Technical Notes

Random variables

\[ \tilde{X}_t = \text{random variable realized at year } t > 0. \]
Realization of \( \tilde{X}_t \) is uncertain at years \( s < t \).
Realization of \( \tilde{X}_t \) is known at years \( u \geq t \).

\( \tilde{X}_t = \tilde{Y}_t \Rightarrow \tilde{X}_t \) and \( \tilde{Y}_t \) have identical distributions.
\( \tilde{X}_t = \alpha \Rightarrow \) the constant \( \alpha \) is the only possible outcome of \( \tilde{X}_t \).

Forecasting

\[ E_0[\tilde{X}_t] = \text{expected value or mean of } \tilde{X}_t. \]

Linearity: \[ E_0[\alpha \tilde{X}_t + \beta \tilde{Y}_t] = \alpha E_0[\tilde{X}_t] + \beta E_0[\tilde{Y}_t]. \]

Alternative notation: \( X_t = E_0[\tilde{X}_t] \).

Suppose \( 0 < t < u \).
\[ E_t[\tilde{X}_u] = \text{conditional expectation of } \tilde{X}_u \text{ at year } t. \]

Assets and capital market

Asset = designated sequence of cash flows \( C_{A0}, \tilde{C}_{A1}, \ldots \tilde{C}_{AT} \).
\( C_{A0} \) is nonrandom, others are random.
\( C_{A0}, \tilde{C}_{At} > 0 \Rightarrow \text{asset owner receives cash.} \)
\( C_{A0}, \tilde{C}_{At} < 0 \Rightarrow \text{asset owner pays cash.} \)

Market values (prices) = \( V_{A0}, \tilde{V}_{A1}, \ldots \tilde{V}_{AT} \).
\( V_{A0} \) is nonrandom, others are random.

Asset sale at year \( t \):
Seller receives \( \tilde{C}_{At} + \tilde{V}_{At} \) at year \( t \).
Buyer pays \( \tilde{V}_{At} \) at year \( t \), receives \( \tilde{C}_{A,t+1} \) at year \( t+1 \).
\( \tilde{C}_{At} < 0 \) or \( \tilde{C}_{A,t+1} < 0 \Rightarrow \text{cash is paid.} \)
\[ \hat{V}_{At} < 0 \Rightarrow \text{seller pays cash to buyer. This defines a liability.} \]

Total cash value or liquidation value at year \( t = \hat{C}_{At} + \hat{V}_{At} \).

Present Value (PV) = total cash value at year 0 = \( C_{A0} + V_{A0} \).

Capital market assumptions

**A1.** Perfectly competitive.
   a. All agents are price takers;
   b. Traded assets may be arbitrarily scaled, and combined into portfolios;
   c. Agents have symmetric information.

**A2.** Arbitrage-free. Agent cannot make trades that yield strictly positive cash flows with certainty.

**A3.** Complete. All risks can be traded.

**A4.** Liquid. Assets can be fully liquidated in any future year.

**Valuation Theorem.** Assume there are no cash flows after year \( T \), and A1-A4 hold. Then there exists a unique collection of random variables \( \tilde{M}_1, ..., \tilde{M}_T \), called pricing factors, such that the current and future market values of any asset are given by

\[
V_{A0} = \sum_{t=1}^{T} E_0[\tilde{M}_t \tilde{C}_{At}],
\]

\[
\hat{V}_{At} = \sum_{t=1}^{T} E_t \left[ \left( \tilde{M}_u / \tilde{M}_t \right) \tilde{C}_{Au} \right], \quad t = 1, ..., T - 1.
\]

and \( \hat{V}_{AT} = 0 \).

**Proof.** See Asset Pricing by John H. Cochrane (Princeton, 2005). \( \square \)

A1-A2 imply the existence of pricing factors and linearity of market values in cash flows. Adding A3 gives uniqueness of the pricing factors and \( \tilde{M}_t \neq 0 \) with probability 1. Adding A4 implies that only one set of pricing factors is applied for all years.

The pricing factors are also called stochastic discount factors.
Returns and OCC

\[ \tilde{r}_{At} = \text{net rate of return at year } t \text{ on a } \text{one-year investment} \text{ in the asset, defined by} \]

\[ \tilde{V}_{A,t-1}(1 + \tilde{r}_{At}) = \tilde{C}_t + \tilde{V}_t. \]

Valuation Theorem \( \Rightarrow \) returns \( \tilde{r}_{A1}, \tilde{r}_{A2}, \ldots, \tilde{r}_{AT} \) exist uniquely whenever \( \tilde{V}_{A,t-1} \neq 0 \). In this case:

\[ \tilde{r}_{At} = \frac{\tilde{C}_t + \tilde{V}_t - \tilde{V}_{A,t-1}}{\tilde{V}_{A,t-1}}. \]

\[ E_{t-1}[\tilde{r}_{At}] = \text{expected return at year } t - 1 = \text{Opportunity Cost of Capital (OCC)} \text{ of the asset at year } t - 1. \]

