Maxmin Auction Design with Known Expected Values*

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Abstract

A single unit of a good is sold to a group of bidders. The seller knows the expectation of each bidder's value and a bound on the support of values, but the seller does not know the correlation structure or bidders' (common prior) beliefs. We construct a strong maxmin solution (Brooks and Du, 2020), consisting of

- a maxmin mechanism that maximizes minimum profit across all correlation and information structures and all equilibria, and
- a minmax correlation and information structure that minimizes maximum profit across all mechanisms and equilibria.

The maxmin mechanism has the feature that bidders with relatively low expected values are excluded from the auction, while bidders with high expected values participate in a "proportional auction": Bidders submit non-negative real numbers, interpreted as the share of the good demanded by each bidder. The seller fills the demands if possible, and otherwise the good is rationed proportional to the bids. We describe a class of transfer rules that, together with the proportional allocation, complete a maxmin mechanism. This class of auctions generalizes those described by Brooks and Du (2020).

KEYWORDS: Mechanism design, information design, optimal auctions, profit maximization, interdependent values, max-min, Bayes correlated equilibrium, direct mechanism.

JEL CLASSIFICATION: C72, D44, D82, D83.

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1 Introduction

This paper studies the design of profit-maximizing mechanisms for the sale of goods when there is ambiguity about both the distribution of bidders' values and bidders' higher order beliefs. Specifically, we consider a setting where there is a fixed set of bidders for a single unit of a good. The seller of the good designs the auction to maximize profit. The bidders' values and higher-order beliefs about values are described by a common-prior information structure. The seller does not know the information structure. Rather, all the seller knows is the expectation of each bidder's value and that there is a common upper bound on bidders' values. We assume the seller wants to maximize his worst-case profit across information structures consistent with this knowledge.

For this environment, we characterize a set of *strong maxmin solutions* (Brooks and Du, 2020a). Each strong maxmin solution is a triple of a mechanism, an information structure, and a strategy profile for the bidders, such that: the strategies are an equilibrium for the mechanism and information structure; for this information structure, the mechanism and equilibrium maximize profit among all mechanisms and equilibria; and for this mechanism, the information structure and equilibrium minimize profit among all information structures and equilibria, where the information structure satisfies the aforementioned bounds on values and the known expected value for each bidder. Thus, the first and second components of the strong maxmin solution are a "max-min" mechanism and a "min-max" information structure: the mechanism maximizes minimum profit across all information structures and equilibria, subject to the constraint on the mechanism that an equilibrium exists at the min-max information. Similarly, the information structure minimizes maximum profit across all mechanisms and equilibria, subject to constraint on the information structure that an equilibrium exists at the max-min mechanism. We refer the profit at the equilibrium in the strong maxmin solution as the *profit guarantee*.[1]

We study a particular class of strong maxmin solutions that have the following structure:[2] Both the signals in the information structure and the actions in the mechanism are non-negative real numbers. In addition, the bidders' strategies are "truthful" (or "obedient"), in that with probability one, each bidder sends a action equal to their signal. In other words, the mechanism is a direct mechanism on the information structure, and the information structure is a Bayes correlated equilibrium (Bergemann and Morris, 2016) of the mechanism. Thus, the solution satisfies the "double revelation principle" as described in Brooks and Du (2020a). In addition, the ex ante distribution of bidders' signals is independent exponential with an arrival rate normalized to one, and only local incentive constraints bind at the profit-maximizing direct mechanism.[3]

Within this class, we identify a particular information structure which is a worst-case for the seller. First, for bidders whose expected values are below a cutoff, their signals are completely uninformative (and in fact we could have specified information so that these

[1] Brooks and Du (2020a,b) have extended discussions on the virtues of this approach to modeling informationally robust auction design.

[2] In a discrete setting, Brooks and Du (2020b) show that this structure is without loss of generality. Similar structure arises in the common-value setting in Brooks and Du (2020a).

[3] As we discuss below, these are equivalent properties.

bidders do not even receive signals). These bidders are always excluded from the allocation. For the remaining "included" bidders, interim expected values depend on a weighted sum of the included bidders' signals. These weights are a parameter of the information structure, and they are chosen to match the known mean of each bidder's value. When the weighted sum is above a cutoff (which we normalize to be equal to one), all included bidders' expected values are at the upper bound on the value. Otherwise, the bidders' interim expected values are exponential functions of the weighted sum. This information structure has the critical feature that whenever the weighted sum of signals is less than one, the seller is indifferent between not allocating the good and allocating to each of the included bidders.

For strong maxmin solutions of the aforementioned class and with this constituent information structure, we have two main results. Theorem 1 gives sufficient conditions on a mechanism for it to complete a strong maxmin solution. Theorem 2 then constructs a mechanism that satisfies the sufficient conditions. The conditions concern (i) the sensitivity of each bidder's allocation to their own action and (ii) the *excess growth* of each bidder's transfer, i.e., the difference between the sensitivity of the bidder's transfer with respect to their own action and the transfer itself. The condition (i) depends on the same weights which parameterize the information structure. In particular, a bidder's allocation sensitivity must be weakly less than their weight, and it must be equal to their weight when the weighted sum of actions is less than one. With respect to (ii), the transfer rule must have a particular aggregate excess growth, which is pinned down by the choice of allocation rule.

In general, there are many allocation rules that satisfy condition (i), and holding fixed the allocation rule, there are many transfer rules satisfying the excess growth equation. The solution we construct in Theorem 2 has a *weighted proportional allocation*: The aggregate allocation is equal to the minimum of the weighted sum and 1, and each bidder's individual allocation is proportional to their weighted action.[4] Finally, we show constructively that the excess growth equation always has a solution, as long as the allocation satisfies the sufficient condition.

An important special case is when the model is symmetric, so that all bidders have the same expected value. In this case, we find that the min-max information structure is one in which bidders have *pure common values*, meaning that the bidders' values are perfectly correlated. At a high level, the degree of correlation in bidders' values has two effects: On the one hand, the more positively correlated are bidders' values, the lower is the efficient surplus. On the other hand, positive correlation reduces bidders' private information about the good, which in principle could lead to lower information rents. In the event, the tradeoff is resolved unambiguously in favor of more correlation, and information rents are derived from the bidders' partial and differential information about their common value.

Thus, all bidders have the same weight, and the information structure is exactly that described by Brooks and Du (2020a). Moreover, there is a max-min mechanism that is a *proportional auction*, which consists of the previously described proportional allocation, and also a *proportional transfer*, in which the aggregate transfer depends only on the sum of the signals, and each bidder's individual transfer is proportional to their action. An equivalent interpretation is that there is a constant price per unit that depends only on

[4]Under a change of units, where each bidder's action is scaled up by their weight, this is the same proportional allocation as identified in Brooks and Du (2020a).

the aggregate action. A key takeaway from this paper is that proportional auctions, which were originally motivated in an environment when values are common, are robustly optimal even when values are non-common and the correlation structure is ambiguous.[5]

In addition to these main theorems, we consider three further topics. First, we consider what happens to the profit guarantee when the number of bidders grows large. When all bidders have the same expected value, the profit guarantee converges to that expected value, which is the efficient surplus when all values are perfectly correlated (as they are in the min-max information). Second, we use our characterizations to show an intuitive comparative static, that the profit guarantee is non-decreasing in the bidders' expected values. Finally, we discuss and give examples of other max-min mechanisms with non-proportional allocation rules.

As a final topic for this introduction, we discuss the related literature. First and foremost, our model can be viewed as a variant of Brooks and Du (2020a). In that paper, bidders have a pure common value that follows a known distribution. In contrast, we allow for non-common values and asymmetry across bidders. At the same time, we only constrain the mean of each bidder's value, and we endogenize the correlation structure according to the worst-case criterion. The structure of the solution shares much of the structure of that in Brooks and Du (2020a), and many of the proofs and analytical techniques carry over. Also related is Brooks and Du (2020b), which shows that there exist "approximate" strong maxmin solutions, consisting of finite mechanisms and information structures, and where all equilibria on the pair have profit that is close to the limit profit guarantee. These approximate strong maxmin solutions exist in a fairly general class of environments, including the one considered here. Moreover, the approximate maxmin mechanism and minmax information structure have much of the same structure as the strong maxmin solutions we construct in the present paper, including one-dimensional signals and actions and the signals are iid censored geometric. Brooks and Du (2020b) also develop a methodology for computing approximate strong maxmin solutions via linear programming. The results of this paper were motivated by such simulations.

He and Li (2020) consider a model of auction design when there is a known marginal distribution of each bidder's value, which is the same for all bidders, but the correlation structure is unknown. Che (2020) considers a similar model but where only the mean of each bidder's value is known. Both papers further assume that each bidder knows their own value. These papers find that as the number of bidders grows large, revenue in the truthful equilibrium of the second-price auction converges to the expectation of a single bidder's value. Relative to He and Li (2020) and Che (2020), we drop the hypothesis that bidders know their own values and we also assume that the seller only knows the expected value. Even so, the seller can still obtain the same asymptotic profit guarantee using proportional auctions, regardless of which equilibrium is played. We also completely characterize max-min auctions with a finite number of bidders within the space of all auctions, both in the symmetric case that they consider as well as when bidders have different expected values.

[5]Indeed, as we argue below, even when bidders have different expected values, the seller could still run a proportional auction and obtain the same profit guarantee as in the symmetric model with the average of the expected values. Thus, asymmetry in expected values benefits the seller relative to the symmetric model with the same average expected value.

More broadly, our work is related to the literature on max-min auction design, including Chung and Ely (2007), Brooks (2013), Bergemann et al. (2016), Carrasco et al. (2018), Du (2018), Chen and Li (2018), and Yamashita and Zhu (2018).

The rest of this paper has the following structure: Section 2 describes our mathematical framework and solution concept. Section 3 gives an informal overview of our results in the special case when there are two bidders. Section 4 exposits our main results. Section 6 presents extensions and applications of the model. Section 7 is a conclusion. Omitted proofs are in the Appendix.

2 Model

There are N bidders for a single unit of a good. The bidders are indexed by $i = 1, \ldots, N$. Bidder i has a value $v_i \in [0, 1]$, and has quasilinear preferences over probability of receiving the good q and and a transfer to the seller t, represented by the utility index $v_i q - t$. The values are uncertain; all that is known is that the expectation of bidder i's value is $\widehat{v}_i \in (0, 1)$. Without loss, we assume that bidders are ordered so that $\widehat{v}_1 \geqslant \widehat{v}_2 \geqslant \cdots \geqslant \widehat{v}_N$.

