t-1} = C_t + K_t + G_t,$$

$$\pi_t = \frac{m_{t-1}}{m_t} \psi_t.$$

These equations determine the endogenous variables C_t, L_t, K_t, m_t, R_t^n, λ_t and π_t given the exogenous variable ψ_t.

The nonstochastic steady state equilibrium is determined by

$$U_C = \lambda,$$

$$-U_L = \lambda F_L,$$

$$\beta \left(F_K + 1 - \delta \right)) = 1,$$

$$\frac{U_m}{\lambda} + \frac{\beta}{\pi} = 1, \tag{3}$$

$$\frac{\beta R^n}{\pi} = 1, \tag{4}$$

$$F - \delta K = C + G,$$

$$\pi = \psi.$$

b. Lifetime utility in the steady state is given by

$$\sum\nolimits_{t=0}^{\infty} \beta^t U\left(C, m_t, L\right) = \frac{U\left(C, m, L\right)}{1-\beta}.$$

The first order necessary condition for maximization of lifetime utility with respect to m is

$$\frac{\partial}{\partial m} \frac{U\left(C, m, L\right)}{1-\beta} = \frac{U_m\left(C, m, L\right)}{1-\beta} = 0,$$

or $U_m = 0$. Using (3), we obtain

$$\frac{U_m}{\lambda} = 1 - \frac{\beta}{\pi} = 1 - \frac{\beta}{\psi} = 0,$$

or $\psi = \beta < 1$. It follows that the optimal policy decreases the money supply, and induces deflation.

Intuitively, the social planner chooses an allocation that equates the social marginal benefit of money, which is proportional to U_m, to the social marginal cost, which is zero. The household is induced to hold the optimal level of real balances by a deflationary policy, so that holding nominal balances generates a positive real return.

c. From (4) we have

$$R^n = \frac{\pi}{\beta} = 1.$$

Thus the optimal policy implies a zero net nominal interest rate.

Combining (3) and (4) gives

$$\frac{U_m}{\lambda} = 1 - \frac{\beta}{\pi} = 1 - \frac{1}{R^n} = \frac{R^n - 1}{R^n}.$$

Thus, if the planner targets $R^n = 1$, then the optimal policy is implemented. This is called the Friedman rule.

d. The steady state equilibrium conditions can now be written as

$$U_C(C, L)V(m) = \lambda,$$

$$-U_L(C, L)V(m) = \lambda F_L(K, L),$$

$$\beta \left(F_K(K, L) + 1 - \delta \right) = 1,$$

$$\frac{U_m(C, m)V(m)}{\lambda} + \frac{\beta}{\pi} = 1,$$

$$\frac{\beta R^n}{\pi} = 1,$$

$$F(K,L) - \delta K = C + G,$$

$$\pi = \psi.$$

Eliminate λ from the system:

$$-U_L(C,L) = U_C(C,L) F_L(K,L), \tag{5}$$

$$\beta \left(F_K(K,L) + 1 - \delta \right) = 1, \tag{6}$$

$$\frac{V_m(m)}{U_C(C,L)V(m)} + \frac{\beta}{\pi} = 1, \tag{7}$$

$$\frac{\beta R^n}{\pi} = 1, \tag{8}$$

$$F(K,L) - \delta K = C + G, \tag{9}$$

$$\pi = \psi. \tag{10}$$

Equations (5), (6) and (9) determine the real variables C, L and K independently from any nominal variable.

Intuitively, the separability of the momentary utility function implies that tradeoffs between the real variables are unaffected by real balances.