Discounted Cash Flow (DCF) formula

**Proposition 2.1.** The current market value of \( \tilde{C}_t \) is given by

\[ V_{A0} = DF_{At} \times C_t, \quad (1) \]

where

\[ DF_{At} = \frac{1}{E_0 [(1 + \tilde{r}_{A1})(1 + \tilde{r}_{A2}) \cdots (1 + \tilde{r}_{At})]}. \quad (2) \]

**Proof.** The Valuation Theorem gives \( \tilde{V}_t = 0 \), since \( \tilde{C}_{At} = 0 \) for \( u > t \). Hence

\[ \tilde{V}_{A,t-1}(1 + \tilde{r}_{At}) = \tilde{C}_t + \tilde{V}_t = \tilde{C}_t, \]

which implies

\[ \tilde{V}_{A,t-1} = \frac{\tilde{C}_t}{1 + \tilde{r}_{At}}. \quad (3) \]

Moreover, since \( \tilde{C}_{A,t-1} = 0 \),

\[ \tilde{V}_{A,t-2}(1 + \tilde{r}_{A,t-1}) = \tilde{C}_{A,t-1} + \tilde{V}_{A,t-1} = \tilde{V}_{A,t-1} = \frac{\tilde{C}_t}{1 + \tilde{r}_{At}}, \]

which implies

\[ \tilde{V}_{A,t-2} = \frac{\tilde{C}_t}{(1 + \tilde{r}_{A,t-1})(1 + \tilde{r}_{At})}. \]

Continue for years \( t - 2, t - 3, \ldots, \) until we reach year 1:

\[ V_{A0} = \frac{\tilde{C}_t}{(1 + \tilde{r}_{A1})(1 + \tilde{r}_{A2}) \cdots (1 + \tilde{r}_{At})}. \quad (4) \]
Now rearrange and take expectation at year 0:

\[ V_{A0}E_0[(1 + \tilde{r}_{A1}) (1 + \tilde{r}_{A2}) \cdots (1 + \tilde{r}_{AT})] = E_0[\tilde{C}_{At}]. \]

Defining \( DF_{At} \) as in (2), and applying the notation \( C_{At} = E_0[\tilde{C}_{At}] \), gives (1). \( \square \)

Combining the Valuation Theorem and Prop. 2.1, it follows that the current market value of the cash flow stream \( \tilde{C}_{A1}, \ldots, \tilde{C}_{AT} \) is given by

\[ V_{A0} = \sum_{t=1}^{T} DF_{At} \times E_0[\tilde{C}_{At}]. \]

A5. Constant OCC. For any asset, the returns \( \tilde{r}_{A1}, \tilde{r}_{A2}, \ldots, \tilde{r}_{AT} \) satisfy

\[ \tilde{r}_{At} = \hat{r}_A + \tilde{\varepsilon}_{At}, \]

where \( E_s[\tilde{\varepsilon}_{At}] = 0 \) for all \( s < t \).

Henceforth assume A1-A5.

The proof of the following proposition makes use of two general properties of conditional expectation:

1. If \( 0 \leq s \leq t \), then \( E_t[\tilde{Y}_s \tilde{X}_u] = \tilde{Y}_s E_t[\tilde{X}_u] \), since \( \tilde{Y}_s \) is a known constant at year \( t \).
2. Law of Iterated Expectations: \( E_s[E_t[\tilde{X}_u]] = E_s[\tilde{X}_u] \).

**Proposition 2.2.** The current market value of \( \tilde{C}_{At} \) is given by

\[ V_{A0} = \frac{C_{At}}{(1 + \hat{r}_A)^t}, \] \( \quad \text{(5)} \)

where \( \hat{r}_A \) is the OCC of \( \tilde{C}_{At} \).