While unknown to the seller, the distribution of bidders' values and the bidders' beliefs are described by an *information structure*, which consists of the following objects: For each bidder i, there is a measurable space of signals S_i, with $S = \times_{i=1}^{N}$. In addition, there is a distribution $\pi \in \Delta(S)$, and an interim expected value function $w : S \to [0, 1]^N$, such that for all i,

$$\int_{s \in S} w_i(s)\pi(ds) = \widehat{v}_i. \tag{1}$$

We denote the information structure by $\mathcal{I} = (S, \pi, w)$. An equivalent representation of an information structure $\mathcal{I} = (S, \pi, w)$ is a joint distribution $\sigma \in \Delta(S \times \{0, 1\}^N)$ whose marginal over S is π and

$$\int_{s \in B} w_i(s)\pi(ds) = \int_{(s,v) \in B \times \{0,1\}^N} v_i \, \sigma(ds, dv), \tag{2}$$

for every measurable subset $B \subseteq S$ and every bidder i.

The seller of the good chooses a *mechanism*. This consists of measurable action spaces A_i for each i and allocation and transfer rules. Let $A = \times_{i=1}^{N} A_i$. The allocation rule is a function $q : A \to \mathbb{R}_+^N$ such that for all $a \in A$,

$$\Sigma q(a) = q_1(a) + \cdots + q_N(a) \leqslant 1.$$

The transfer rule is a function $t : A \to \mathbb{R}$. We denote the mechanism by $\mathcal{M} = (A, q, t)$. We say that the mechanism is *participation secure* if for every i, there exists an action $0 \in A_i$ such that $t_i(0, a_{-i}) = 0$ for all $a_{-i} \in A_{-i}$.

A mechanism and information structure together define a simultaneous-move Bayesian game $(\mathcal{M}, \mathcal{I})$. Bidder i's strategies in this game are measurable mappings $\beta_i : S_i \to \Delta(A_i)$.

A profile of strategies β is identified with a mapping $\beta : S \to \Delta(A)$, where $\beta(s)$ is simply the product measure $\times_{i=1}^{N}\beta_i(s_i)$. Bidder i's expected utility under the strategy profile β is

$$U_i(\beta) = \int_{s\in S}\int_{a\in A} (w_i(s)q_i(a) - t_i(a))\,\beta(da|s)\pi(ds)$$

(where we suppress the dependence of the utility on the mechanism and information structure). The strategy profile β is a *(Bayes Nash) equilibrium* if $U_i(\beta) \geqslant U_i(\beta_i', \beta_{-i})$ for all i and strategies β_i'. The set of equilibria is denoted $B(\mathcal{M}, \mathcal{I})$.

Profit of the seller under the strategy profile β is

$$\Pi(\mathcal{M}, \mathcal{I}, \beta) = \int_{s\in S}\int_{a\in A} \Sigma t(a)\beta(da|s)\pi(ds).$$

The solution concept employed in this paper is the *strong maxmin solution* (Brooks and Du, 2020a), which is a triple $(\mathcal{M}, \mathcal{I}, \beta)$ that satisfies the following conditions:

1. For all $\mathcal{M}'$ and $\beta' \in B(\mathcal{M}', \mathcal{I})$, $\Pi(\mathcal{M}, \mathcal{I}, \beta) \geqslant \Pi(\mathcal{M}', \mathcal{I}, \beta')$;
2. For all $\mathcal{I}'$ and $\beta' \in B(\mathcal{M}, \mathcal{I}')$, $\Pi(\mathcal{M}, \mathcal{I}, \beta) \leqslant \Pi(\mathcal{M}, \mathcal{I}', \beta')$;
3. $\beta \in B(\mathcal{M}, \mathcal{I})$.

Thus, the mechanism $\mathcal{M}$ in the solution (the *max-min mechanism*) guarantees the seller a profit of at least $\Pi(\mathcal{M}, \mathcal{I}, \beta)$ across all information structures and all equilibria, while the information structure $\mathcal{I}$ in the solution (the *min-max information structure*) guarantees that the seller earns a profit of at most $\Pi(\mathcal{M}, \mathcal{I}, \beta)$ across all mechanisms and all equilibria. We call $\Pi(\mathcal{M}, \mathcal{I}, \beta)$ the *profit guarantee* of the strong maxmin solution $(\mathcal{M}, \mathcal{I}, \beta)$

As discussed in Brooks and Du (2020a), the strong maxmin solution can be interpreted as an equilibrium-selection-invariant Nash equilibrium: Consider the simultaneous move game between seller and Nature, where the seller chooses the mechanism and Nature chooses the information structure. If no equilibrium exists, both players' payoffs are $-\infty$. Otherwise, there is a fixed equilibrium selection rule. Suppose $B(\mathcal{M}, \mathcal{I}) \neq \varnothing$. Then $(\mathcal{M}, \mathcal{I})$ is a Nash equilibrium of the mechanism design/information design game for all equilibrium selection rules if and only if there exists a β such that $(\mathcal{M}, \mathcal{I}, \beta)$ is a strong maxmin solution.

Finally, given a function $f : \mathbb{R}^N \to \mathbb{R}^N$, we say that f is *own-right-differentiable* if for every i, the limit

$$\nabla_i f(x_i, x_{-i}) \equiv \lim_{\epsilon\downarrow 0} \frac{f_i(x_i + \epsilon, x_{-i}) - f_i(x_i, x_{-i})}{\epsilon}$$

exists and is finite at every x. We let $\nabla f(x)$ denote the vector whose ith element is $\nabla_i f(x)$. We also denote by $\nabla \cdot f(x) = \sum_{i=1}^{N} \nabla_i f(x)$.

3 An illustration of the results

In this section we give an intuitive and heuristic derivation of our results.

We start from the *double revelation principle*, namely the max-min mechanism is a profit maximizing direct mechanism on the min-max information structure, and the min-max information structure is a profit minimizing correlated equilibrium on the max-min mechanism. Thus we look for a strong maxmin solution $(\overline{\mathcal{M}}, \overline{\mathcal{I}}, \overline{\beta})$ where the signal space and the action space coincide and are normalized to be $\overline{S} = \overline{A} = \mathbb{R}_+^N$, and $\overline{\beta}$ is the truth-telling strategy profile. We can operationalize the double revelation principle by optimizing the following Lagrangian for the seller and Nature's respective problems:

$$\begin{aligned}\mathcal{L}(q,t,\sigma;\alpha,\gamma,\lambda) = &\sum_{i=1}^N \int_{\overline{A}\times\{0,1\}^N} t_i(a)\sigma(da,dv) \\ &+ \sum_{i=1}^N \int_{\overline{A}\times\{0,1\}^N} \alpha_i(a_i)\left[v_i \nabla_i q(a) - \nabla_i t(a)\right]\sigma(da,dv) \\ &+ \int_{\overline{A}} \gamma(a)\left[1 - \Sigma q(a)\right] da \\ &+ \sum_{i=1}^N \int_{\overline{A}\times\{0,1\}^N} \lambda_i(v_i)\left[\mu_i(dv_i) - \sigma(da,dv)\right]\end{aligned} \tag{3}$$

where $\mu_i \in \Delta(\{0,1\})$, $\mu_i(\{1\}) = \widehat{v}_i$ and $\mu_i(\{0\}) = 1 - \widehat{v}_i$. Clearly, the marginal distribution of σ on v_i is μ_i (enforced by the fourth line in (3)) if and only if the ex ante expected value for i in σ is $\widehat{v}_i$.

For a given information structure σ the seller maximizes the profit over direct mechanism (q,t), subject to the constraint that there is no incentive to deviate locally from truth-telling $(a_i = s_i)$ in the mechanism given a realized signal s_i drawn according σ. This constrained maximization problem is equivalent to maximizing the Lagrangian in equation (3) over (q,t), where $\alpha_i(a_i) \geqslant 0$ is the multiplier on the local incentive constraint, and $\gamma(a) \geqslant 0$ is the multiplier on the feasibility constraint for the allocation. For a fixed information structure σ, the fourth line in the Lagrangian is a constant and is in fact zero.

On the other hand, for a given mechanism (q,t) Nature minimizes the profit over Bayes correlated equilibrium σ, which satisfies local incentive constraints (the second line in (3)) as well as the constraints that the marginal of σ over v_i is μ_i (the fourth line in (3), where $\lambda_i(v_i)$ is the multiplier). This constrained minimization problem is equivalent to minimizing the Lagrangian in equation (3) over σ; the third line is a constant for a fixed allocation q and is in fact zero given an optimal γ multiplier (since $\gamma(x) > 0$ only if $\Sigma q(x) = 1$).

Thus, a strong maxmin solution is equivalent to a saddle point (q,t,σ) where (q,t) maximizes the Lagrangian given σ while σ minimizes the Lagrangian given (q,t).

For the seller's profit maximization problem, the derivatives in the second line of (3) must be interpreted as left-derivatives, since the multiplier $\alpha_i(a_i)$ is non-negative, so the corresponding local incentive constraint is the limit of

$$\int_{\overline{A}_{-i}\times\{0,1\}^N}\left[v_i\frac{q_i(a_i,a_{-i}) - q_i(a_i-\epsilon,a_{-i})}{\epsilon} - \frac{t_i(a_i,a_{-i}) - t_i(a_i-\epsilon,a_{-i})}{\epsilon}\right]\sigma(da,dv) \geqslant 0$$

as $\epsilon \searrow 0$. On the other hand, for Nature's profit minimization problem, a non-negative multiplier $\alpha_i(a_i)$ corresponds to the local incentive constraint

$$\int_{\overline{A}_{-i}\times\{0,1\}^N} \left[v_i \frac{q_i(a_i+\epsilon, a_{-i}) - q_i(a_i, a_{-i})}{\epsilon} - \frac{t_i(a_i+\epsilon, a_{-i}) - t_i(a_i, a_{-i})}{\epsilon} \right] \sigma(da, dv) \leqslant 0$$

as $\epsilon \searrow 0$, so the derivatives in the second line of (3) must be interpreted as right-derivatives. In our heuristic discussion we will assume the subtle distinction between left and right derivatives are immaterial; see Brooks and Du (2020b) for a formal argument.

