Bargaining and equilibrium: The core of a market economy

Set $X^i = \mathbb{R}_+^N$, all $i$.

Each $i \in H$ has an endowment $r^i \in \mathbb{R}_+^N$ and a preference quasi-ordering $\succeq^i$ defined on $\mathbb{R}_+^N$.

An allocation is an assignment of $x^i \in \mathbb{R}_+^N$ for each $i \in H$. A typical allocation, $x^i \in \mathbb{R}_+^N$ for each $i \in H$, will be denoted $\{x^i, i \in H\}$. An allocation, $\{x^i, i \in H\}$, is feasible if $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$, where the inequality holds coordinatewise.

We assume preferences fulfill weak monotonicity (C.IV*), continuity (C.V), and strict convexity (C.VI(SC)).

The core of a pure exchange economy

Definition A coalition is any subset $S \subseteq H$. Note that every individual comprises a (singleton) coalition.

Definition An allocation $\{x^i, h \in H\}$ is blocked by $S \subseteq H$ if there is a coalition $S \subseteq H$ and an assignment $\{y^i, i \in S\}$ so that:

(i) $\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$ (where the inequality holds coordinatewise),
(ii) $y^i \succeq^i x^i$, for all $i \in S$, and
(iii) $y^h \succ^h x^h$, for some $h \in S$

Definition The core of the economy is the set of feasible allocations that are not blocked by any coalition $S \subseteq H$. 
• Any allocation in the core must be individually rational. That is, if \( \{x^i, i \in H\} \) is a core allocation then we must have \( x^i \succeq_h r^i \), for all \( i \in H \).

• Any allocation in the core must be Pareto efficient.

(i) The competitive equilibrium is always in the core (Theorem 21.1).

Theorems 22.2 and 22.3 say that

(ii) For a large economy, the set of competitive equilibria and the core are virtually identical. All core allocations are (nearly) competitive equilibria.

The competitive equilibrium allocation is in the core

Definition \( p \in \mathbb{R}^N_+, p \neq 0, x^i \in \mathbb{R}^N_+, \) for each \( i \in H \), constitutes a competitive equilibrium if

(i) \( p \cdot x^i \leq p \cdot r^i \), for each \( i \in H \),

(ii) \( x^i \succeq_i y \), for all \( y \in \mathbb{R}^N_+ \), such that \( p \cdot y \leq p \cdot r^i \), and

(iii) \( \sum_{i \in H} x^i \leq \sum_{i \in H} r^i \) (the inequality holds coordinatewise) with \( p_k = 0 \) for any \( k = 1, 2, \ldots, N \) so that the strict inequality holds.

Theorem 21.1 Let the economy fulfill C.II, C.IV*, C.VI(SC) and let \( X^i = \mathbb{R}^N_+ \). Let \( p, x^i, i \in H \), be a competitive equilibrium. Then \( \{x^i, i \in H\} \) is in the core of the economy.

Proof We will present a proof by contradiction. Suppose the theorem were false. Then there would be a blocking coalition
$S \subseteq H$ and a blocking assignment $y^i, i \in S$. We have
\[ \sum_{i \in S} y^i \leq \sum_{i \in S} r^i \] (attainability, the inequality holds coordinatewise)
\[ y^i \succeq_i x^i, \quad \text{for all } i \in S, \text{ and} \]
\[ y^h \succ_h x^h, \quad \text{some } h \in S. \]

But $x^i$ is a competitive equilibrium allocation. That is, for all $i \in H$, $p \cdot x^i = p \cdot r^i$ (recalling Lemma 17.1), and $x^i \succeq_i y$, for all $y \in \mathbb{R}^N_+$ such that $p \cdot y \leq p \cdot r^i$.

Note that $\sum_{i \in S} p \cdot x^i = \sum_{i \in S} p \cdot r^i$. Then for all $i \in S$, $p \cdot y^i \geq p \cdot r^i$. That is, $x^i$ represents $i$’s most desirable consumption subject to budget constraint. $y^i$ is at least as good under preferences $\succeq_i$ fulfilling C.II, C.IV*, C.VI(SC), (local non-satiation). Therefore, $y^i$ must be at least as expensive. Furthermore, for $h$, we must have $p \cdot y^h > p \cdot r^h$. Therefore, we have
\[ \sum_{i \in S} p \cdot y^i > \sum_{i \in S} p \cdot r^i. \]

Note that this is a strict inequality. However, for coalitional feasibility we must have
\[ \sum_{i \in S} y^i \leq \sum_{i \in S} r^i. \]

But since $p \geq 0, p \neq 0$, we have $\sum_{i \in S} p \cdot y^i \leq \sum_{i \in S} p \cdot r^i$. This is a contradiction. The allocation \{\(y^i, i \in S\)\} cannot simultaneously be smaller or equal to the sum of endowments $r^i$ coordinatewise and be more expensive at prices $p, p \geq 0$. The contradiction proves the theorem. QED

**Convergence of the core of a large economy**

**Replication; a large economy**

In replication, the economy keeps cloning itself.
duplicate to triplicate, ..., to $Q$-tuplicate, and so on, the set of core allocations keeps getting smaller, although it always includes the set of competitive equilibria (per Theorem 21.1).