**Proof.** The proof is by induction. Rearrange (3) and take expectation at year \( t - 1 \):

\[ E_{t-1}[\tilde{V}_{At,t-1}(1 + \tilde{r}_{At})] = E_{t-1}[\tilde{C}_{At}]. \]

\( \tilde{V}_{At,t-1} \) is a known constant at year \( t - 1 \). Thus:

\[ E_{t-1}[\tilde{V}_{At,t-1}(1 + \tilde{r}_{At})] = \tilde{V}_{At,t-1}E_{t-1}[(1 + \tilde{r}_{At})]. \]

Now apply A5 and use the linearity of expected value:

\[ \tilde{V}_{At,t-1}E_{t-1}[(1 + \tilde{r}_{At})] = \tilde{V}_{At,t-1}E_{t-1}[(1 + \hat{r}_A + \tilde{\varepsilon}_{At})] \]
\[ \tilde{V}_{A,t-1}(1 + r_A + E_t-1[\tilde{\epsilon}_A]) = \tilde{V}_{A,t-1}(1 + r_A). \]

Substitute into (6) and rearrange to obtain
\[ \tilde{V}_{A,t-1} = \frac{E_t-1[\tilde{C}_A]}{1 + r_A}. \]

Now suppose the market value at year \( t - k \) satisfies
\[ \tilde{V}_{A,t-k} = \frac{E_{t-k}[\tilde{C}_A]}{(1 + r_A)^k}. \]

Since \( \tilde{C}_{t-k-1} = 0 \), the year \( t - k \) return is determined by
\[ \tilde{V}_{A,t-k-1}(1 + \tilde{r}_{A,t-k}) = \tilde{V}_{A,t-k} = \frac{E_{t-k}[\tilde{C}_A]}{(1 + r_A)^k}. \]

Take expectation of the preceding equation at year \( t - k - 1 \):
\[ E_{t-k-1}[\tilde{V}_{A,t-k-1}(1 + \tilde{r}_{A,t-k})] = \frac{E_{t-k-1}[E_{t-k}[\tilde{C}_A]]}{(1 + r_A)^k}. \tag{7} \]

Apply the steps used above:
\[ E_{t-k-1}[\tilde{V}_{A,t-k-1}(1 + \tilde{r}_{A,t-k})] = \tilde{V}_{A,t-k-1} E_{t-k-1}[(1 + r_A + \tilde{\epsilon}_{A,t-k})] \]
\[ = \tilde{V}_{A,t-k-1}(1 + r_A + E_{t-k-1}[\tilde{\epsilon}_{A,t-k}]) = \tilde{V}_{A,t-k-1}(1 + r_A). \]

Moreover, the Law of Iterated Expectations gives
\[ E_{t-k-1}[E_{t-k}[\tilde{C}_A]] = E_{t-k-1}[\tilde{C}_A]. \]

Substitute into (7) and rearrange to obtain
\[ \tilde{V}_{A,t-k-1} = \frac{E_{t-k-1}[\tilde{C}_A]}{(1 + r_A)^k}. \]

Working backward to year \( t - k - 1 = 0 \), and applying the notation \( C_A = \tilde{C}_A \),
gives (5).

Combining the Valuation Theorem and Prop. 2.2, and letting \( T \to \infty \), it follows that the current market value of the infinite-horizon cash flow stream \( \tilde{C}_{A1}, \tilde{C}_{A2} \ldots \) is given by
\[ V_{A0} = \sum_{t=1}^{\infty} \frac{C_A}{(1 + r_A)^t}, \]

provided the limits exist. Call this the Valuation Formula.

5
Portfolios and Value Additivity

Assume A1-A4. Consider a portfolio, called Asset $P$, composed of Assets $j \in J$:

$$\tilde{C}_P t = \sum_{j \in J} \tilde{C}_j t, \quad t = 0, 1, ..., T.$$  

**Proposition 2.3.** The market values of the portfolio consisting of Assets $j \in J$ are given by

$$\tilde{V}_P t = \sum_{j \in J} \tilde{V}_j t, \quad t = 0, 1, ..., T.$$  

**Proof.** The Valuation Theorem gives

$$V_{P0} = \sum_{t=1}^{T} E_0 [\tilde{M}_t \tilde{C}_P t]$$

$$= \sum_{t=1}^{T} E_0 \left[ \tilde{M}_t \sum_{j \in J} \tilde{C}_j t \right] = \sum_{t=1}^{T} \sum_{j \in J} E_0 \left[ \tilde{M}_t \tilde{C}_j t \right]$$

$$= \sum_{j \in J} \sum_{t=1}^{T} E_0 \left[ \tilde{M}_t \tilde{C}_j t \right] = \sum_{j \in J} V_{j0}.$$  

Similarly for future market values. \qed

Thus, the value of a portfolio of assets is the sum of the values of the assets making up the portfolio. This property is called Value Additivity.