For the seller's profit maximization problem, let us suppose for simplicity that the given informaiton structure σ has a differentiable density $h(a)$ for the marginal distribution over a. Collecting the terms involving t_i in (3), we have

$$\begin{aligned} &\int_{\overline{A}\times\{0,1\}^N} t_i(a)\sigma(da, dv) - \int_{\overline{A}\times\{0,1\}^N} \alpha_i(a_i)\nabla_i t(a)\sigma(da, dv) \\ =&\int_{\overline{A}} t_i(a)h(a)da - \int_{\overline{A}} \alpha_i(a_i)\nabla_i t(a)h(a)da \\ =&\int_{\overline{A}} t_i(a)h(a)da + \int_{\overline{A}} t_i(a)\frac{\partial}{\partial a_i}[\alpha_i(a_i)h(a)]da, \end{aligned}$$

where we integrated by parts in the second line and assumed that $t_i(0, a_{-i}) = 0$ for all a_{-i}. The first order condition for maximizing the Lagrangian with respect to $t_i(a)$ is

$$h(a) = -\frac{\partial}{\partial a_i}[\alpha_i(a_i)h(a)], \tag{4}$$

which must hold for all i and $a \in \overline{A}$. This is obviously equivalent to

$$\alpha_i(a_i) = \frac{\int_{a_i'=a_i}^{\infty} h(a_i', a_{-i})da_i'}{h(a)}.$$

Thus, the signals must be independently distributed (since the hazard rate of i's signal distribution does not depend on others' signals), and moreover the multiplier on the local incentive constraint must be the inverse hazard rate of the signal distribution.

Following the convention in Brooks and Du (2020a,b), we set hereafter

$$\alpha_i(a_i) = 1$$

for all i and $a_i \in \overline{A}_i$, which implies independent, exponential distribution:

$$h(a) = \exp(-\Sigma a).$$

The terms involving q_i in (3) are

$$\begin{aligned}
&\int_{\overline{A}\times\{0,1\}^N} v_i \nabla_i q(a)\sigma(da,dv) - \int_{\overline{A}} \gamma(a) q_i(a) da \\
&= \int_{\overline{A}} w_i(a)\nabla_i q(a) h(a) da - \int_{\overline{A}} \gamma(a) q_i(a) da \\
&= \left[- \int_{\overline{A}} q_i(a) \frac{\partial}{\partial a_i}[w_i(a)h(a)] da \right] - \int_{\overline{A}} \gamma(a) q_i(a) da \\
&= \int_{\overline{A}} q_i(a)[w_i(a) - \nabla_i w(a)] h(a) da - \int_{\overline{A}} \gamma(a) q_i(a) da
\end{aligned}$$

where in the second line w_i is the interim expected value (cf. equation (2)), the third line follows from integration by parts (assuming $q_i(0, a_{-i}) = 0$), and the fourth line follows since h is exponential. Clearly, $w_i(a) - \nabla_i w(a)$ is bidder i's virtual value at signal profile a. Thus, the first order condition for maximizing the Lagrangian with respect to $q_i(a)$ is

$$w_i(a) - \nabla_i w(a) \leqslant \gamma(a)/h(a), \tag{5}$$

which holds for all i and $a \in \overline{A}$. If an optimal allocation is positive for bidder i at a ($q_i(a) > 0$), then the above condition must hold with an equality, i.e., bidder i must have the highest virtual value. Moreover, an optimal $\Sigma q(a) < 1$ only if the multiplier $\gamma(a)$ is zero, i.e., all virtual values $w_i(a) - \nabla_i w(a) \leqslant 0$.

Turning to Nature's profit minimization problem for a given mechanism (q, t), the terms involving σ in (3) are

$$\int_{\overline{A}\times\{0,1\}^N} \left(\Sigma t(a) - \nabla \cdot t(a) + v \cdot \nabla q - \sum_i \lambda_i(v_i) \right) \sigma(da, dv).$$

Thus, the first order condition for minimizing the Lagrangian with respect to $\sigma(da, dv)$ is

$$\Sigma t(a) - \nabla \cdot t(a) + v \cdot \nabla q(a) - \sum_i \lambda_i(v_i) \geqslant 0 \tag{6}$$

which must hold all $a \in \overline{A}$ and $v \in \{0,1\}^N$. Moreover, the support of an optimal σ must be concentrated on (a, v) for which (6) holds with an equality.

In summary, one can interpret the transfer $t_i(a)$ as a multiplier on the constraint (4), the allocation $q_i(a)$ as a multiplier on the constraint (5), and the probability $\sigma(da, dv)$ as a multiplier on the constraint (6). The necessary conditions for (q, t, σ) to constitute a strong maxmin solution is *complementary slackness*: First, all of these constraints are satisfied; moreover, $q_i(a) > 0$ only if constraint (5) binds, $\gamma(a) > 0$ only if $\Sigma q(a) = 1$, and $\sigma(da, dv) > 0$ only if constraint (6) binds. In Theorem 1 we will prove that *complementary slackness* is also sufficient for strong maxmin solution.

3.1 Symmetric expected value

We first observe that when all expected values are the same ($\widehat{v}_1 = \widehat{v}_2 = \cdots = \widehat{v}_N$), the strong maxmin solution for the common, binary value special case of Brooks and Du (2020a) satisfies the complementary slackness conditions and is hence a strong maxmin solution in the present setting with a known expected value.

In the min-max information structure $\overline{\mathcal{I}}$ of Brooks and Du (2020a), signals are independently and exponentially distributed, the interim value is common for all i:

$$\overline{w}_i(s) = \begin{cases} \exp(\Sigma s - 1/\eta) & \Sigma s < 1/\eta, \\ 1 & \Sigma s \geqslant 1/\eta, \end{cases} \tag{7}$$

where the parameter $\eta \in \mathbb{R}_+$ ensures that the interim value has the correct expectation:

$$\int_{\overline{S}} \overline{w}(s) \exp(-\Sigma s) ds = \widehat{v}_i.$$

The max-min mechanism $\overline{\mathcal{M}}$ of Brooks and Du (2020a) is the proportional auction:

$$\begin{aligned} \overline{q}_i(a) &= \begin{cases} \eta a_i & \Sigma a < 1/\eta, \\ \frac{a_i}{\Sigma a} & \Sigma a \geqslant 1/\eta, \end{cases} \qquad \overline{t}_i(a) = \overline{q}_i(a) \cdot \overline{T}(\Sigma a), \\ \overline{T}(x) &= \begin{cases} 0 & x = 0, \\ \frac{1}{g_N(x)} \int_{y=0}^{x} \overline{\Xi}(y) g_N(y) dy & x > 0, \end{cases} \end{aligned} \tag{8}$$

where $g_N(x) = \frac{x^{N-1}e^{-x}}{(N-1)!}$ is the density for the random variable Σs, and $\overline{\Xi}$ will be specified in equation (9).

The proportional auction $\overline{\mathcal{M}}$ maximizes profit on the information structure $\overline{\mathcal{I}}$, since all bidders have the same virtual value $\overline{w}_i(s) - \nabla_i \overline{w}_i(s)$ which is 0 in the low region ($\Sigma s < 1/\eta$) and 1 in the high region ($\Sigma s \geqslant 1/\eta$), so condition (5) is always binding. Moreover the proportional allocation $\overline{q}$ fully allocates the good on the high region where the common virtual value is 1. Thus, the complementary slackness between constraint (5) and allocation $\overline{q}$ is satisfied.

As discussed in the introduction, it is intuitive that a common value information structure is the worst case information structure when only the expected value is known. Indeed, $\overline{\mathcal{I}}$ minimizes the profit on the proportional auction $\overline{\mathcal{M}}$, which is a consequence of the complementary slackness between the constraint (6) and the information structure underlying the interim value $\overline{w}$. Applying the explicit formula in (8), it is easy to verify

$$\nabla \cdot \overline{t}(a) - \Sigma \overline{t}(a) = \overline{\Xi}(\Sigma a).$$

Since $\overline{w}_i(a) \in (0,1)$ on the low region and $\overline{w}_i(a) = 1$ on the high region, we must have condition (6) bind for all $v \in \{0,1\}^N$ when a is in the low region, but bind only for $v = \mathbf{1}$ when a is in the low region.[6] This uniquely pins down $\overline{\Xi}$:

$$\overline{\Xi}(\Sigma a) = \nabla \cdot \overline{q}(\Sigma a) - \sum_{i=1}^{N} \overline{\lambda}_i(1), \tag{9}$$

[6] We use $\mathbf{1}$ (respectively, $\mathbf{0}$) to denote a vector v where $v_i = 1$ (respectively, $v_i = 0$) for every i.

for all $a \in \overline{A}$, where using (8) it is easy to see that $\nabla \cdot \overline{q}(a)$ depends only on Σa and is $N\eta$ on the low region and $\frac{N-1}{\Sigma a}$ on the high region. Moreover, since $\nabla_i \overline{q}(a) = \eta$ on the low region, condition (6) will bind for all $v \in \{0,1\}^N$ and a in the low region if

$$\overline{\lambda}_i(1) - \overline{\lambda}_i(0) = \eta.$$

Finally, condition (6) holds for all $v \in \{0,1\}^N$ for a in the high region since it is easy to check that $\overline{q}_i(a) \leqslant \eta$ on the high region.

3.2 Asymmetric expected value

Now suppose we have asymmetric expected values $\hat{v}_i$. We posit that in the general case the signal/action space is still divided into a low region and a high region, where the bidders' virtual values are all tied and are zero on the low region and one on the high region.

As before, we must have condition (6) bind for all $v \in \{0,1\}^N$ when a is in the low region, but bind only for $v = 1$ when a is in the high region, since we need $\overline{w}_i(a) \in (0,1)$ to induce a virtual value of zero on the low region, and $\overline{w}_i(a) = 1$ to induce a virtual value of one on the high region. Exactly as in the symmetric case, this complementary slackness implies

$$\nabla \cdot \bar{t}(a) - \Sigma \bar{t}(a) = \nabla \cdot \overline{q}(a) - \sum_{i=1}^{N} \overline{\lambda}_i(1),$$

for every $a \in \overline{A}$,

$$\nabla_i \overline{q}(a) = \overline{\lambda}_i(1) - \overline{\lambda}_i(1) \equiv \eta_i$$

for a in the low region, and

$$\nabla_i \overline{q}(a) \leqslant \eta_i$$

for a in the high region.

