Monotone Comparative Statics

- Loyalty to Stanford
- Comparative Statics Without Calculus
- Optimizer Set Valued
- No concavity
- No differentiability
Motivating Example

\[ x^*(\theta) \equiv \arg \max f(x, \theta), \text{ subject to } \theta \in \Theta; x \in S(\theta) \]

This problem is equivalent to

\[ x^*(\theta) \equiv \arg \max \phi(f(x, \theta)), \text{ subject to } \theta \in \Theta; x \in S(\theta) \]

for any strictly increasing \( \phi(\cdot) \).
\( \phi(\cdot) \) may destroy smoothness or concavity properties of the objective function.
Formulation

- Begin with problems in which $S(\theta)$ is independent of $\theta$ and both $x$ and $\theta$ are real variables.
- Assume existence of a solution.
- Don’t assume uniqueness.
- Generalize Notion of Increasing.
Strong Set Order

**Definition**
For two sets of real numbers $A$ and $B$, we say that $A \geq_s B$ ("$A$ is greater than or equal to $B$ in the strong set order") if for any $a \in A$ and $b \in B$, $\min\{a, b\} \in B$ and $\max\{a, b\} \in A$. 
1. According to this definition $A = \{1, 3\}$ is not greater than or equal to $B = \{0, 2\}$.
2. Includes the standard definition when sets are singletons.
3. $x^*(\cdot)$ is non-decreasing in $\mu$ if and only if $\mu < \mu'$ implies that $x^*(\mu') \geq_s x^*(\mu)$.
4. If $x^*(\cdot)$ is nondecreasing and $\min x^*(\theta)$ exists for all $\theta$, then $\min x^*(\theta)$ is non decreasing.
5. An analogous statement holds for $\max x^*(\cdot)$. 

Comments
Supermodular

Definition
The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is supermodular or has increasing differences in $(x; \mu)$ if for all $x' > x$, $f(x'; \mu) - f(x; \mu)$ is nondecreasing in $\mu$.

- If $f$ is supermodular in $(x; \mu)$, then the incremental gain to choosing a higher $x$ is greater when $\mu$ is higher.
- Supermodularity is equivalent to the property that $\mu' > \mu$ implies that $f(x; \mu') - f(x; \mu)$ is nondecreasing in $x$. 
When $f$ is smooth, supermodularity has a characterization in terms of derivatives.

**Lemma**

A twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is supermodular in $(x; \mu)$ if and only if $D_{12}f(x; \mu) \geq 0$ for all $(x; \mu)$. The inequality in the definition of supermodularity is just the discrete version of the mixed-partial condition in the lemma.
Topkis’s Monotonicity Theorem

Supermodularity is sufficient to draw comparative statics conclusions in optimization problems.

Theorem (Topkis’s Monotonicity Theorem)

If $f$ is supermodular in $(x; \mu)$, then $x^*(\mu)$ is non-decreasing.
Proof.
Suppose $\mu' > \mu$ and that $x \in x^*(\mu)$ and $x' \in x^*(\mu')$.

1. $x \in x^*(\mu)$ implies $f(x; \mu) - f(\min\{x, x'\}; \mu) \geq 0$.
2. This implies that $f(\max\{x, x'\}; \mu) - f(x'; \mu) \geq 0$ (you need to check two cases, $x > x'$ and $x' > x$).
3. By supermodularity, $f(\max\{x, x'\}; \mu') - f(x'; \mu') \geq 0$.
4. Hence $\max\{x, x'\} \in x^*(\mu')$.
5. $x' \in x^*(\mu')$ implies that $f(x'; \mu') - f(\max\{x, x'\}, \mu) \geq 0$.
6. or equivalently $f(\max\{x, x'\}, \mu) - f(x'; \mu') \leq 0$.
7. This implies that $f(\max\{x, x'\}; \mu') - f(x'; \mu') \geq 0$.
8. which by supermodularity implies
   $f(x; \mu) - f(\min\{x, x'\}; \mu) \leq 0$
9. and so $\min\{x, x'\} \in x^*(\mu)$.

\[\square\]
Comment

Don’t be surprised.
Theorem follows from the IFT whenever the standard full-rank condition in the IFT holds.
At a maximum, if $D_{11}f(x^*, \mu) \neq 0$, if must be negative (by the second-order condition), hence the IFT tells you that $x^*(\mu)$ is strictly increasing.
A monopolist solves $\max p(q)q - c(q, \mu)$ by picking quantity $q$. $p(\cdot)$ is the price function and $c(\cdot)$ is the cost function, parametrized by $\mu$.