**Corollary 2.1.** The return at year $t > 0$ on the portfolio consisting of Assets $j \in J$ is given by

$$\tilde{V}_{P,t-1} \tilde{r}_P t = \sum_{j \in J} \tilde{V}_{j,t-1} \tilde{r}_j t.$$  

**Proof.** The definition of a return gives

$$\tilde{V}_{P,t-1} \tilde{r}_P t = \tilde{C}_P t + \tilde{V}_P t - \tilde{V}_{P,t-1}$$

$$= \sum_{j \in J} \left( \tilde{C}_j t + \tilde{V}_j t - \tilde{V}_{j,t-1} \right) = \sum_{j \in J} \tilde{V}_{j,t-1} \tilde{r}_j t.$$ \qed

**Growing Annuities**

**Growing Annuity** = asset whose expected cash flow stream satisfies

$$C_{At} = \begin{cases} B(1 + g)^{t-1}, & t = 1, 2, ..., T \\ 0, & t = T + 1, T + 2, ... \end{cases}$$

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Proposition 2.4. The current market value of a Growing Annuity is given by

\[ V_{A0} = \begin{cases} \frac{B}{r_A-g} \left[ 1 - \left( \frac{1+g}{1+r_A} \right)^T \right], & g \neq r_A \\ \frac{TB}{1+r_A}, & g = r_A \end{cases} \]

Proof. Suppose \( g \neq r_A \). For \( T = 1 \) we have, using the Valuation Formula,

\[
V_{A0} = \frac{B}{1 + r_A} + \sum_{t=2}^{\infty} \frac{0}{(1 + r_A)^t} = \frac{B}{1 + r_A} \times \frac{r_A - g}{r_A - g} = \frac{B}{1 + r_A} \left[ \frac{r_A - g}{r_A - g} \left[ \frac{1}{1 + r_A} + \frac{1}{1 + r_A} \right] \right] = \frac{B}{1 + r_A} \left[ 1 - \frac{1 + g}{1 + r_A} \right].
\]

Now fix \( T > 1 \) and suppose the result holds for \( 1, 2, \ldots, T - 1 \). Again using the Valuation Formula, we have

\[
V_{A0} = \frac{B}{1 + r_A} \sum_{t=1}^{T} \frac{(1 + g)^{t-1}}{(1 + r_A)^t} + \sum_{t=T+1}^{\infty} \frac{0}{(1 + r_A)^t} = \frac{B}{1 + r_A} \sum_{t=1}^{T-1} \frac{(1 + g)^{t-1}}{(1 + r_A)^t} + \frac{B(1 + g)^{T-1}}{(1 + r_A)^T} \times \frac{r_A - g}{r_A - g} = \frac{B}{1 + r_A} \left[ 1 - \left( \frac{1 + g}{1 + r_A} \right)^{T-1} \times \frac{1 + r_A}{1 + r_A} \right] = \frac{B}{1 + r_A} \left[ 1 - \frac{(1 + g)^{T-1}(1 + r_A) - (1 + g)^{T-1}(r_A - g)}{(1 + r_A)^T} \right] = \frac{B}{1 + r_A} \left[ 1 - \frac{(1 + g)^T}{1 + r_A} \right].
\]

Thus by induction the result is valid for all \( T \).

For \( g = r_A \):

\[
V_{A0} = \sum_{t=1}^{\infty} \frac{B(1 + r_A)^{t-1}}{(1 + r_A)^t} + \sum_{t=T+1}^{\infty} \frac{0}{(1 + r_A)^t} = \sum_{t=1}^{T} \frac{B}{1 + r_A} = \frac{TB}{1 + r_A}.
\]

\( \square \)
Proposition 2.5. If \( g < r_A \), then the current market value of a Growing Annuity satisfies

\[
\lim_{T \to \infty} V_{A0} = \frac{B}{r_A - g}.
\]

Proof. Define the constant \( \alpha \) by

\[
\alpha = \frac{1 + g}{1 + r_A}.
\]

Then \(-1 < g < r_A\) implies \(0 < \alpha < 1\), so that \(\lim_{T \to \infty} \alpha^T = 0\). Moreover, the Growing Annuity Formula may be written as

\[
V_{A0} = \frac{B}{r_A - g} \left[ 1 - \alpha^T \right].
\]

Since this is a continuous function of \(\alpha^T\), we have

\[
\lim_{T \to \infty} V_{A0} = \frac{B}{r_A - g} \left[ 1 - \lim_{T \to \infty} \alpha^T \right] = \frac{B}{r_A - g}.
\]

Alternate Proof. Using the Valuation Formula, we have

\[
V_{A0} = \sum_{t=1}^{T} \frac{B(1 + g)^{t-1}}{(1 + r_A)^t} = \frac{B}{(1 + r_A)} \sum_{t=1}^{T} \frac{(1 + g)^{t-1}}{(1 + r_A)^{t-1}}
\]

\[
= \frac{B}{(1 + r_A)} \sum_{t=1}^{T} \alpha^{t-1} = \frac{B}{(1 + r_A)} \sum_{u=0}^{T-1} \alpha^u,
\]

where the substitution \(u = t - 1\) has been made. Since \(0 < \alpha < 1\), the Geometric Series Theorem yields

\[
\lim_{T \to \infty} \sum_{u=0}^{T-1} \alpha^u = \frac{1}{1 - \alpha}.
\]