Since the virtual value of every bidder is 1 on the high region, by complementary slackness we must have $\Sigma \overline{q}(a) = 1$ on the high region. If the allocation $\overline{q}$ is continuous, then we must also have $\Sigma \overline{q}(a) = 1$ on the boundary between high and low regions; since $\overline{q}_i(a) = \eta_i a_i$ on the low region, by the continuity of $\overline{q}$ such a boundary must be defined by $\eta \cdot a = 1$. Thus, the low region is the set of a such that $\eta \cdot a \leqslant 1$, and the high region is the set of a such that $\eta \cdot a \geqslant 1$.

Finally, solving the differential equation $\overline{w}_i(a) - \nabla_i \overline{w}(a) = 0$ gives

$$\overline{w}_i(a) = C(a_{-i}) \exp(a_i)$$

on the low region of $\eta \cdot a \leqslant 1$. Together with the boundary condition that $\overline{w}_i(a) = 1$ when $\eta \cdot a = 1$, we get

$$\overline{w}_i(a) = \begin{cases} \exp\left(\frac{\eta \cdot a - 1}{\eta_i}\right) & \eta \cdot a < 1, \\ 1 & \eta \cdot a \geqslant 1. \end{cases}$$

Thus, the complementary slackness conditions let us derive information structure and mechanisms that naturally generalize those from the common value model. Our Theorem 1 shows that these characterize a set of strong maxmin solutions.

4 Strong maxmin solutions

4.1 Minmax information

We construct an information structure as follows. Let $\overline{S}_i = \mathbb{R}_+$ for all i, and let

$$\overline{\pi}(ds) = \exp(-\Sigma s)ds \tag{10}$$

i.e., the signals are independent and exponentially distributed random variables with an arrival rate of 1. The interim value function has the following form: Fix parameters $\eta \in \mathbb{R}^N_+$. Then define

$$\overline{w}_i(s) = \begin{cases} \min\{(\exp(\eta \cdot s - 1))^{1/\eta_i}, 1\} & \text{if } \eta_i > 0; \\ 0 & \text{if } \eta_i = 0 \text{ and } \eta \cdot s < 1; \\ \frac{\widehat{v}_i}{\int_{\{s \in \overline{S} | \eta \cdot s \geqslant 1\}} \exp(-\Sigma s) ds} & \text{if } \eta_i = 0 \text{ and } \eta \cdot s \geqslant 1. \end{cases} \tag{11}$$

A preliminary result is that there exists a vector η such that (1) is satisfied:

Lemma 1. *There exists a $\eta \in \mathbb{R}^N_+$ such that for $\overline{\pi}$ and $\overline{w}$ defined by* (10) *and* (11)*, $\overline{\mathcal{I}} = (\overline{S}, \overline{\pi}, \overline{w})$ is a well-defined information structure. In particular, it satisfies the moment conditions* (1)

Proof. Let $\overline{\eta} \in \mathbb{R}_+$ such that

$$\int_{s_1=0}^{\infty} \min\{(\exp(\overline{\eta} s_1 - 1))^{1/\overline{\eta}}, 1\} \exp(-s_1) ds_1 \geqslant \widehat{v}_1.$$

Such a $\overline{\eta}$ exists because as $\overline{\eta} \to \infty$, the integrand converges monotonically pointwise to 1, so the Dominated Convergence Theorem converges monotonically to 1, which is strictly greater than the right-hand side.

Now, let us define the mapping $G_i : [0, \overline{\eta}]^N \to \mathbb{R}$ according to

$$G_i(\eta) = \begin{cases} \int_{s \in \mathbb{R}^N_+} \min\left\{(\exp(\eta \cdot s - 1)^{1/\eta_i}, 1\right\} \exp(-\Sigma s) ds & \text{if } \eta_i > 0; \\ \int_{\{s \in \mathbb{R}^N_+ | \eta_{-i} \cdot s_{-i} \geqslant 1} \exp(-\Sigma s_{-i}) ds_{-i} & \text{if } \eta_i = 0. \end{cases}$$

Note that G_i is continuous and strictly increasing in η_i for $\eta_i > 0$. Moreover, the Dominated Convergence Theorem implies that

$$\lim_{\eta_i \to 0} G_i(\eta_i, \eta_{-i}) = G_i(0, \eta_{-i}),$$

so that G_i is continuous at $\eta_i = 0$.

Define the mapping $F : [0, \overline{\eta}]^N \to [0, \overline{\eta}]^N$ as follows: For fixed $\eta \in [0, \overline{\eta}]^N$, we define $F_i(\eta)$ as the solution $\eta_i' \in [0, \overline{\eta}]$ to

$$G_i(\eta_i', \eta_{-i}) = \max\{\widehat{v}_i, G_i(0, \eta_{-i})\}. \tag{12}$$

Note $G_i(\eta)$ is strictly increasing in η_i, so if a solution to (12) exists, it is unique. Moreover, G_i is increasing in η_{-i}, so from how we have defined $\overline{\eta}$, there exists a $\eta_i' > 0$ that satisfies (12) as an equality if and only if $G_i(0, \eta_{-i})$ is weakly less than $\widehat{v}_i$. Otherwise, the unique solution is $\eta_i' = 0$.

Since the left-hand side of (12) is strictly increasing in η_i', the Implicit Function Theorem in Kumagai (1980) implies that $F_i(\eta)$ is continuous. The Brouwer Fixed-Point Theorem then implies that F has a fixed point, which necessarily solves the system (12).

We next claim that for any η that is a fixed point of F. Moreover, $\eta_i = 0$ if and only if

$$\int_{\{s \in \mathbb{R}_+^N | \eta \cdot s \geqslant 1\}} \exp(-\Sigma s) ds \geqslant \widehat{v}_i. \tag{13}$$

For if this condition is satisfied and $\eta_i > 0$, then $G_i(\eta)$ is strictly greater than the left-hand side of (13), which is in turn weakly greater than $G_i(0, \eta_{-i})$. Thus, $G_i(\eta)$ is strictly greater than both terms on the right-hand side of (12), which contradicts the hypothesis that η satisfies (12). (Note that $G_i(0) = 0$, so there must be at least one i for which $\eta_i > 0$.)

Finally, we can define $\overline{w}$ according to any fixed point of F. Clearly, $\overline{w}_i$ satisfies (1) for all i such that $\eta_i > 0$. And since (13) is satisfied for any i such that $\eta_i = 0$, $\overline{w}_i(s) \in [0, 1]$ for all s, and also satisfies (1). □

Throughout the rest of our analysis, we fix a η as in Lemma 1.

4.2 Characterization of maxmin mechanisms

Our main theorem will characterize maxmin mechanisms of a particular form. For a $v_i \in \{0, 1\}$, let us define

$$\overline{\lambda}_i(v_i) \equiv \frac{1}{N} \int_{\{s \in \mathbb{R}_+^N | \eta \cdot s \geqslant 1\}} \exp(-\Sigma s) ds + (\mathbb{I}_{v_i = 1} - \widehat{v}_i)\eta_i \tag{14}$$

and

$$\overline{\lambda}(v) \equiv \sum_{i=1}^{N} \overline{\lambda}_i(v_i).$$

We will consider maxmin mechanisms for which the space of actions is $\overline{A}_i = \mathbb{R}_+$.

Given an own-right-differentiable allocation rule, let us define

$$\overline{\Xi}(a; q) = \nabla \cdot q(a) - \overline{\lambda}(\mathbf{1}).$$

Our main theorem is the following:

Theorem 1. *Suppose that* $\overline{\mathcal{M}} = (\overline{A}, \overline{q}, \overline{t})$ *satisfying the following conditions:*

1. $\overline{A}_i = \mathbb{R}_+$ *for all* i*;*
2. $\overline{q}$ *is own-right-differentiable,* $\overline{q}_i(0, a_{-i}) = 0$*,* $\nabla_i \overline{q}(a)$ *is right-continuous in* a_i *and* $\nabla \overline{q}(a) \geqslant 0$ *for all* a*.*

3. $\nabla\overline{q}(a) = \eta$ *if* $\eta \cdot a < 1$, $\nabla\overline{q}(a) \leqslant \eta$ *if* $\eta \cdot a \geqslant 1$, *and* $\Sigma\overline{q}(a) = 1$ *if* $\eta \cdot a \geqslant 1$.

4. $\overline{t}$ *is own-right-differentiable,* $\overline{t}$ *and* $\nabla\overline{t}$ *are bounded,* $\overline{t}_i(0, a_{-i}) = 0$ *for all* i *and* a_{-i}, *and for all* a,

$$\nabla \cdot \overline{t}(a) - \Sigma\overline{t}(a) = \overline{\Xi}(a; \overline{q}).$$

Define $\overline{\beta}$ *to be the truthful strategy profile such that* $\overline{\beta}_i(\{s_i\}|s_i) = 1$ *for all* i *and* s_i. *Then* $(\overline{\mathcal{M}}, \overline{\mathcal{I}}, \overline{\beta})$ *is a strong maxmin solution. Moreover, the profit guarantee of this solution is*

$$\overline{\Pi} = \int_{\{s \in \mathbb{R}^N_+ | \eta \cdot s \geqslant 1\}} \exp(-\Sigma s) ds.$$

A leading example of an allocation satisfying the hypotheses of Theorem 1 is the proportional allocation:

$$\overline{q}_i(a) = \frac{\eta_i a_i}{\min\{1, \eta \cdot a\}}. \tag{15}$$

Conditions 1 and 2 and the first part of condition 3 clearly hold for the above allocation rule. For the second part of condition 3, we calculate that whenever $\eta \cdot a \geqslant 1$,

$$\nabla_i \overline{q}(a) = \frac{\eta_i(\eta_{-i} \cdot a_{-i})}{(\eta \cdot a)^2} \leqslant \eta_i,$$

since $\frac{\eta_{-i} \cdot a_{-i}}{\eta \cdot a} \leqslant 1$.