$Q$-fold replica economy, denoted $Q$-$H$. $Q = 1, 2, \ldots$

$\#H \times Q$ agents.

$Q$ agents with preferences $\succeq 1$ and endowment $r^1$,

$Q$ agents with preferences $\succeq 2$ and endowment $r^2$, $\ldots$, and $Q$ agents with preferences $\succeq \#H$ and endowment $r^{\#H}$. Each household $i \in H$ now corresponds to a household type. There are $Q$ individual households of type $i$ in the replica economy $Q$-$H$.

Competitive equilibrium prices in the original $H$ economy will be equilibrium prices of the $Q$-$H$ economy. Household $i$'s competitive equilibrium allocation $x^i$ in the original $H$ economy will be a competitive equilibrium allocation to all type $i$ households in the $Q$-$H$ replica economy. Agents in the $Q$-$H$ replica economy will be denoted by their type and a serial number. Thus, the agent denoted $i, q$ will be the $q$th agent of type $i$, for each $i \in H, q = 1, 2, \ldots, Q$.

Equal treatment

Theorem 22.1 (Equal treatment in the core) Assume C.IV, C.V, and C.VI(SC). Let $\{x^{i,q}, i \in H, q = 1, \ldots, Q\}$ be in the core of $Q$-$H$, the $Q$-fold replica of economy $H$. Then for each $i, x^{i,q}$ is the same for all $q$. That is, $x^{i,q} = x^{i,q'}$ for each $i \in H, q \neq q'$.

Proof of Theorem 22.1 Recall that the core allocation must be feasible. That is,

$$\sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} \sum_{q=1}^{Q} r^i.$$
Equivalently,
\[
\frac{1}{Q} \sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} r^i.
\]
Suppose the theorem to be false. Consider a type \( i \) so that \( x^{i,q} \neq x^{i,q'} \). For each type \( i \), we can rank the consumptions attributed to type \( i \) according to \( \succeq_i \).

For each \( i \), let \( x^{i*} \) denote the least preferred of the core allocations to type \( i \), \( x^{i,q}, q = 1, \ldots, Q \). For some types \( i \), all individuals of the type will have the same consumption and \( x^{i*} \) will be this expression. For those in which the consumption differs, \( x^{i*} \) will be the least desirable of the consumptions of the type. We now form a coalition consisting of one member of each type: the individual from each type carrying the worst core allocation, \( x^{i*} \).

Consider the average core allocation to type \( i \), to be denoted \( \bar{x}^i \).

\[
\bar{x}^i = \frac{1}{Q} \sum_{q=1}^{Q} x^{i,q}.
\]

We have, by strict convexity of preferences (C.VI(SC)),

\[
\bar{x}^i = \frac{1}{Q} \sum_{q=1}^{Q} x^{i,q} \succ_i x^{i*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are not identical,}
\]

and

\[
x^{i,q} = \bar{x}^i = \frac{1}{Q} \sum_{q=1}^{Q} x^{i,q} \sim_i x^{i*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are identical.}
\]

From feasibility, above, we have that

\[
\sum_{i \in H} \bar{x}^i = \sum_{i \in H} \frac{1}{Q} \sum_{q=1}^{Q} x^{i,q} = \frac{1}{Q} \sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} r^i.
\]

In other words, a coalition composed of one of each type (the worst off of each) can achieve the allocation \( \bar{x}^i \). However, for each agent in the coalition, \( \bar{x}^i \succeq_i x^{i*} \) for all \( i \) and \( \bar{x}^i \succ_i x^{i*} \) for
some $i$. Therefore, the coalition of the worst off individual of each type blocks the allocation $x^{i,q}$. The contradiction proves the theorem. QED

$$\text{Core}(Q) = \{x^i, i \in H\}$$ where $x^{i,q} = x^i, q = 1, 2, \ldots, Q$, and the allocation $x^{i,q}$ is unblocked.

Core convergence in a large economy

As $Q$ grows there are more blocking coalitions, and they are more varied. Any coalition that blocks an allocation in $Q-H$ still blocks the allocation in $(Q+1)-H$, but there are new blocking coalitions and allocations newly blocked in $(Q+1)-H$.

Recall the Bounding Hyperplane Theorem:

Theorem 8.1, Bounding Hyperplane Theorem (Minkowski) Let $K$ be convex, $K \subseteq \mathbb{R}^N$. There is a hyperplane $H$ through $z$ and bounding for $K$ if $z$ is not interior to $K$. That is, there is $p \in \mathbb{R}^N, p \neq 0$, so that for each $x \in K, p \cdot x \geq p \cdot z$.

Theorem 22.2 (Debreu-Scarf) Assume C.IV*, C.V, C.VI(SC), and let $X^i = \mathbb{R}^N_+$. Let $\{x^{o_i}, i \in H\} \in \text{core}(Q)$ for all $Q = 1, 2, 3, 4, \ldots$. Then $\{x^{o_i}, i \in H\}$ is a competitive equilibrium allocation for $Q-H$, for all $Q$.

Proof TBA QED