Thus,

\[
\lim_{T \to \infty} V_{A0} = \frac{B}{(1 + r_A)} \times \frac{1}{1 - \alpha} = \frac{B}{(1 + r_A)} \times \frac{1}{1 - \frac{1 + g}{1 + r_A}} = \frac{B}{r_A - g}.
\]

Future market values

Corollary 2.2. The market value at year \(t\) of the stream \(\tilde{C}_{A,t+1}, \tilde{C}_{A,t+2}, \ldots\) satisfies

\[
V_{At} = \sum_{u=t+1}^\infty \frac{C_{Au}}{(1 + r_A)^{u-t}},
\]

provided the limits exist.
Proof. The proof of Prop. 2.2 showed that the value of $\tilde{C}_{Au}$ at year $t < u$ is

$$\tilde{V}_A = \frac{E_t[\tilde{C}_{Au}]}{(1 + r_A)^{u-t}}.$$ 

Thus, the value of the stream $\tilde{C}_{A,t+1}, \tilde{C}_{A,t+2}, ..., \tilde{C}_{AT}$ at year $t$ is

$$\tilde{V}_A = \sum_{u=t+1}^{T} E_t[\tilde{C}_{Au}] (1 + r_A)^{u-t}. \quad (9)$$

Next, take expectation of (9) at year 0,

$$E_0[\tilde{V}_A] = E_0\left[ \sum_{u=t+1}^{T} E_t[\tilde{C}_{Au}] (1 + r_A)^{u-t} \right].$$

Linearity of expected value and the Law of Iterated Expectations give

$$E_0\left[ \sum_{u=t+1}^{T} E_t[\tilde{C}_{Au}] (1 + r_A)^{u-t} \right] = \sum_{u=t+1}^{T} E_0[E_t[\tilde{C}_{Au}]]$$

$$= \sum_{u=t+1}^{T} E_0[\tilde{C}_{Au}] (1 + r_A)^{u-t}. \quad (10)$$

Finally, (8) is obtained by applying the notation $V_A = E_0[\tilde{V}_A], C_{Au} = E_0[\tilde{C}_{Au}]$ and taking the limit as $T \to \infty$.

\[ \square \]

Corollary 2.3. The current market value of $\tilde{V}_A$ is given by

$$\frac{V_A}{(1 + r_A)^T}.$$ 

Proof. Consider an asset such that $\tilde{C}_{Au} = 0$ at each year $u \leq t$. The Valuation Formula and Cor. 2.2 give

$$V_{A0} = \sum_{u=t+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u}, \quad (10)$$

$$= \frac{1}{(1 + r_A)^T} \sum_{u=t+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^{u-t}} = \frac{V_A}{(1 + r_A)^T}.$$

\[ \square \]
Corollary 2.4. The Valuation Formula converges as \( T \to \infty \) if and only if
\[
\lim_{t \to \infty} \frac{V_{At}}{(1 + r_A)^t} = 0. \tag{11}
\]

**Proof.** The Valuation Formula converges if and only if
\[
\lim_{t \to \infty} \left( \sum_{u=t+1}^{\infty} \frac{C_{Au}}{(1 + r_A)^u} \right) = 0.
\]

In view of (10), this is equivalent to (11). \( \Box \)

Cor. 2.4 shows that convergence of the Valuation Formula places a bound on the growth or decline of future market values. (This rules out "explosive nonfundamental pricing bubbles.")

**Internal Rate of Return**

**IRR Equation:**

\[
\sum_{t=1}^{T} \frac{C_{Bt}}{(1 + r)^t} - I_0 = 0.
\]

Any solution \( r_I > 0 \) is called an *Internal Rate of Return* (IRR).

**IRR Regularity Conditions:**

**Condition 1.** \( \sum_{t=1}^{T} C_{Bt} > I_0 > 0 \).

**Condition 2.** \( \sum_{t=1}^{T} C_{Bt}/(1 + r)^t \) is a strictly decreasing function of \( r \) for \( r > 0 \).

**Proposition 2.6.** Suppose \( C_{B1}, C_{B2}, \ldots \) and \( I_0 \) satisfy IRR Regularity Conditions 1 and 2. Then the IRR exists uniquely, and the IRR Rule is equivalent to the NPV Rule.

**Proof.** Define the function \( V_B(r) \) by
\[
V_B(r) = \sum_{t=1}^{T} \frac{C_{Bt}}{(1 + r)^t} - I_0, \quad r > -1.
\]

It follows that \( r_I \) satisfies the IRR Equation if and only if \( V_B(r_I) = 0 \), i.e., \( V_B \) intersects the \( r \)-axis at \( r_I \).