For any allocation rule $\overline{q}$ that satisfies the hypotheses of Theorem 1, there is a canonical way to define the transfer as follows. Let Z denote the set of permutations of $\{1, \ldots, N\}$ with a typical element ζ. We denote by

$$[\zeta \leqslant k] = \{j \mid \zeta(j) \leqslant k\},$$

and analogously define $[\zeta > k]$. Next, let

$$\tau_{\zeta,k}(a; \overline{q}) = \int_{\mathbb{R}^{N-k}_+} \overline{\Xi}(a_{[\zeta \leqslant k]}, x_{[\zeta > k]}; \overline{q}) \exp(-\Sigma x_{[\zeta > k]}) dx_{[\zeta > k]}, \tag{16}$$

and

$$\overline{\xi}_i(a; \overline{q}) = \frac{1}{N!} \sum_{\zeta \in Z} \left[\tau_{\zeta,\zeta(i)}(a; \overline{q}) - \tau_{\zeta,\zeta(i)-1}(a; \overline{q})\right]. \tag{17}$$

Finally, define the transfer rule:

$$\overline{t}_i(a) = \exp(a_i) \int_{x_i=0}^{a_i} \overline{\xi}_i(x_i, a_{-i}; \overline{q}) \exp(-x_i) dx_i. \tag{18}$$

We have the following second main result:

Theorem 2. *There exists a $\overline{q}$ that satisfies conditions 1–3 of Theorem 1. For any such $\overline{q}$, let $\overline{t}$ be defined by (16)–(18), and let $\overline{\mathcal{M}} = (\overline{A}, \overline{q}, \overline{t})$. Then $(\overline{\mathcal{M}}, \overline{\mathcal{I}}, \overline{\beta})$ is a strong maxmin solution. In particular, a strong maxmin solution exists.*

In the symmetric case where $\widehat{v}_1 = \widehat{v}_2 = \cdots = \widehat{v}_N$, we clearly have $\eta_1 = \eta_2 = \cdots = \eta_N$. In this case the allocation rule in (15) defines the proportional auction. Let us also define the proportional transfer rule:

$$\begin{aligned} \overline{t}_i(a) &= \overline{q}_i(a) \cdot \overline{T}(\Sigma a), \\ \overline{T}(x) &= \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{g_N(x)} \int_{y=0}^{x} \overline{\Xi}(y; \overline{q}) g_N(y) dy & \text{if } x > 0, \end{cases} \end{aligned} \tag{19}$$

where $g_N(x) = \frac{x^{N-1}e^{-x}}{(N-1)!}$. Brooks and Du (2020a) show that the hypotheses of Theorem 1 are satisfied for the above mechanism $(\overline{q}, \overline{t})$; hence we have

Corollary 1. *Suppose $\widehat{v}_1 = \widehat{v}_2 = \cdots = \widehat{v}_N$. Let $\overline{\mathcal{M}} = (\overline{A}, \overline{q}, \overline{t})$ be the proportional auction where $\overline{A}_i = \mathbb{R}_+$, $\overline{q}$ is given by (15) and $\overline{t}$ is given by (19). Then $(\overline{\mathcal{M}}, \overline{\mathcal{I}}, \overline{\beta})$ is a strong maxmin solution.*

5 Proof of Theorems 1 and 2

5.1 Proof of Theorem 1

Proposition 1. *$\overline{\mathcal{I}}$ is a well-defined information structure. For all mechanisms $\mathcal{M}$ and equilibria β of $(\mathcal{M}, \overline{\mathcal{I}})$, $\Pi(\mathcal{M}, \overline{\mathcal{I}}, \beta) \leqslant \overline{\Pi}$.*

Proof of Proposition 1. Fix an incentive compatible and individually rational direct mechanism (q, t) and define

$$U_i(s_i, s_i') \equiv \int_{\overline{S}_{-i}} (\overline{w}_i(s_i, s_{-i}) q_i(s_i'.s_{-i}) - t_i(s_i', s_{-i})) exp(-\Sigma s_{-i}) ds_{-i},$$

and $U_i(s_i) \equiv U_i(s_i, s_i)$. Incentive compatibility says that for all i, s_i, and s_i',

$$U_i(s_i) \geqslant U_i(s_i, s_i') = U_i(s_i') + \int_{\overline{S}_{-i}} (\overline{w}_i(s_i, s_{-i}) - \overline{w}_i(s_i', s_{-i})) q_i(s_i', s_{-i}) \exp(-\Sigma s_{-i}) ds_{-i}.$$

and individual rationality says that $U_i(s_i) \geqslant 0$. Thus, for all $\Delta \geqslant 0$,

$$\begin{aligned} U_i &\equiv \int_{\overline{S}_i} U_i(s_i) \exp(-s_i) ds_i \\ &\geqslant \int_{\{s \in \overline{S} | s_i \geqslant \Delta\}} [U_i(s_i - \Delta) + (\overline{w}_i(s_i, s_{-i}) - \overline{w}_i(s_i - \Delta, s_{-i})) q_i(s_i - \Delta, s_{-i}))] \exp(-\Sigma s) ds \\ &= \exp(-\Delta) \left(U_i + \int_{\overline{S}} (\overline{w}_i(s_i + \Delta, s_{-i}) - \overline{w}_i(s)) q_i(s) \exp(-\Sigma s) ds \right). \end{aligned}$$

Rearranging, we have

$$U_i \geqslant \frac{1}{\exp(\Delta) - 1} \int_{\overline{S}} (\overline{w}_i(s_i + \Delta, s_{-i}) - \overline{w}_i(s)) q_i(s) \exp(-\Sigma s) ds.$$

Since total surplus is

$$\sum_{i=1}^{N} \int_{\overline{S}} \overline{w}_i(s) q_i(s) \exp(-\Sigma s) ds,$$

we conclude that an upper bound on profit is

$$\sum_{i=1}^{N} \int_{\overline{S}} \left[\overline{w}_i(s) - \frac{1}{\exp(\Delta) - 1} (\overline{w}_i(s_i + \Delta, s_{-i}) - \overline{w}_i(s)) \right] q_i(s) \exp(-\Sigma s) ds.$$

To apply the Dominated Convergence Theorem and take $\Delta \to 0$, we just need to show that the discrete derivative is bounded:

$$\begin{aligned}
&\max_{s \in \overline{S}} \frac{1}{\exp(\Delta) - 1} (\overline{w}_i(s_i + \Delta, s_{-i}) - \overline{w}_i(s))) \\
&\leqslant \max_{\{s \in \overline{S} | \eta \cdot s \leqslant 1\}} \frac{1}{\exp(\Delta) - 1} (\exp((\eta \cdot s)/\eta_i + \Delta) - \exp((\eta \cdot s)/\eta_i) \\
&= \max_{\{s \in \overline{S} | \eta \cdot s \leqslant 1\}} \exp((\eta \cdot s)/\eta_i) \\
&= \exp(1/\eta_i).
\end{aligned}$$

Thus, the limit of the profit upper bound as $\Delta \to 0$ is

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\overline{S}} \left[\overline{w}_i(s) - \nabla_i \overline{w}_i(s) \right] q_i(s) \exp(-\Sigma s) ds \\
&= \int_{\{s \in \overline{S} | \eta \cdot s \geqslant 1\}} \sum_{i=1}^{N} q_i(s) \exp(-\Sigma s) ds \\
&\leqslant \int_{\{s \in \overline{S} | \eta \cdot s \geqslant 1\}} \exp(-\Sigma s) ds = \overline{\Pi}.
\end{aligned}$$

□

Proposition 2. *Suppose that $\overline{\mathcal{M}}$ satisfies the hypotheses of Theorem 1. Then for any information structure $\mathcal{I}$ and equilibrium β of $(\overline{\mathcal{M}}, \mathcal{I})$, $\Pi(\overline{\mathcal{M}}, \mathcal{I}, \beta) \geqslant \overline{\Pi}$.*

Lemma 2. *Suppose that $\overline{\mathcal{M}}$ satisfies the hypotheses of Theorem 1. Then for any information structure $\mathcal{I}$ and equilibrium β of $(\overline{\mathcal{M}}, \mathcal{I})$,*

$$\int_S \int_{\overline{A}} \left[w(s) \cdot \nabla \overline{q}(a) - \nabla \cdot \overline{t}(a) \right] \beta(da|s) \pi(ds) \leqslant 0. \tag{20}$$

Proof of Lemma 2. For all $\Delta > 0$, the fact that β is an equilibrium implies that

$$\int_S \int_{\overline{A}} \sum_{i=1}^N [w_i(s)(\overline{q}_i(a_i+\Delta, a_{-i}) - \overline{q}_i(a)) - (\overline{t}_i(a_i+\Delta) - \overline{t}_i(a))]\beta(da|s)\pi(ds) \leqslant 0.$$

Since $\nabla_i \overline{q}$ and $\nabla_i \overline{t}$ are bounded, we conclude that the integrand in left-hand side of the preceding inequality is bounded by $K\Delta\beta(da|s)\pi(ds)$ for some constant K. We can divide through by Δ. The limit of the left-hand side as $\Delta \to 0$ must be non-positive as well. Finally, the Dominated Convergence Theorem implies that the limit is precisely the left-hand side of (20). □

Proof of Proposition 2. First we show that

$$\overline{\lambda}(v) = \sum_{i=1}^N \overline{\lambda}_i(v_i) \leqslant v \cdot \nabla \overline{q}(a) - \overline{\Xi}(a; \overline{q}) \tag{21}$$

for all $v \in \{0,1\}^N$ and $a \in \overline{A}$. When $\eta \cdot a < 1$, we have $\nabla \cdot \overline{q}(a) = \eta$, $\overline{\Xi}(a;\overline{q}) = -\overline{\lambda}(0)$, and $\overline{\lambda}_i(1) = \overline{\lambda}_i(0) + \eta_i$ for every i, so (21) clearly holds with an equality for all $v \in \{0,1\}^N$. When $\eta \cdot a \geqslant 1$, (21) holds with an equality for $v = \mathbf{1}$ since $\overline{\Xi}(a;\overline{q}) = \mathbf{1} \cdot \nabla \overline{q}(a) - \overline{\lambda}(\mathbf{1})$; as each v_i changes from 1 to 0, the left-hand side of (21) is decreased by η_i, while the right-hand side of (21) is decreased by $\nabla_i \overline{q}(a)$ which is assumed to be less than η_i; inductively this implies that (21) holds for all $v \in \{0,1\}^N$.

