"[An] important and difficult question...[is] not answered by the approach taken here: the integration of money in the theory of value..."

------ Gerard Debreu, Theory of Value (1959)

No two-commodity model and no two-agent model can provide a foundation for the role of money in the economy. At least three agents and three commodities are required. It's going to be hard to diagram; it's necessarily a 'general equilibrium' (many commodity) problem.
Prof. Frank Hahn notes

”The most serious challenge that the existence of money poses to the theorist is this: the best developed model of the economy cannot find room for it. The best developed model is, of course, the Arrow-Debreu version of a Walrasian general equilibrium. A first, and...difficult...task is to find an alternative construction without...sacrificing the clarity and logical coherence ... of Arrow-Debreu.”
Carl Menger (1892)

“It is obvious ... that a commodity should be given up by its owner ... for another more useful to him. But that every[one] ... should be ready to exchange his goods for little metal disks apparently useless as such ... or for documents representing [them] ... is...mysterious....

why...is...economic man ...ready to accept a certain kind of commodity, even if he does not need it, ... in exchange for all the goods he has brought to market[?] 

The problem ... consists in giving an explanation of a general, homogeneous, course of action ... which ... makes for the common interest, and yet which seems to conflict with the ... interests of contracting individuals.

[Call] goods ... more or less saleable, according to the ... facility with which they can be disposed of ... at current purchasing
prices or with less or more diminution...Men ... exchange goods ... for other goods ... more saleable....[which] become generally acceptable media of exchange [emphasis in original]."

Interpretation: A good is very saleable (liquid) if the price at which a household can sell it (the bid price) is very near the price at which it can buy (the ask price). Hence, Menger suggests that liquid goods, those with narrow spreads between bid and ask prices, become principal media of exchange, money: Liquidity creates monetization.

Menger gives us two big ideas here: a functional definition of liquidity (saleability) and the notion that through repeated use, specialization of the medium of exchange will become standardized. In Tobin's language, the designation of a medium of
exchange is "self-justifying" through the liquidity that is a consequence of large-scale usage. This comes about in two ways: scale economies and learning by doing. If there are scale economies in transaction costs, then the designation of a common medium of exchange creates a natural monopoly; once sufficient scale in use of the medium of exchange is achieved, it acquires a monopoly on the medium of exchange function simply because that large scale implies very low marginal and average transaction costs. Learning by doing is the same phenomenon dynamically; as traders become increasing accustomed to a medium of exchange, the transaction costs of using it decline.
Hicks (1935)

"This, as I see it, is really the central issue in the pure theory of money. Either we have to give an explanation of the fact that people do hold money when rates of interest are positive, or we have to evade the difficulty somehow."

“ It was marginal utility that really made sense of the theory of value; and to come to a branch of economics which does without marginal utility altogether! No wonder there are such difficulties and such differences! What is wanted is a "marginal revolution"! That is my suggestion. But I know that it will meet with apparently crushing objections. I shall be told that the suggestion has been tried out before. It was tried by Wicksell ... It was tried by Mises ... The suggestion has a history, and its history is not encouraging."
... I think we have to look ... frictions in the face, and see if they are really so refractory after all. This will, of course., mean that we cannot allow them to go to sleep under so vague a title. The most obvious sort of friction, and undoubtedly one of the most important, is the cost of transferring assets from one form to another. “
Prof. Tobin (1961) comments:

“The intellectual gulf between economists' theory of the values of goods and services and their theories of the value of money is well known and periodically deplored. Twenty-five years after Hicks's eloquent call for a marginal revolution in monetary theory our students still detect that their mastery of the presumed fundamental, theoretical apparatus of economics is put to very little test in their studies of monetary economics and aggregative models. As Hicks complained, anything seems to go in a subject where propositions do not have to be grounded in someone's optimizing behavior and where shrewd but casual empiricisms and analogies to mechanics or thermodynamics take the place of inferences from utility and profit maximization.
From the other side of the chasm, the student of monetary phenomena can complain that pure economic theory has never delivered the tools to build a structure of Hicks's brilliant design. The utility maximizing individual and the profit maximizing firm know everything relevant about the present and future and about the consequences of their decisions. They buy and sell, borrow and lend, save and consume, work and play, live and let live, in a frictionless world; information, transactions, and decisions are costless. Money holdings have no place in that world, unless possession of green pieces of paper and yellow pieces of metal satisfies some ultimate miserly or numismatic taste.”
G. F. Knapp's *Staatliche Theorie des Geldes* (1923),

"Money is a creature of law."

"First and foremost, money frees us from our debts toward the state; for the state in emitting it, acknowledges that, in receiving, it will accept his means of payment."

Or as Knapp's student Kaulla ( ) put it, "the note debt of the state stands against a corresponding quantity of demands by the state which can be unconditionally satisfied by the notes."
As Lerner (1947):

“The modern state can make anything it chooses generally acceptable as money and thus establish its value quite apart from any connection, even of the most formal kind, with gold or with backing of any kind. It is true that a simple declaration that such and such is money will not do, even if backed by the most convincing constitutional evidence of the state's absolute sovereignty. But if the state is willing to accept the proposed money in payment of taxes and other obligations to itself the trick is done ... On the other hand if the state should decline to accept some kind of money in payment of obligations to itself, it is difficult to believe that it would retain much of its general acceptability.”
Jevons (1875) describes it this way:

“The earliest form of exchange must have consisted in giving what was not wanted directly for that which was wanted. This simple traffic we call barter ... and distinguish it from sale and purchase in which one of the articles exchanged is intended to be held only for a short time, until it is parted with in a second act of exchange. The object which thus temporarily intervenes in sale and purchase is money. At first sight it might seem that the use of money only doubles the trouble, by making two exchanges necessary where one was sufficient; but a slight analysis of the difficulties inherent in simple barter shows that the balance of trouble lies quite in the opposite direction. ... The first difficulty in barter is to find two persons whose disposable possessions mutually suit each other's wants. There may be many people wanting, and many possessing those things wanted; but to allow of an act of barter, there must be
a double coincidence, which will rarely happen.... Sellers and purchasers can only be made to fit by the use of some commodity, ... which all are willing to receive for a time, so that what is obtained by sale in one case, may be used in purchase in another.”
Explicit recognition that the foundations of money as the medium of exchange depends on a 3 by 3 example comes from Wicksell (1898):

“Let us suppose, to take the simplest case, that commodity (A) is desired only by the owners of commodity (B), that commodity (B) is desired by the owners, not of commodity (A), but of a third commodity (C), which, in its turn, is demanded by the possessors of commodity (A) and by no others. It is then obvious that no direct exchange can take place. only an indirect exchange is possible.

For instance, the possessors of (A) might exchange their commodity for commodity (B) with the intention, not of consuming it, but of offering it to the owners of commodity (C), and so of acquiring this commodity (C) which is the one that they desire.
But this kind of intermediate trade would soon prove too clumsy and troublesome for any developed economic system unless it were conducted on organised lines. It has therefore become an immemorial custom among all nations to hold stocks of some commodity for which there is a universal demand and to employ it as a medium of exchange (in the narrower sense of the term). A commodity is particularly suitable for this purpose if it can be easily transported and if it is not susceptible to rapid decay, so that everyone willingly accepts quantities that are in excess of his immediate requirements.

Let us call such a commodity, (M). Then in our example the possessors of commodity (A), assuming that they were provided with a sufficient supply of (M) would obtain the commodity (C), which they desire, in direct exchange for a certain quantity of (M). Then the owners of (C) can use the quantity of (M) which they
acquire in this way to buy the commodity (B), and the owners of (B) can then use it to buy the commodity (A).

**Fiat Money**

Money is typically held as an inventory.

The next step in economizing on the capital tied up in backing the currency is to use a fiat money.

Adam Smith (1776)

“The substitution of paper in the room of gold and silver money, replaces a very expensive instrument of commerce with one much less costly, and sometimes equally convenient .... When paper is
substituted in the room of gold and silver money, the quantity of ...capital ... may be increased by the whole value of gold and silver .... The operation ... resembles that of the undertaker of some great work, who, in consequence of some improvement in mechanics, takes down his old machinery, and adds the difference between its price and that of the new to his ... capital.

The gold and silver money which circulates in any country ... is ... all dead stock. It is a very valuable part of the capital of the country, which produces nothing to the country. The judicious operations of banking, by substituting paper in the room of a great part of this gold and silver, enable the country to convert a great part of this dead stock into active and productive stock. ”
Paradox of Positivity of Value of Fiat Money

Worthless paper printed with the name of the government remains worthless paper. Prof. Lerner (1947) notes "a simple declaration that such and such is money will not do, even if backed by the most convincing constitutional evidence of the state's absolute sovereignty." Thus, for a fiat money there is always the possibility that it will not be able to serve its function because it may have no value in trade. Equivalently, the price level denominated in fiat money may become infinitely high if participants in the economy are unsure that the currency has a positive value.

Taxation and the Value of Fiat Money

Adam Smith (1776) writes, "A prince, who should enact that a certain proportion of his taxes be paid in a paper money of a
certain kind, might thereby give a certain value to this paper money"

Then says Prof. Lerner (1947),
“If the state is willing to accept the proposed money in payment of
taxes and other obligations to itself the trick is done. Everyone who
has obligations to the state will be willing to accept the pieces of
paper with which he can settle the obligations, and all other people
will be willing to accept these pieces of paper because they know
that the taxpayers, etc., will be willing to accept them in turn.”
A Price Theory of Money

Menger and Hicks, focusing on transaction costs, recommend a research strategy allowing price theory to formulate a theory of the medium of exchange. That theory should be able to provide a foundation for four anomalies in the structure of transactions:

What puzzles must a price-theoretic fundamental model of money resolve?

- Trade is monetary. One side of almost all transactions is the economy's common medium of exchange.

- Money is (virtually) unique. Though 'money' differs among economies, almost all the transactions in most places most of the time use a single common medium of exchange.
Even transactions suitable for barter resolution, displaying a double coincidence of wants, are transacted with money.

’Money’ is government-issued fiat money, trading at a positive value though it conveys directly no utility or production.

General equilibrium models typically model complete markets and a fully articulated price system. Using the general equilibrium approach allows us to pursue a parsimonious theory: What is a minimal set of market imperfections so that money arises endogenously?
Appendix: Summary for the General Equilibrium Theorist

Arrow-Debreu model with two additional structures:

transactions take place at commodity-pairwise trading posts;

trade is resource-using, requiring transaction costs.

Prices at the trading posts are characterized as bid and ask rates of exchange between the goods traded at the post; the bid/ask spread prices transaction costs. At each commodity-pairwise trading post, budget constraints apply separately for each transaction. In an economy of \( N \) commodities, there are \( \frac{N(N - 1)}{2} \) trading posts and \( N(N - 1) \) distinct prices. Market equilibrium occurs at prices
so that each trading post clears and market makers cover their costs.

The most liquid (lowest bid/ask spread) good is the natural money, consistent with Menger (1892)'s observations. Segmentation by commodity-pairwise trading posts is a convenient and arbitrary device. The essential point is that it is not simultaneously possible (or economically attractive) to trade all goods together in a single grand trade (as the Arrow-Debreu model posits).

Scale economies in transaction costs create a natural monopoly in the monetary instrument. A class of examples with scale economies demonstrates the commonplace observation that

- Money is (virtually) unique.