Using Condition 1, we have
\[
V_B(0) = \sum_{t=1}^{T} C_{Bt} - I_0 > 0,
\]
Thus $V_B(r)$ must cross the $r$-axis at least once for $r > 0$. Since $V_B(r)$ is a continuous function, it must intersect the $r$-axis, which means $V_B(r) = 0$ for some $r > 0$. Conclude that an IRR exists. Moreover, Condition 2 implies that $V_B(r)$ is strictly decreasing in $r$, so it cannot intersect the $r$-axis at more than one point. Conclude that the IRR is unique.

Next, using Condition 2, we have that $r_I > r_B$ holds if and only if $0 = V_B(r_I) < V_B(r_B) = \text{NPV}$, whereas $r_I < r_B$ holds if and only if $0 = V_B(r_I) > V_B(r_B) = \text{NPV}$. This proves that the IRR and NPV Rules are equivalent.

\[ \lim_{r \to -\infty} V_B(r) = -I_0 < 0. \]

Debt valuation

**Lemma 4.1.** For any $l$ and $t$, the market value at year $t$ of type $l$ bonds issued at years $t + 1, t + 2, \ldots$ is zero.

**Proof.** Let $F^l(t + v, T)$ be the face value of type $l$ bonds of maturity $T$ issued at year $t + v > t$, and let $\tilde{D}^l_{t+u}(t + v, T)$ be the market value of these bonds at year $t + v$. Net payments to holders of these bonds at years $u \geq t + v$ are as follows:

\[
\tilde{C}^l_{Du}(t + v, T) = \begin{cases} 
-D^l_{t+u}(t + v, T), & u = t + v, \\
 r^l_C F^l(t + v, T), & u = t + v + 1, \ldots, t + v + T - 1, \\
r^l_C F^l(t + v, T), & u = t + v + T, \\
0, & u = t + v + T + 1, t + v + T + 2, \ldots.
\end{cases}
\]

From equation (9) in the proof of Corollary 2.2, it follows that the market value of the bonds at year $t + v$ is

\[
\tilde{D}^l_{t+u}(t + v, T) = \sum_{u=t+v+1}^{\infty} E_{t+v} \left[ \tilde{C}^l_{Du}(t + v, T) \right] \frac{1}{(1 + r^l_D)^{u-(t+v)}}.
\]

Thus, the market value of the bonds at year $t$ is

\[
\tilde{D}^l_{t+u}(t + v, T) = -E_t \left[ \tilde{D}^l_{t+u}(t + v, T) \right] + \sum_{u=t+v+1}^{\infty} E_t \left[ \tilde{C}^l_{Du}(t + v, T) \right] \frac{1}{(1 + r^l_D)^{u-t}}
\]

\[
= \frac{1}{(1 + r^l_D)^v} E_t \left[ -\tilde{D}^l_{t+u}(t + v, T) + \sum_{u=t+v+1}^{\infty} E_{t+v} \left[ \tilde{C}^l_{Du}(t + v, T) \right] \frac{1}{(1 + r^l_D)^{u-(t+v)}} \right]
\]

\[
= \frac{1}{(1 + r^l_D)^v} \cdot E_t[ 0 ] = 0.
\]
It follows that the market value at year $t$ of type $l$ bonds issued in all future years at all maturities is given by

$$\sum_{v=1}^{\infty} \sum_{T=1}^{\infty} \tilde{D}_t^l(t + v, T) = 0.$$ 

\[\square\]

**Proposition 4.1.** If type $l$ debt satisfies $r_C^l = r_D^l$, then $\tilde{D}_t^l = \tilde{F}_t^l$ for all $t$.

**Proof.** Fix $T > 0$, and let $\tilde{F}_t^l(T)$ denote the face value of type $l$ bonds outstanding at year $t$ that mature at year $t + T$. Net payments to holders of these bonds at years $u > t$ are as follows:

$$\tilde{C}_{Du}^l(T) = \begin{cases} r_C^l \tilde{F}_t^l(T), & u = t + 1, \ldots, t + T - 1, \\ r_C^l \tilde{F}_t^l(T) + \tilde{F}_t^l(T), & u = t + T, \\ 0, & u = t + T + 1, t + T + 2, \ldots. \end{cases}$$