Now fix an information structure $\mathcal{I} = (S, \pi, w)$ and equilibrium β of $(\overline{\mathcal{M}}, \mathcal{I})$. Equation (21) implies

$$\sum_{i=1}^N \left[w_i(s)\overline{\lambda}_i(1) + (1 - w_i(s))\overline{\lambda}_i(0)\right] \leqslant w(s) \cdot \nabla \overline{q}(a) - \overline{\Xi}(a;\overline{q})$$

for all $s \in S$ and $a \in \overline{A}$. Moreover, from Lemma 2, we know that (20) must be satisfied. As a result,

$$\begin{aligned}
\int_S \int_{\overline{A}} \Sigma \overline{t}(a)\beta(da|s)\pi(ds) &\geqslant \int_S \int_{\overline{A}} \left[\Sigma \overline{t}(a) + w(s) \cdot \nabla \overline{q}(a) - \nabla \cdot \overline{t}(a)\right] \beta(da|s)\pi(ds) \\
&= \int_S \int_{\overline{A}} \left[w(s) \cdot \nabla \overline{q}(a) - \overline{\Xi}(a;\overline{q})\right] \beta(da|s)\pi(ds) \\
&\geqslant \int_S \int_{\overline{A}} \sum_{i=1}^N \left[(1 - w_i(s))\overline{\lambda}_i(0) + w_i(s)\overline{\lambda}_i(1)\right] \beta(da|s)\pi(ds).
\end{aligned}$$

From (1), we conclude that this is at least

$$\sum_{i=1}^N \left[(1 - \widehat{v}_i)\overline{\lambda}_i(0) + \widehat{v}_i \overline{\lambda}_i(1)\right] = \overline{\Pi}$$

as desired. □

Proposition 3. *Suppose that $\overline{\mathcal{M}}$ satisfies the hypotheses of Theorem 1. Then the truthful strategy profile $\overline{\beta}$ is an equilibrium of $(\overline{\mathcal{M}}, \overline{\mathcal{I}})$.*

Proof of Proposition 3. We first derive an expression for the interim expected transfer in terms of the allocation (equation (24)). Define the individual excess growth as

$$\overline{\xi}_j(a) = \nabla_j \overline{t}(a) - \overline{t}_j(a).$$

With the assumption that $\overline{t}_j(0, a_{-j}) = 0$, the above equation is equivalent to

$$\overline{t}_j(a) = \exp(a_j) \int_{s_j=0}^{a_j} \overline{\xi}_j(s_j, a_{-j}) \exp(-s_j) ds_j. \tag{22}$$

Therefore, we can write the interim expected transfer of bidder i in $(\overline{\mathcal{M}}, \overline{\mathcal{I}})$ as

$$\begin{aligned}
\overline{t}_i(a_i) &= \int_{\overline{A}_{-i}} \overline{t}_i(a_i, s_{-i}) \exp(-\Sigma s_{-i}) ds_{-i} \\
&= \int_{\overline{A}_{-i}} \exp(a_i) \int_{s_i=0}^{a_i} \overline{\xi}_i(s_i, s_{-i}) \exp(-s_i) ds_i \exp(-\Sigma s_{-i}) ds_{-i}.
\end{aligned}$$

Since $\overline{t}_j$ is bounded in equation (22), it must be that

$$\int_{s_j=0}^{\infty} \overline{\xi}_j(s_j, s_{-j}) \exp(-s_j) ds_j = 0 \tag{23}$$

for all j and s_{-j}. Hence, we can rewrite the interim expected transfer as

$$\begin{aligned}
\overline{t}_i(a_i) &= -\int_{\overline{A}_{-i}} \exp(a_i) \int_{s_i=a_i}^{\infty} \overline{\xi}_i(s_i, s_{-i}) \exp(-s_i) ds_i \exp(-\Sigma s_{-i}) ds_{-i} \\
&= -\int_{\overline{A}} \overline{\xi}_i(a_i + s_i, s_{-i}) \exp(-\Sigma s) ds \\
&= -\int_{\overline{A}} \left[\overline{\xi}_i(a_i + s_i, s_{-i}) + \Sigma \overline{\xi}_{-i}(a_i + s_i, s_{-i})\right] \exp(-\Sigma s) ds \\
&= -\int_{\overline{A}} \overline{\Xi}_i(a_i + s_i, s_{-i}; \overline{q}) \exp(-\Sigma s) ds,
\end{aligned}$$

where we applied equation (23) to each $j \neq i$ in the third line, and used the assumption of $\Sigma \overline{\xi} = \overline{\Xi}$ in the fourth line.

Using the definition of $\overline{\Xi}$, we get

$$\begin{aligned}
\overline{t}_i(a_i) &= -\int_{\{\eta \cdot s + \eta_i a_i \geqslant 1\}} \left(\nabla \cdot \overline{q}(a_i + s_i, s_{-i}) - \overline{\lambda}(1)\right) e^{-\Sigma s} ds - \int_{\{\eta \cdot s + \eta_i a_i < 1\}} \left(-\overline{\lambda}(0)\right) e^{-\Sigma s}\, ds \\
&= -\int_{\{\eta \cdot s + \eta_i a_i \geqslant 1\}} \left(\nabla \cdot \overline{q}(a_i + s_i, s_{-i}) - \Sigma \eta\right) e^{-\Sigma s} ds + \overline{\lambda}(0),
\end{aligned}$$

where in the second line we used the fact that $\overline{\lambda}(1) = \Sigma \eta + \overline{\lambda}(0)$, and $\{\eta \cdot s + \eta_i a_i \geqslant 1\}$ is a shorthand for $\{s \in \overline{S} \mid \eta \cdot s + \eta_i a_i \geqslant 1\}$.

Integrating by parts, we have

$$\begin{aligned}
&\int_{\{\eta\cdot s+\eta_i a_i\geqslant 1\}} \nabla\cdot\overline{q}(a_i+s_i,s_{-i})e^{-\Sigma s}ds\\
&=\sum_{j=1}^{N}\int_{\overline{A}_{-j}}\int_{s_j=\frac{(1-\eta_i s_i-\eta_{-j}\cdot s_{-j})^+}{\eta_j}}^{\infty}\nabla_j\overline{q}(a_i+s_i,s_{-i})e^{-\Sigma s}\,ds_j\,ds_{-j}\\
&=\int_{\overline{A}_{-i}}\Bigg[-\overline{q}_i\left(a_i+\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i},s_{-i}\right)e^{-\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i}-\Sigma s_{-i}}\\
&\qquad\qquad+\int_{s_i=\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i}}^{\infty}\overline{q}_i(a_i+s_i,s_{-i})e^{-\Sigma s}ds_i\Bigg]ds_{-i}\\
&\quad+\sum_{j\neq i}\int_{\overline{A}_{-j}}\Bigg[-\overline{q}_j\left(a_i+s_i,\frac{(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+}{\eta_j},s_{-i-j}\right)e^{-\frac{(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+}{\eta_j}-\Sigma s_{-j}}\\
&\qquad\qquad\qquad+\int_{s_j=\frac{(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+}{\eta_j}}^{\infty}\overline{q}_j(a_i+s_i,s_{-i})e^{-\Sigma s}ds_j\Bigg]ds_{-j}\\
&=-\int_{\overline{A}_{-i}}\overline{q}_i\left(a_i+\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i},s_{-i}\right)e^{-\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i}-\Sigma s_{-i}}ds_{-i}\\
&\quad-\sum_{j\neq i}\int_{\overline{A}_{-j}}(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+e^{-\frac{(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+}{\eta_j}-\Sigma s_{-j}}\,ds_{-j}+\int_{\{\eta\cdot s+\eta_i a_i\geqslant 1\}}e^{-\Sigma s}ds,
\end{aligned}$$

where in the last line, we used the facts that $\overline{q}_j\left(a_i+s_i,\frac{(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+}{\eta_j},s_{-i-j}\right)=(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+$ and $\sum_{j=1}^N\overline{q}_j(a_i+s_i,s_{-i})=1$ whenever $\eta\cdot s+\eta_i a_i\geqslant 1$.

Therefore, we have the following expression for the interim expected transfer:

$$\begin{aligned}
\overline{t}_i(a_i)=&\underbrace{\int_{\overline{A}_{-i}}\overline{q}_i\left(a_i+\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i},s_{-i}\right)e^{-\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i}-\Sigma s_{-i}}ds_{-i}}_{X}\\
&+\underbrace{\sum_{j\neq i}\int_{\overline{A}_{-j}}(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+e^{-\frac{(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})^+}{\eta_j}-\Sigma s_{-j}}ds_{-j}}_{Y}\\
&-\underbrace{\int_{\eta\cdot s+\eta_i a_i\geqslant 1}(1-\Sigma\eta)e^{-\Sigma s}ds}_{Z}+\overline{\lambda}(0).
\end{aligned}\tag{24}$$

Next, we show there is no incentive to locally deviate from truthtelling (equation (25)). We calculate

$$\begin{aligned}
\frac{\partial X}{\partial a_i}=&\int_{\{\eta_{-i}\cdot s_{-i}+\eta_i a_i\geqslant 1\}}\nabla_i\overline{q}(a_i,s_{-i})e^{-\Sigma s_{-i}}ds_{-i}\\
&+\int_{\{\eta_{-i}\cdot s_{-i}+\eta_i a_i<1\}}(1-\eta_{-i}\cdot s_{-i})e^{-\frac{1-\eta_i a_i-\eta_{-i}\cdot s_{-i}}{\eta_i}-\Sigma s_{-i}}ds_{-i},
\end{aligned}$$

where we used the fact that $\overline{q}_i\left(a_i + \frac{(1-\eta_i a_i - \eta_{-i}\cdot s_{-i})^+}{\eta_i}, s_{-i}\right) = 1-\eta_{-i}\cdot s_{-i}$ if $\eta_{-i}\cdot s_{-i}+\eta_i a_i < 1$. Likewise,

$$\begin{aligned}\frac{\partial Y}{\partial a_i} =& \sum_{j\neq i}\Bigg[\int_{\{\eta_{-j}\cdot s_{-j}+\eta_i a_i<1\}}(-\eta_i)e^{-\frac{1-\eta_i a_i-\eta_{-j}\cdot s_{-j}}{\eta_j}-\Sigma s_{-j}}ds_{-j}\\ &+\int_{\{\eta_{-j}\cdot s_{-j}+\eta_i a_i<1\}}(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})e^{-\frac{1-\eta_i a_i-\eta_{-j}\cdot s_{-j}}{\eta_j}-\Sigma s_{-j}}\frac{\eta_i}{\eta_j}ds_{-j}\Bigg]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial Z}{\partial a_i} =& \frac{\partial}{\partial a_i}\left[\int_{\overline{A}_{-i}}\left(\int_{s_i=\frac{(1-\eta_i a_i-\eta_{-i}\cdot s_{-i})^+}{\eta_i}}^{\infty}(1-\Sigma\eta)e^{-s_i}ds_i\right)e^{-\Sigma s_{-i}}ds_{-i}\right]\\ =& \int_{\{\eta_{-i}\cdot s_{-i}+\eta_i a_i<1\}}(1-\Sigma\eta)e^{-\frac{1-\eta_i a_i-\eta_{-i}\cdot s_{-i}}{\eta_i}-\Sigma s_{-i}}ds_{-i}.\end{aligned}$$