Let $q^*(\mu)$ be the monopolist’s optimal quantity choice. If $-c(q, \mu)$ is supermodular in $(q, \mu)$ then the entire objective function is.

It follows that $q^*$ is nondecreasing as long as the marginal cost of production decreases in $\mu$. 
It is sometimes useful to “invent” an objective function in order to apply the theorem. For example, if one wishes to compare the solutions to two different maximization problems, $\max_{x \in S} g(x)$ and $\max_{x \in S} h(x)$, then we can apply the theorem to an artificial function, $f$

$$f(x, \mu) = \begin{cases} 
  g(x) & \text{if } \mu = 0 \\
  h(x) & \text{if } \mu = 1 
\end{cases}$$

so that if $f$ is supermodular ($h(x) - g(x)$ nondecreasing), then the solution to the second problem is greater than the solution to the first.
Single-Crossing

Definition
The function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the *single-crossing condition* in $(x; \mu)$ if for all $x' > x$, $\mu' > \mu$

$$f(x'; \mu) - f(x; \mu) \geq 0 \text{ implies } f(x'; \mu') - f(x; \mu') \geq 0$$

and

$$f(x'; \mu) - f(x; \mu) > 0 \text{ implies } f(x'; \mu') - f(x; \mu') > 0.$$
Theorem

If $f$ is single crossing in $(x; \mu)$, then $x^*(\mu) = \arg \max_{x \in S(\mu)} f(x; \mu)$ is nondecreasing. Moreover, if $x^*(\mu)$ is nondecreasing in $\mu$ for all choice sets $S$, then $f$ is single-crossing in $(x; \mu)$. 
Unconstrained Extrema of Real-Valued Functions

Definition
Take $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

$x^*$ is a **local maximizer** $\iff \exists \delta > 0$ such that $\forall x \in B_\delta(x^*)$, $f(x) \leq f(x^*)$

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$x^*$ is a **global maximizer** $\iff \forall x \in \mathbb{R}^n$, we have $f(x) \leq f(x^*)$

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Theorem (First Order Conditions)

If \( f \) is differentiable at \( x^* \), and \( x^* \) is a local maximizer or minimizer then

\[ Df(x) = 0. \]

That is

\[ \frac{\partial f}{\partial x_i}(x^*) = 0, \]

\( \forall \ i = 1, 2, \ldots, n. \)
Define $h: \mathbb{R} \rightarrow \mathbb{R}$ by

$$h(t) \equiv f(x^* + tv)$$

for any $v \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Take the case of a maximizer:
Fix a direction $v$ ($\|v\| \neq 0$).
We have

$$f(x^*) \geq f(x),$$

\forall x \in B_\delta(x^*), \text{ for some } \delta > 0. \text{ In particular for } t \text{ small } (t < \delta \|v\|) \text{ we have}

$$f(x^* + tv) = h(t) \leq f(x^*)$$

Thus, $h$ is maximized locally by $t^* = 0$. 
Our F.O.C. from the $\mathbb{R} \rightarrow \mathbb{R}$ case

$$\implies h'(0) = 0$$

So

$$\implies \nabla f(x^*) \cdot v = 0$$

And since this must hold for every $v \in \mathbb{R}^n$, this implies that

$$\nabla f(x^*) = 0$$

We know that if $f$ is differentiable, then $Df$ is represented by the matrix of partial derivatives. Hence $Df(x^*) = 0$.

**Definition**

If $x^*$ satisfies $Df(x^*) = 0$, then it is a *critical point* of $f$. 
Intuition

1. Like one-variable theorem.
2. If $x^*$ is a local maximum, then the one variable function you obtain by restricting $x$ to move along a fixed line through $x^*$ (in the direction $v$) also must have a local maximum.
3. Hence all directional derivatives are zero.
4. The first-derivative test cannot distinguish between local minima and local maxima, but an examination of the proof tells you that at local maxima derivatives decrease in the neighborhood of a critical point.
5. Critical points may fail to be minima or maxima.
6. One variable case: a function decreases if you reduce $x$ (suggesting a local maximum) and increases if you increase $x$ (suggesting a local minimum).
7. Generalization: this behavior could happen in any direction.
8. Also: the function restricted to direction has a local maximum, but it has a local minimum with respect to another direction.
9. Conclude: It is “hard” for critical point of a multivariable function to be a local extremum in the many variable case.