Equation (9) in the proof of Corollary 2.2 implies that the market value of the bonds at year $t$ is

$$\tilde{D}_t^l(T) = \sum_{u=t+1}^{\infty} \frac{r_C^l \tilde{F}_t^l(T)}{(1 + r_D^l)^{u-t}} = \sum_{u=t+1}^{t+T} \frac{r_C^l \tilde{F}_t^l(T)}{(1 + r_D^l)^{u-t}} + \frac{\tilde{F}_t^l(T)}{(1 + r_D^l)^T}$$

$$= \frac{r_C^l \tilde{F}_t^l(T)}{r_D^l} \left[ \frac{1}{1 - \frac{1}{(1 + r_D^l)^T}} \right] + \frac{\tilde{F}_t^l(T)}{(1 + r_D^l)^T}.$$

Since $r_C^l = r_D^l$, we have

$$\tilde{D}_t^l(T) = \tilde{F}_t^l(T) \left[ \frac{1}{1 - \frac{1}{(1 + r_D^l)^T}} \right] + \frac{\tilde{F}_t^l(T)}{(1 + r_D^l)^T} = \tilde{F}_t^l(T).$$

Moreover, Lemma 4.1 shows that the market value at year $t$ of type $l$ bonds of all maturities issued at years $t + 1, t + 2, \ldots$ is zero. Thus,

$$\tilde{D}_t^l = \sum_{T=1}^{\infty} \tilde{D}_t^l(T) = \sum_{T=1}^{\infty} \tilde{F}_t^l(T) = \tilde{F}_t^l.$$ 

\[\square\]

**Proposition 4.2.** If type $l$ debt satisfies $r_C^l = r_D^l$ and $\tilde{F}_t^l = F_0^l(1 + g^l)^t$ for all $t > 0$, then the value of ITS is given by

$$V_{0,\text{ITS}}^l = \frac{r_D^l D_0^l \tau}{r_D^l - g^l}.$$
Proof. In view of Prop. 4.1, \( r_C^l = r_D^l \) implies \( F_0^l = D_0^l \). Thus, we have \( \tilde{F}_t^l = D_0^l(1 + g^l)^t \), and the Growing Perpetuity Formula yields

\[
V_{ITS,0}^l = \sum_{t=1}^{\infty} \frac{r_C^l(1 + r_D^l)^{t-1}}{(1 + r_D^l)^t} = \sum_{t=1}^{\infty} \frac{r_D^l D_0^l (1 + g^l)^{t-1}}{(1 + r_D^l)^t} = \frac{r_D^l D_0^l}{r_D^l - g^l}.
\]

\[\square\]

**Proposition 4.3.** Suppose for every \( l \), \( r_C^l = r_D^l \) and \( \bar{F}_t^l = F_0^l \) for all \( t > 0 \). Then the value and OCC of ITS are given by

\[
V_{ITS,0} = V_{D,0} \tau, \quad r_{ITS} = r_D.
\]

**Proof.** Under the assumptions, Prop. 4.2 implies, for all \( l \)

\[
V_{0,ITS}^l = \frac{r_D^l D_0^l \tau}{r_D^l - 0} = D_0^l \tau.
\]

Thus,

\[
V_{ITS,0} = \sum_{l=1}^{L} V_{ITS,0}^l = \sum_{l=1}^{L} D_0^l \tau = \left( \sum_{l=1}^{L} D_0^l \right) \tau = D_0 \tau,
\]

\[
r_{ITS} = \sum_{l=1}^{L} \frac{V_{ITS,0}^l}{V_{ITS,0}} r_D^l = \sum_{l=1}^{L} \frac{D_0^l \tau}{D_0^l} r_D^l = \sum_{l=1}^{L} D_0^l \frac{r_D^l}{D_0^l} = r_D.
\]

\[\square\]

**Corollary 4.1.** Under the conditions of Proposition 4.3, expected net payouts to debtholders for all \( t > 0 \) are given by

\[
C_{Dt} = \text{Int}_t = r_D D_0.
\]

**Proof.** Let \( \bar{F}_{Mt}^l \) denote the face value of type \( l \) debt that matures at year \( t \), and let \( \bar{F}_{Nt}^l \) and \( \bar{D}_{Nt}^l \) denote the face value and market value, respectively, of newly-issued type \( l \) debt at year \( t \). The face value of type \( l \) debt at year \( t + 1 \) is determined by

\[
\bar{F}_{t+1}^l = \bar{F}_t^l - \bar{F}_{Mt}^l + \bar{F}_{Nt}^l.
\]

We have \( \bar{F}_{t+1}^l = \bar{F}_t^l = F_0^l \) by hypothesis, so the preceding equation implies \( \bar{F}_{Mt}^l = \bar{F}_{Nt}^l \).