In the expression for $\frac{\partial Y}{\partial a_i}$, we change the variables from $s_{-j} = (s_{-i-j}, s_i)$ to $s_{-i} = (s_{-i-j}, s_j)$ by leaving s_{-i-j} unchanged and defining $s_j = \frac{1-\eta_i a_i-\eta_i s_i-\eta_{-i-j}\cdot s_{-i-j}}{\eta_j}$. This change of variable implies:

$$\begin{aligned}&\int_{\{\eta_{-j}\cdot s_{-j}+\eta_i a_i<1\}}(-\eta_i)e^{-\frac{1-\eta_i a_i-\eta_{-j}\cdot s_{-j}}{\eta_j}-\Sigma s_{-j}}ds_{-j}\\ =& \int_{\{\eta_{-i}\cdot s_{-i}+\eta_i a_i<1\}}(-\eta_j)e^{-\frac{1-\eta_i a_i-\eta_{-i}\cdot s_{-i}}{\eta_i}-\Sigma s_{-i}}ds_{-i}\end{aligned}$$

and

$$\begin{aligned}&\int_{\{\eta_{-j}\cdot s_{-j}+\eta_i a_i<1\}}(1-\eta_i a_i-\eta_{-j}\cdot s_{-j})e^{-\frac{1-\eta_i a_i-\eta_{-j}\cdot s_{-j}}{\eta_j}-\Sigma s_{-j}}\frac{\eta_i}{\eta_j}ds_{-j}\\ =& \int_{\{\eta_{-i}\cdot s_{-i}+\eta_i a_i<1\}}\eta_j s_j e^{-\frac{1-\eta_i a_i-\eta_{-i}\cdot s_{-i}}{\eta_i}-\Sigma s_{-i}}ds_{-i}.\end{aligned}$$

Combining the above expressions of $\frac{\partial X}{\partial a_i}$, $\frac{\partial Y}{\partial a_i}$ and $\frac{\partial Z}{\partial a_i}$ with equation (24), we get

$$\begin{aligned}\overline{t}_i'(a_i) =& \int_{\{\eta_{-i}\cdot s_{-i}+\eta_i a_i\geqslant 1\}}\nabla_i\overline{q}(a_i, s_{-i})e^{-\Sigma s_{-i}}ds_{-i}\\ &+\int_{\{\eta_{-i}\cdot s_{-i}+\eta_i a_i<1\}}\eta_i e^{-\frac{1-\eta_i a_i-\eta_{-i}\cdot s_{-i}}{\eta_i}-\Sigma s_{-i}}ds_{-i}\\ =& \int_{\overline{A}_{-i}}\nabla_i\overline{q}(a_i, s_{-i})\overline{w}_i(a_i, s_{-i})e^{-\Sigma s_{-i}}ds_{-i},\end{aligned}\tag{25}$$

where in the second equality we used the fact that $\nabla_i\overline{q}(a_i, s_{-i}) = \eta_i$ and $\overline{w}_i(a_i, s_{-i}) = e^{\frac{\eta_i a_i+\eta_{-i}\cdot s_{-i}-1}{\eta_i}}$ if $\eta_{-i}\cdot s_{-i}+\eta_i a_i < 1$, and $\overline{w}_i(a_i, s_{-i}) = 1$ if $\eta_{-i}\cdot s_{-i}+\eta_i a_i \geqslant 1$.

Finally, suppose a bidder receives a signal s_i in $\overline{\mathcal{I}}$; by bidding s_i' instead of s_i in $\overline{\mathcal{M}}$, his interim expected transfer is changed by

$$
\begin{aligned}
\bar{t}_i(s_i') - \bar{t}_i(s_i) &= \int_{a_i=s_i}^{s_i'} \bar{t}_i'(a_i)da_i \\
&= \int_{\overline{A}_{-i}} \int_{a_i=s_i}^{s_i'} \nabla_i\overline{q}(a_i, s_{-i})\overline{w}_i(a_i, s_{-i})da_i\, e^{-\Sigma s_{-i}}ds_{-i} \\
&\geqslant \int_{\overline{A}_{-i}} \int_{a_i=s_i}^{s_i'} \nabla_i\overline{q}(a_i, s_{-i})\overline{w}_i(s_i, s_{-i})da_i\, e^{-\Sigma s_{-i}}ds_{-i} \\
&= \int_{\overline{A}_{-i}} (\overline{q}_i(s_i', s_{-i}) - \overline{q}_i(s_i, s_{-i}))\overline{w}_i(s_i, s_{-i})e^{-\Sigma s_{-i}}ds_{-i}
\end{aligned}
$$

where we applied (25) and exchanged the order of integration in the second line, and the inequality in the third line follows because $\nabla_i\overline{q}(a_i, s_{-i}) \geqslant 0$ and $\overline{w}_i(a_i, s_{-i})$ increases with a_i. This shows that the truthtelling $\overline{\beta}$ is an equilibrium of $(\overline{\mathcal{M}}, \overline{\mathcal{I}})$. □

Proof of Theorem 1. Fix a tuple $(\overline{\mathcal{M}}, \overline{\mathcal{I}}, \overline{\beta})$ that satisfies the hypotheses of Theorem 1. Proposition 1 implies condition 1 for $(\overline{\mathcal{M}}, \overline{\mathcal{I}}, \overline{\beta})$ to be a strong maxmin solution, Proposition 2 implies condition 2, and Proposition 3 implies condition 3. □

5.2 Proof of Theorem 2

Lemma 3. *Suppose that $\overline{q}$ satisfies the hypotheses of Theorem 1. Then*

$$\int_{\overline{A}} \overline{\Xi}(a; \overline{q}) \exp(-\Sigma a)da = 0.$$

Proof of Lemma 3. The paragraph following equation (21) implies that

$$\sum_{i=1}^{N} \left[\overline{w}_i(a)\overline{\lambda}_i(1) + (1 - \overline{w}_i(a))\overline{\lambda}_i(0)\right] = \overline{w}(a) \cdot \nabla\overline{q}(a) - \overline{\Xi}(a; \overline{q})$$

for all $a \in \overline{A}$, since $\overline{w}(a) = 1$ whenever $\eta \cdot a \geqslant 1$.

The ex ante expectation of $\overline{\Xi}$ is therefore the sum over i of the integrals

$$\int_{\overline{A}} \overline{w}_i(a)\nabla_i\overline{q}_i(a) \exp(-\Sigma a)da - \hat{v}_i\overline{\lambda}_i(1) - (1 - \hat{v}_i)\overline{\lambda}_i(0).$$

Integrating by parts and using the fact that $\overline{q}_i(0, a_{-i}) = 0$ and the definition of $\overline{\lambda}$, this is

$$\int_{\overline{A}_{-i}} \left(\int_{\overline{A}_i} (\overline{w}_i(a) - \nabla_i\overline{w}(a))\overline{q}_i(a) \exp(-a_i)da_i\right) \exp(-\Sigma a_{-i})da_{-i} - \frac{\overline{\Pi}}{N}.$$

Summing across i, we get

$$\int_{\overline{A}} \sum_{i=1}^{N} (\overline{w}_i(a) - \nabla_i \overline{w}(a)) \overline{q}_i(a) \exp(-\Sigma a) da$$
$$= \int_{\{a \in \overline{A} | \eta \cdot a \geqslant 1\}} \sum_{i=1}^{N} \overline{q}_i(a) \exp(-\Sigma a) da - \overline{\Pi} = 0,$$

since $\overline{w}_i(a) - \nabla_i \overline{w}(a) = 0$ when $\eta \cdot a < 1$, and $\Sigma \overline{q}(a) = 1$ when $\eta \cdot a \geqslant 1$. □

Proof of Theorem 2. We show that condition 4 of Theorem 1 is satisfied. Equation (18) implies that

$$\overline{\xi}_i(a; \overline{q}) = \nabla_i \overline{t}(a) - \overline{t}_i(a)$$

for all $a \in \overline{A}$. Given the definition of $\overline{\xi}_i(a; \overline{q})$ in (17), $\nabla \cdot \overline{t}(a) - \Sigma \overline{t}_i(a) = \overline{\Xi}(a; \overline{q})$ follows by telescoping the summation over i for each fixed permutation ζ and noticing that $\tau_{\zeta,N}(a; \overline{q}) = \overline{\Xi}(a; \overline{q})$ and $\tau_{\zeta,0}(a; \overline{q}) = 0$ (by Lemma 3). Finally, to show that $\overline{t}$ is bounded, by equation (18) and the fact that $\overline{\Xi}$ is bounded it suffices to show that

$$\int_{x_i=0}^{\infty} \overline{\xi}_i(x_i, a_{-i}; \overline{q}) \exp(-x_i) dx_i = 0,$$

for every $a_{-i} \in \overline{A}_{-i}$. The above equation follows from the definition of $\overline{\xi}$ in (17) since it is easy to see that

$$\int_{x_i=0}^{\infty} \tau_{\zeta,\zeta(i)}(x_i, a_{-i}; \overline{q}) \exp(-x_i) dx_i = \tau_{\zeta,\zeta(i)-1}(a_i, a_{-i}; \overline{q}),$$

where the right-hand side does not depend on a_i. □

6 Discussion

In this section, we discuss three further topics: What happens as the number of bidders grows large, how the profit guarantee varies with the bidders' expected values, and the set of maxmin mechanisms.

6.1 The many-bidder limit

Consider the symmetric model, in which all bidders have the same expected value, equal to $\widehat{v}_1$. What happens to the profit guarantee as we take the number of bidders to infinity? Theorem 2 shows that for every N, the min-max information structure has pure common values. In fact it is the min-max information structure for the pure common value model in which all bidders have a value of 0 with probability $1 - \widehat{v}_1$ and a value of 1 with probability $\widehat{v}_1$. Proposition 7 of Brooks and Du (2020a) shows that as the number of bidders grows large, optimal profit in this information structure converges to the expected value, which is $\widehat{v}_1$. A fortiori, in the present model where we only know each bidder's expected value, the profit guarantee must also converge to $\widehat{v}_1$. Since the model is symmetric, this asymptotic guarantee is obtained by proportional auctions.

Note that this bound is unimprovable: Clearly, it is always possible for Nature to pick an information structure such that bidders' values are perfectly correlated, in which case the efficient surplus is $\widehat{v}_1$. For such an information structure, optimal profit can never be greater than $\widehat{v}_1$, so this is an upper bound on the profit guarantee.

This finding is closely related to results of He and Li (2020) and Che (2020). Both of these papers consider max-min auction design when the correlation between bidders values is ambiguous. In contrast to the present paper, they assume that values are private, i.e., every bidder has perfect information about his own value. He and Li (2020) assume a fixed and symmetric marginal distribution of each bidder's interim expected value, whereas Che (2020) only constrains the expectation of each bidder's value (as in the present paper). These papers conclude that the truthful equilibrium of the second price auction asymptotically attains the optimal profit of $\widehat{v}_1$.[7] In comparison, the proportional auctions have the same asymptotic profit guarantee, but this guarantee is attained in all equilibria and even if values are not private.

6.2 Varying expected values

Suppose that the mechanism $(\overline{q}, \overline{t})$ is a maxmin mechanism for the profile of expected values $\widehat{v} = (\widehat{v}_1, \ldots, \widehat{v}_N)$. We may ask, how would this mechanism perform if instead the expected values changed to $\widehat{v}'$? As previously observed in Brooks and Du (2020a,b), there is a simple way to bound the performance of the mechanism as we change the expected values. Careful examination of the proof of Proposition 2 shows that the argument goes through when the expected values are $\widehat{v}'$, but we arrive at the lower bound on profit

$$\sum_{i=1}^{N} \left[(1 - \widehat{v}_i') \overline{\lambda}_i(0) + \widehat{v}_i' \overline{\lambda}_i(1) \right], \tag{26}$$

where $\overline{\lambda}$ is defined by (14) using $\widehat{v}$. This expression is a continuous and linear function of $\widehat{v}'$. Thus, the profit guarantee for $(\overline{q}, \overline{t})$ varies smoothly as we vary $\widehat{v}$.

This argument has a further implication for how the profit guarantee varies with $\widehat{v}$. Suppose $(\overline{q}, \overline{t})$ is the maxmin mechanism with value multiplier $\overline{\lambda}$. Now suppose that the expected values increase to $\widehat{v}' \geqslant \widehat{v}$. Since $\overline{\lambda}_i(1) \geqslant \overline{\lambda}_i(0)$ for each i, it must be that the lower bound on profit in $(\overline{q}, \overline{t})$ is higher at expected values $\widehat{v}'$ than at $\widehat{v}$. A fortiori, maxmin

[7] A subtle difference is that He and Li (2020) use the second-price auction without reserve price, whereas Che (2020) uses a random reserve price.

profit at $\widehat{v}'$ must be greater than (26).[8] We therefore conclude that the profit guarantee is non-decreasing in $\widehat{v}$.

The value multiplier $\overline{\lambda}$ plays a similar role in the common value case studied in Brooks and Du (2020a). In that setting as well, $\overline{\lambda}$ is a non-decreasing of the common value, so that the profit guarantee is non-decreasing in the distribution of the common value in the sense of first-order stochastic dominance. More broadly, we conjecture that the profit guarantee is non-decreasing in the value distribution.

6.3 Other maxmin auctions

In the discussion preceding Theorem 2, we constructed a particular strong maxmin solution in which the mechanism has the weighted proportional form, and the transfer is given by (16)–(18). As Corollary 1 shows, there are generally other solutions to the excess growth equation, and in fact the proportional transfer is a distinct solution to that defined by (16)–(18). Moreover, there also exist strong maxmin solution with distinct allocation rules. In fact, the argument in Theorem 2 shows that as long as the allocation rule $\overline{q}$ satisfies conditions 2 and 3 of Theorem 1, then the transfer rule defined by (16)–(18) will complete a mechanism that is part of a strong maxmin solution.

An example of such an allocation is the following *Shapley rule*: Each bidder submits a message a_i. Bidders are then randomly ordered, with all orders being equally likely. Let us denote by i_k the kth bidder in the realized order. Then bidder i_k's allocation is equal to

$$\min\left\{\eta_i a_i, \max\left\{1 - \sum_{k'<k} \eta_{i_{k'}} a_{i_{k'}}, 0\right\}\right\}.$$

In words, each bidder i "requests" $\eta_i a_i$ units of the good. Bidders are "served" in order, and a bidder either receives the lesser of their request and the remaining amount of the good. Clearly, if $\eta \cdot a < 1$, then all bidders demands are met, regardless of the order, and $\nabla \overline{q} = \eta$. If $\eta \cdot a \geqslant 1$, then under every order, some bidder will not receive their demanded amount. When this happens, the bidder's allocation is insensitive to their action. Hence, for every action profile, $\nabla \overline{q}(a) \leqslant \eta$.

It is interesting to note that when all η_i's are the same the Shapley allocation is part of a maxmin mechanism in the common value model. This was shown by Bergemann, Brooks, and Morris (2016) when there are two bidders, and generalized to many bidders in an early working paper version of Brooks and Du (2020a) (available from the authors upon request). It is also an immediate corollary of Theorem 1, since the information structure $\overline{\mathcal{I}}$ in Theorem 1 is a common value information structure when all η_i's are the same.

[8] To make this statement rigorous, we need to show that the mechanism $(\overline{q}, \overline{t})$ has an equilibrium at the min-max information structure for $\widehat{v}'$. A minor complication is that the action and signal spaces are non-compact. It is straightforward to compactify the signal space in the min-max information structure, since the interim expected value is constant when $\eta \cdot s \geqslant 1$. The max-min auction is only slightly more subtle, since the proportional rule used in Theorem 2 is not continuous at infinity (although the transfer rule is continuous at infinity). However, as we observe in Section 6.3, there are other max-min allocations that satisfy the hypotheses of Theorem 1 and are continuous at infinity, so that equilibrium existence follows immediately from the results of Milgrom and Weber (1985).

Another example is the "consistent" rule of Aumann and Maschler (1985), which reduces to the Shapley rule when $N = 2$ but differs for $N > 2$. In particular, if we let $f_i(d_1, \ldots, d_N)$ denote the share of agent i under the consistent rule when there is a unit surplus to be divided among the N agents, and each agent i demands d_i. As shown by Aumann and Maschler, $\partial f_i(d)/\partial d_i \in \{0, 1/2, 1\}$. Thus, if we define the allocation rule $\overline{q}_i(a) = f_i(\eta_1 a_1, \ldots, \eta_N a_N)$, then $\nabla_i \overline{q}(a \in \{0, \eta_i/2, \eta_i\}$, as required by Theorem 1.

Brooks and Du (2020a) also shows that when the distribution of the value does not have an atom at the top, we select for the proportional rule as the unique max-min allocation when actions are sufficiently large. Whether it is possible to select for the proportional allocation in the present setting is an interesting question for future work.

7 Conclusion

This paper has considered optimal auction design according to a notion of profit maximization that is robust to both the bidders' information and the correlation between their values. In contrast to prior work on informationally-robust auction design, we have assumed that bidders have arbitrary interdependent values, with the only restriction being that each bidder's valuation for the good has a known expectation. We have constructed an information structure that minimizes maximum equilibrium profit. We have also characterized and constructed mechanisms that maximize minimum equilibrium profit across all information structures. These statements remain true regardless of how an equilibrium is selected.

In previous work, we identified the novel class of proportional auctions. We showed that these mechanisms are max-min optimal when bidders have common values. A notable conclusion of the present paper is that proportional auctions continue to be max-min mechanisms when bidders have arbitrary interdependent values, as long as all bidders have the same expected value. This is a strong argument in favor of the robust optimality of proportional auctions.

More broadly, we have characterized maxmin mechanisms when bidders are asymmetric. The maxmin allocation rules are essentially the same as those in the symmetric case, except that we weight each bidder's action. In extreme cases, bidders are given zero weight, in which case they are excluded from the auction altogether. Otherwise, this is merely a change of units for actions. The transfer rules are different. For example, the proportional transfer is no longer part of a maxmin mechanism, even under a change of units. An important direction for future work is to identify simple and tractable maxmin transfer rules when bidders are asymmetric.

References

Aumann, R. J. and M. Maschler (1985): "Game Theoretic Analysis of a Bankruptcy Problem from the Talmud," *Journal of Economic Theory*, 36, 195–213.

Bergemann, D., B. Brooks, and S. Morris (2016): "Informationally Robust Optimal Auction Design," Tech. rep., Princeton University and the University of Chicago and Yale University, working paper.

Bergemann, D. and S. Morris (2016): "Bayes Correlated Equilibrium and the Comparison of Information Structures in Games," *Theoretical Economics*, 11, 487–522.

Brooks, B. (2013): "Surveying and Selling: Belief and Surplus Extraction in Auctions," Working paper.

Brooks, B. and S. Du (2020a): "Optimal Auctions with Common Values: An Informationally Robust Approach," Tech. rep., The University of Chicago and University of California-San Diego, working paper.

——— (2020b): "A Strong Minimax Theorem for Informationally-Robust Auction Design," Tech. rep., The University of Chicago and University of California-San Diego, working paper.

Carrasco, V., V. F. Luz, N. Kos, M. Messner, P. Monteiro, and H. Moreira (2018): "Optimal selling mechanisms under moment conditions," *Journal of Economic Theory*, 177, 245–279.

Che, E. (2020): "Distributionally Robust Optimal Auction Design under Mean Constraints," *arXiv*, arXiv–1911.

Chen, Y.-C. and J. Li (2018): "Revisiting the Foundations of Dominant-Strategy Mechanisms," *Journal of Economic Theory*, 178, 294–317.

Chung, K.-S. and J. C. Ely (2007): "Foundations of Dominant-Strategy Mechanisms," *The Review of Economic Studies*, 74, 447–476.

Du, S. (2018): "Robust mechanisms under common valuation," *Econometrica*, 86, 1569–1588.

He, W. and J. Li (2020): "Correlation-Robust Auction Design," .

Kumagai, S. (1980): "An implicit function theorem: Comment," *Journal of Optimization Theory and Applications*, 31, 285–288.

Milgrom, P. R. and R. J. Weber (1985): "Distributional Strategies for Games with Incomplete Information," *Mathematics of Operations Research*, 10, 619–632.

Yamashita, T. and S. Zhu (2018): "On the foundations of ex post incentive compatible mechanisms," .