Net payouts to type \( l \) debtholders at year \( t \) are given by

\[
\bar{C}_{Dt} = \bar{\text{Int}}_t + \bar{F}_{Mt}^l - \bar{D}_{Nt}^l = r_C^l \bar{F}_t^l + \bar{F}_{Mt}^l - \bar{D}_{Nt}^l.
\]

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Since $r_C^l = r_D^l$, applying the argument of the proof of Prop. 4.1 to newly-issued debt gives $F_{Nt}^l = D_{Nt}^l$. Substituting for $D_{Nt}^l$ and taking current expectation gives

$$C_{Dt}^l = r_C^l F_1^l + F_{Mt}^l - F_{Nt}^l = r_C^l F_0^l = r_C^l D_0^l.$$  

Expected net payouts to all debtholders are obtained as follows:

$$C_{Dt} = \sum_{l=1}^L C_{Dt}^l = \sum_{l=1}^L r_D^l D_0^l \times \frac{D_0}{D_0} = \sum_{l=1}^L r_D^l \frac{D_0^l}{D_0} \times D_0 = r_D D_0.$$

\[\square\]

Equity valuation

**Proposition 4.4.** The Value of Equity equals the value of the net payout stream:

$$E_0 = E_0^{NP}.$$  

**Proof.** Since each share has a cash value of $\tilde{\text{Div}}_1 + \tilde{P}_1$ at year 1, the portfolio of all currently outstanding shares has a cash value of $(\tilde{\text{Div}}_1 + \tilde{P}_1)K_0$ at year 1. Moreover, the definition of $\tilde{C}_{E1}$ may be rearranged to obtain

$$(\tilde{\text{Div}}_1 + \tilde{P}_1)K_0 = \tilde{C}_{E1} + \tilde{P}_1 \tilde{K}_1.$$  

Thus the cash value of the portfolio of shares at year 1 equals $\tilde{C}_{E1} + \tilde{P}_1 \tilde{K}_1$. It follows that the Value of Equity must satisfy

$$E_0 = \frac{C_{E1}}{1 + r_E^{NP}} + \frac{P_1 K_1}{1 + r_E^1},$$

where $r_E^{NP}$ is the OCC of the stream $\tilde{C}_{E1}, \tilde{C}_{E2}, \ldots$; and $r_E^1$ is the OCC of $\tilde{P}_1 \tilde{K}_1$.

Next, since each share has a cash value of $\tilde{\text{Div}}_2 + \tilde{P}_2$ at year 2, the portfolio of all shares outstanding at year 1 has a cash value of $(\tilde{\text{Div}}_2 + \tilde{P}_2)\tilde{K}_1$ at year 2. Moreover, the definition of $\tilde{C}_{E2}$ may be rearranged to obtain

$$(\tilde{\text{Div}}_2 + \tilde{P}_2)\tilde{K}_1 = \tilde{C}_{E2} + \tilde{P}_2 \tilde{K}_2.$$  

Since $\tilde{P}_1 \tilde{K}_1$ is the market value of the portfolio at year 1, it follows that the current market value of $\tilde{P}_1 \tilde{K}_1$ must satisfy

$$\frac{P_1 K_1}{1 + r_E^1} = \frac{C_{E2}}{(1 + r_E^{NP})^2} + \frac{P_2 K_2}{(1 + r_E^2)^2},$$

where $r_E^2$ is the OCC of $\tilde{P}_2 \tilde{K}_2$. Substituting into equation (12) gives

$$E_0 = \frac{C_{E1}}{1 + r_E^{NP}} + \frac{C_{E2}}{(1 + r_E^{NP})^2} + \frac{P_2 K_2}{(1 + r_E^2)^2}.$$
Proceeding in this way for years $t = 2, 3, ..., $ we obtain

$$E_0 = \sum_{t=1}^{\infty} \frac{C_{E_t}}{(1 + r_{NP})^t} = E_0^{NP}. $$

\[ \square \]

**Proposition 4.5.** The Return on Equity equals the OCC of the net payout stream:

$$r_E = r_{NP}^E.$$

**Proof.** From the definition of $\tilde{C}_{E1}$ we have

$$(\tilde{Div}_1 + \tilde{P}_1)K_0 = \tilde{C}_{E1} + \tilde{E}_1.$$ 

Moreover, Prop. 4.4 gives $E_0 = E_0^{NP}$, and the argument of the proof may be used to obtain $\tilde{E}_1 = \tilde{E}_1^{NP}$. Thus,

$$\bar{r}_E = \frac{\tilde{Div}_1 + \tilde{P}_1 - P_0}{P_0} = \frac{(\tilde{Div}_1 + \tilde{P}_1)K_0 - P_0K_0}{P_0K_0} = \frac{\tilde{C}_{E1} + \tilde{E}_1 - E_0}{E_0} = \frac{\tilde{C}_{E1} + \tilde{E}_1^{NP} - E_0^{NP}}{E_0^{NP}} = \bar{r}_{NP}.$$ 

Taking current expectation gives the result.  

\[ \square \]